

Improved lattice operators: Case of the topological charge density

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We analyze the properties of a class of improved lattice topological charge density operators, constructed by a smearinglike procedure. By optimizing the choice of the parameters introduced in their definition we find operators having (i) a better statistical behavior as estimators of the topological charge density on the lattice, i.e., less noisy, (ii) a multiplicative renormalization much closer to one, and (iii) a large suppression of the perturbative tail (and other unphysical mixings) in the corresponding lattice topological susceptibility.

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I. INTRODUCTION

In QCD an important role is played by topological properties. By the axial anomaly, matrix elements or correlation functions involving the topological charge density operator $q(x)$ can be related to relevant quantities of hadronic phenomenology. We mention the topological susceptibility χ , which is determinant in the explanation of the $U_A(1)$ problem [1], and the on-shell nucleon matrix element of $q(x)$, which can be related to the so-called spin content of the nucleon [2].

Lattice techniques represent our best source of nonperturbative calculations; however, investigating the topological properties of QCD on the lattice is a nontrivial task. In a lattice theory the field is defined on a discretized set and therefore the associated topological properties are strictly trivial. One relies on the fact that the physical continuum topological properties should be recovered in the continuum limit.

From a field theoretical point of view, i.e., considering the lattice as a regulator, difficulties may come from unphysical divergences proportional to powers of the cutoff, which must be removed and therefore make the extraction of the physical signal hard. In order to get reliable quantitative estimates of physical quantities, one should control the unphysical cutoff-dependent corrections even when they disappear in the continuum limit, given that numerical simulations are performed at finite lattice spacings, i.e., at finite values of the cutoff. Such corrections may be relevant, in that the typical values of the bare coupling g_0^2 where simulations are usually performed are actually not small, but $g_0^2 \approx 1$, thus few terms in perturbation theory are not always reliable.

Considering a lattice version of $q(x)$, $q_L(x)$, the classical continuum limit must be in general corrected by including a renormalization function. In pure QCD, where $q(x)$ is renormalization-group invariant [3],

$$q_L(x) \rightarrow a^4 Z(g_0^2) q(x) + O(a^6), \quad (1)$$

where $Z(g_0^2)$ is a finite function of the bare coupling g_0^2 going to one in the limit $g_0^2 \rightarrow 0$, but at $g_0^2 \approx 1$ it may be very different from one. The finite renormalization of the widely used lattice operator [4]

$$q_L(x) = -\frac{1}{2^9 \pi^2} \sum_{\mu\nu\rho\sigma=\pm 1}^{\pm 4} \epsilon_{\mu\nu\rho\sigma} \text{Tr}[\Pi_{\mu\nu} \Pi_{\rho\sigma}] \quad (2)$$

[$\Pi_{\mu\nu}(x)$ is the product of link variables $U_\mu(x)$ around a 1×1 plaquette] is quite non-negligible: for $SU(3)$ $Z(g_0^2=1) \approx 0.18$ [5].

The relation of the zero-momentum correlation of two $q_L(x)$ operators,

$$\chi_L = \sum_x \langle q_L(x) q_L(0) \rangle, \quad (3)$$

with the topological susceptibility χ , is further complicated by an unphysical background term, which eventually becomes dominant in the continuum limit. (We recall that the definition of χ requires also a prescription to define the product of operators [6].) Indeed

$$\chi_L(g_0^2) = a^4 Z(g_0^2)^2 \chi + M(g_0^2). \quad (4)$$

Neglecting terms $O(a^6)$, the background term $M(g_0^2)$ can be written in terms of mixings with the unity operator (so-called perturbative tail scaling as $\sim a^0$) and with the trace of the energy-momentum (scaling as $\sim a^4$). In the case of the operator (2) and for $SU(3)$, $M(g_0^2)$ is already dominant at $g_0^2 \approx 1$: it is about 85% of χ_L at $g_0^2 = 1$ [7]. As a consequence the uncertainty on χ can be hardly made smaller than $\approx 10\%$ by using the operator (2) and the heating method to evaluate $Z(g_0^2)$ and $M(g_0^2)$ [8–10].

Another problem, which has come up in some studies concerning the lattice determination of the on-shell proton matrix element of $q(x)$ [11,5], is that the lattice operator (2) is very noisy, requiring very accurate statistics and therefore expensive simulations in order to get a reasonable uncertainty on the final result. In view of a full QCD lattice calculation the search for a better estimator appears a necessary step.

We study, within the field theoretical approach, the possibility of improving the lattice estimator of $q(x)$ with respect to all the problems listed above, that is we look for local versions of $q(x)$ which are less noisy, have a multiplicative renormalization closer to one, and whose corresponding χ_L is

not dominated by the unphysical background signal $M(g_0^2)$ in the region $g_0^2 \approx 1$. [Any χ_L defined from a local $q_L(x)$ will eventually be dominated by its perturbative tail in the continuum limit. For the purpose of evaluating χ it would suffice to have a small tail at $g_0^2 \approx 1$, which should be already in the scaling region.]

II. IMPROVED TOPOLOGICAL CHARGE DENSITY OPERATORS

Inspired by the widely used smearing techniques we consider the following set of operators defined in terms of smeared links $V_\mu^{(i)}(x)$:

$$q_L^{(i)}(x) = -\frac{1}{2^9 \pi^2} \sum_{\mu\nu\rho\sigma = \pm 1}^{\pm 4} \epsilon_{\mu\nu\rho\sigma} \text{Tr}[\Pi_{\mu\nu}^{(i)} \Pi_{\rho\sigma}^{(i)}], \quad (5)$$

where $\Pi_{\mu\nu}^{(i)}$ is the product of smeared links $V_\mu^{(i)}(x)$ around a 1×1 plaquette. Such smeared links are constructed by the procedure

$$\begin{aligned} V_\mu^{(0)}(x) &\equiv U_\mu(x), \\ \hat{V}_\mu^{(i)}(x) &= (1-c)V_\mu^{(i-1)}(x) + \frac{c}{6} \sum_{\pm\nu, \nu \neq \mu} V_\nu^{(i-1)}(x) \\ &\quad \times V_\mu^{(i-1)}(x+\nu) V_\nu^{(i-1)}(x+\mu)^\dagger, \\ V_\mu^{(i)}(x) &= \frac{\hat{V}_\mu^{(i)}(x)}{\left[\frac{1}{N} \text{Tr} \hat{V}_\mu^{(i)}(x)^\dagger \hat{V}_\mu^{(i)}(x) \right]^{1/2}}, \end{aligned} \quad (6)$$

where $V_{-\nu}^{(i)}(x) = V_\nu^{(i)}(x-\nu)^\dagger$. $V_\mu^{(i)}(x)$ and therefore $q_L^{(i)}(x)$ depend on the parameter c , which can be tuned to optimize the properties of $q_L^{(i)}(x)$. All these operators have the correct classical continuum limit, i.e., for $a \rightarrow 0$, $q_L^{(i)}(x) \rightarrow a^4 q(x)$.

Notice that the size of $q_L^{(i)}(x)$ increases with increasing the integer parameter i . Nevertheless $q_L^{(i)}(x)$ can be still considered as local operators when keeping i fixed while approaching the continuum limit. Also, as we shall see, by optimizing the choice of the parameter c , a good improvement with respect to $q_L^{(0)}(x) \equiv q_L(x)$ is already achieved for small values of i . For $SU(2)$ the procedure (6) keeps the smeared links $V_\mu^{(i)}(x)$ belonging to the $SU(2)$ group, and it is equivalent to the smearing procedures proposed in Ref. [12]. For $N \geq 3$ the smeared links no longer belong to the $SU(N)$ group.

The procedure (6) may be used to improve any local operator involving link variables. Smearing methods to improve lattice estimators have been already widely employed in the study of long-distance correlations, such as large Wilson loops and hadron source operators.

One often adopts an equivalent ‘‘Schrödinger picture’’ of smearing, whereby lattice operators retain their original definition, while all links in the configuration undergo transformation (6). Full consistency of this picture would then require that $V_\mu^{(i)}(x)$ be unitary. [As it stands, $V_\mu^{(i)}(x)$ is only unitary in the case of $SU(2)$.] Projecting a matrix V onto

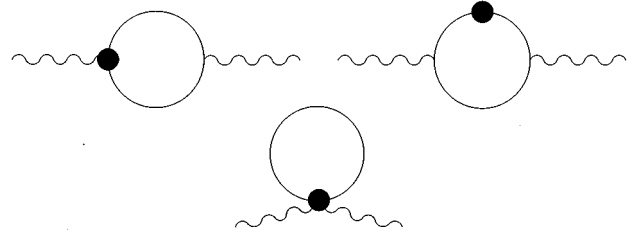


FIG. 1. One-loop diagrams contributing to the multiplicative renormalization of $q_L^{(i)}(x)$.

$SU(N)$ amounts to finding $X \in SU(N)$ which minimizes $\text{Tr}[(X^\dagger - V^\dagger)(X - V)]$, or equivalently maximizes $\text{Tr}(X^\dagger V + V^\dagger X)$. The solution is given by

$$X = i\alpha V^{-1} + (V^\dagger V - \alpha^2 I)^{1/2} V^{-1}, \quad (7)$$

where α is the real root of the equation:

$$\prod_i ((d_i^2 - \alpha^2)^{1/2} + i\alpha) = \det V \quad (8)$$

and $d_i \geq 0$ are the eigenvalues of $(V^\dagger V)^{1/2}$. It can be verified that the lower loop results for $Z(g_0^2)$ and the perturbative tail $P(g_0^2)$ (see Sec. III) are not modified by rendering $V_\mu^{(i)}(x)$ unitary as above. It is worth mentioning at this point that abrupt cooling leads to exactly the same unitary links X , for $c=1$. Indeed, cooling reassigns to each link a new value, $X_\mu(x)$ in a way as to minimize the action, i.e., maximize: $\text{Tr}[X_\mu(x) V_\mu^{(i)}(x)^\dagger + X_\mu(x)^\dagger V_\mu^{(i)}(x)]$ at $c=1$, which coincides with Eq. (7).

For $N \geq 3$, instead of projecting back onto the $SU(N)$ group we propose last step of the procedure (6), which is simpler and should retain most of the advantages of the standard smearing procedure.

III. PERTURBATIVE ANALYSIS

We have calculated $Z^{(1)}(g_0^2)$ to one loop for the once-smearing operator $q_L^{(1)}(x)$ with the Wilson action. To carry out this calculation, $q_L^{(1)}(x)$ is expanded in a Taylor series in the gauge field $A_\mu(x)$, where $U_\mu(x) = \exp[ig_0 A_\mu(x)]$. In Fig. 1 we show the three diagrams contributing to $Z^{(1)}$. We find

$$\begin{aligned} Z^{(1)} &= 1 + z_1 g_0^2 + O(g_0^4), \\ z_1 &= N \left[\frac{1}{4N^2} - \frac{1}{8} - \frac{1}{2\pi^2} - I_0 + c \left(0.67789 - \frac{0.24677}{N^2} \right) \right. \\ &\quad \left. + c^2 \left(-0.48436 + \frac{0.03991}{N^2} \right) \right], \end{aligned} \quad (9)$$

where $I_0 = 0.15493$. At $c=0$ we recover the nonsmeared results [3]:

$$\begin{aligned} Z &= 1 - 0.5362 g_0^2 + O(g_0^4) \quad \text{for } N=2, \\ Z &= 1 - 0.9084 g_0^2 + O(g_0^4) \quad \text{for } N=3, \end{aligned} \quad (10)$$

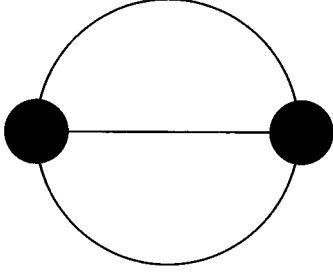


FIG. 2. Diagram contributing to the lowest order term of the perturbative tail.

which do not lead to a reliable estimate of $Z(g_0^2 \approx 1)$. As c varies, the following extreme values of Z are obtained:

$$Z = 1 - 0.1360g_0^2 + O(g_0^4) \quad (c = 0.6495), \quad N = 2, \quad (11)$$

$$Z = 1 - 0.2472g_0^2 + O(g_0^4) \quad (c = 0.6774), \quad N = 3.$$

In both cases, Z is quite close to unity for typical values of g_0^2 , making the one loop estimate more reliable. It is noteworthy that the last step in the smearing procedure (6) turns out to be essential to make Z approach one for $c \geq 0$.

For $q_L^{(1)}(x)$ we have also calculated the lowest perturbative contribution to the mixing with the unity operator $P(g_0^2)$, which is the dominant part of the background term $M(g_0^2)$ in the continuum limit. The corresponding diagram is shown in Fig. 2 and leads to the result

$$P(g_0^2) = g_0^6 \frac{3N(N^2 - 1)}{128\pi^4} p(c) + O(g_0^8),$$

$$p(c) = 0.002867 - 0.017685c + 0.048665c^2 - 0.075362c^3 + 0.068526c^4 - 0.034433c^5 + 0.007445c^6. \quad (12)$$

The minimum of this everywhere-concave polynomial is $p(c = 0.872) = 1.4 \times 10^{-5}$. Thus, for all N , the leading order of $P(g_0^2)$ diminishes by more than 2 orders of magnitude compared to its nonsmeared value ($c = 0$).

In the presence of dynamical fermions one should take into account the fact that, unlike pure gauge theory, the topological charge density mixes under renormalization with $\partial_\mu j_\mu^5$, where j_μ^5 is the singlet axial vector current. The nonrenormalizability property of the anomaly in the modified minimal subtraction (MS) scheme means that the anomaly equation should take exactly the same form in terms of bare or renormalized quantities. However the renormalization of $\partial_\mu j_\mu^5(x)$ and $q(x)$ is nontrivial [13]. A renormalization-group analysis leads to the following relation valid for all matrix elements of a lattice version $q_L(x)$ of $q(x)$ in the chiral limit [14]:

$$\langle i2N_f q_L \rangle = Y(g_0^2) \langle R \rangle, \quad (13)$$

where $Y(g_0^2)$ is a finite function of g_0^2 , and

$$\langle R \rangle \equiv \langle \partial_\mu j_\mu^5(x) \rangle_{R_{\overline{\text{MS}}}} \exp \int_{g(\mu)}^0 \frac{\tilde{\gamma}(\tilde{g})}{\beta_{\overline{\text{MS}}}(\tilde{g})} d\tilde{g} \quad (14)$$

is a renormalization-group invariant quantity; $\partial_\mu j_\mu^5(x)_{R_{\overline{\text{MS}}}}$ indicates the operator $\partial_\mu j_\mu^5(x)$ renormalized in the MS scheme, and the function $\tilde{\gamma}(g)$ is related to the anomalous dimension of the continuum operators $q(x)$, $\partial_\mu j_\mu^5(x)$ in the MS scheme: $\tilde{\gamma}(g) = (1/16\pi^4)(3c_f/2)N_f g^4 + O(g^6)$. Notice that $\langle R \rangle$ is what can be naturally extracted also from experimental data.

In perturbation theory one finds $Y(g_0^2) = 1 + (z_1 + y_1)g_0^2 + O(g_0^4)$, where z_1 is the coefficient of the $O(g_0^2)$ term of the finite renormalization of q_L in the pure gauge theory [cf. Eq. (9)], and y_1 turns out to be a small number: $y_1 = -0.0486$ for $N = 3$ and $N_f = 4$ [14].

IV. NONPERTURBATIVE ANALYSIS BY THE HEATING METHOD

Estimates of the multiplicative renormalizations of the operators $q_L^{(i)}(x)$ and of the background term in the corresponding χ_L can be obtained using the numerical heating method [8,9], without any recourse to perturbation theory. This method relies on the idea that the multiplicative renormalization $Z(g_0^2)$ and the background term $M(g_0^2)$ is produced by short-ranged fluctuations at the scale of the cutoff a . Therefore, when using a standard local algorithm (for example, Metropolis or heat bath) to reach statistical equilibrium, the modes contributing to Z and M should not suffer from critical slowing down, unlike global quantities, such as the topological charge, which should experience a severe form of critical slowing [10].

We applied the heating method to the operators $q_L^{(i)}(x)$ for $i = 1, 2$ and for a number of values of c in the region $0 \leq c \leq 1$. We restricted our analysis to the SU(2) pure gauge theory, expecting no substantial differences for $N = 3$. The measurements were performed at $\beta = 2.6$ ($g_0^2 = 1.5384 \dots$), which is a typical value for the SU(2) simulations with the Wilson action. The local updating was performed using the heat-bath algorithm.

An estimate of Z can be obtained by heating a configuration \mathcal{E}_0 which is an approximate minimum of the lattice action and carries a definite topological charge $Q_{L,0}$. Such a configuration has been constructed by discretizing an instanton solution in the singular gauge

$$A_\mu(x) = \frac{\rho^2}{x^2 + \rho^2} \frac{i}{2} (s_\mu^\dagger s_\nu - s_\nu^\dagger s_\mu) \frac{x_\nu}{x^2}, \quad (15)$$

where $s_4 = 1$ and $s_k = i\sigma_k$, and exponentiating it to define link variables $U_\mu(x) = \exp[iA_\mu(x)]$. Then a few cooling steps (about 5) were performed to make the configuration smoother. On a lattice 14^4 and choosing $\rho = 6$ we obtained an instantonlike configuration carrying a topological charge $Q_{L,0} \approx 0.96$ (all improved operators we considered gave approximately the same value for \mathcal{E}_0).

One then constructs ensembles \mathcal{E}_n of many independent configurations obtained by heating \mathcal{E}_0 for the same number

TABLE I. We present $Z^{(i)}(\beta=2.6)$ for $i=1,2$ and various values of c , as obtained by the heating method for a number of ≈ 750 trajectories. The errors displayed include both a statistical error (determined by the typical errors of data in the plateau) and a systematic error related to the stability of the background configuration. The numbers in this table should be compared with the value $Z(\beta=2.6)=0.25(2)$ for the standard operator (2) [10].

i	$c=0.6$	$c=0.8$	$c=1.0$
1	0.52(2)	0.57(2)	0.54(2)
2	0.68(2)	0.75(2)	0.68(2)

n of updating steps, averaging $Q_L^{(i)} = \sum_x q_L^{(i)}(x)$ over \mathcal{E}_n at fixed n . Let us define $Q_{L,n}^{(i)} \equiv \langle Q_L^{(i)} \rangle_{\mathcal{E}_n}$, i.e., the average on the ensemble \mathcal{E}_n . Fluctuations of length $l \approx a$ contributing to Z should rapidly thermalize, while the topological structure of the initial configuration is left (approximately) unchanged for a long time. After a few heating steps where the short-ranged modes contributing to Z get thermalized, $Q_{L,n}$ should show a plateau approximately at $ZQ_{L,0}$. The estimates of $Z^{(i)}(\beta=2.6)$ from the plateaus observed in the heating procedure are reported in Table I, and should be compared with the value $Z(\beta=2.6)=0.25(2)$ for the standard operator (2) [10]. The plateaus formed by the ratios $Q_{L,n}^{(i)}/Q_{L,0}^{(i)}$ starting from $n \approx 6$ are clearly observed in Fig. 3, where data for $i=1,2$ and $c=0.8$ are plotted versus n . Checks of the stability of the background topological structure of the initial configuration were performed at $n=8,10$, by cooling back the configurations (locally minimizing the action) finding $Q_L \approx Q_{L,0}$ after few cooling steps.

This analysis confirms the one-loop perturbative calculations, that is the improved operators we considered have a

multiplicative renormalization closer to one than that of the initial operator $q_L(x)$. From $Z(\beta=2.6) \approx 0.25$ of $q_L(x)$, we pass, by roughly optimizing with respect to the parameter c , to $Z^{(1)}(c=0.8, \beta=2.6) \approx 0.57$ by one improving step, and $Z^{(2)}(c=0.8, \beta=2.6) \approx 0.75$ by two improving steps. For larger i we expect to get $Z^{(i)}$ closer and closer to one, as also suggested from the results of the cooling method [15]. On the other hand we should not forget that increasing the number of improving steps the size of the operator $q_L^{(i)}(x)$ increases. One should find a reasonable compromise taking into account the size of the lattice one can afford in the simulations.

A comparison of the above results for $i=1$ with the one-loop calculation (9) shows that the contribution of the higher perturbative orders is still non-negligible, but not so relevant as in the case of the operator without improving.

Another important property of the improved operators we can infer from the heating method results is that they are much less noisy than $q_L(x)$ at fixed background. In other words, in the Monte Carlo determinations of the matrix elements of $q(x)$ the contribution of short-ranged fluctuations to the error is largely suppressed. A quantitative idea of this fact may come from the quantity $e^{(i)} \equiv \Delta Z^{(i)}/Z^{(i)}$, where $\Delta Z^{(i)}$ is the typical error of the data in the plateau during the heating procedure described above. We indeed found for $c \approx 1.0$ and for an equal number of measurements

$$\frac{e^{(0)}}{e^{(1)}} \approx 6, \quad \frac{e^{(0)}}{e^{(2)}} \approx 15. \quad (16)$$

An estimate of the background signal $M(g_0^2)$ can be obtained by measuring $\chi_{L,n} = \langle Q_L^2 \rangle_{\mathcal{E}_n} / V$ on ensembles of configurations \mathcal{E}_n constructed by heating the flat configuration

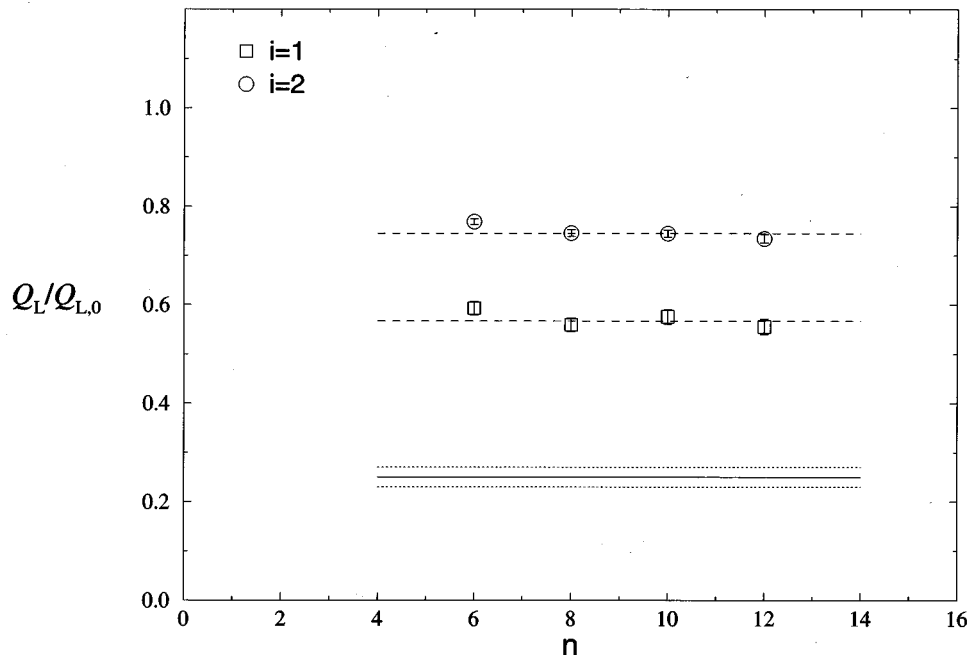


FIG. 3. $Q_{L,n}/Q_{L,0}$ versus n in the heating procedure of an instanton configuration. Data for $i=1,2$ at $c=0.8$ are shown. For comparison the solid line represents the estimate of Z for the standard operator (2).

TABLE II. We present $M^{(i)}(\beta=2.6)$ for $i=1,2$ and various values of c , as obtained by the heating method (1000 trajectories), i.e., from the value of χ_L at the plateau, observed around $n \approx 16$ for all operators considered. The numbers in this table should be compared with the value $M(\beta=2.6) = 2.10(5) \times 10^{-5}$ for the standard operator (2).

i	$c=0.6$	$c=0.8$	$c=1.0$
1	$0.60(2) \times 10^{-5}$	$0.37(2) \times 10^{-5}$	$0.27(2) \times 10^{-5}$
2	$0.23(2) \times 10^{-5}$	$0.13(2) \times 10^{-5}$	$0.07(2) \times 10^{-5}$

for the same number n of updating steps [9,10]. Measurements were performed on a 12^4 lattice. The plateau showed after few heating steps ($n \approx 14$ in this case) by the data of $\chi_{L,n}^{(i)}$ should be placed approximately at the value of $M^{(i)}(g_0^2)$, since no topological activity is detected there, i.e., the background is still flat (this is checked by cooling back the heated configurations), while the other modes contributing to $M(g_0^2)$ should be already approximately thermalized (for a discussion of this issue see [10]). The estimates of $M^{(i)}(\beta=2.6)$ from the plateaus observed during heating are given in Table II, and should be compared with the value $M(\beta=2.6) = 2.10(5) \times 10^{-5}$ relative to the standard operator (2). In Fig. 4 we plot $\chi_{L,n}^{(i)}$ for $i=1,2$ and $c=1.0$ as a function of the heating step n , and compare with the corresponding data for the standard operator. The expected plateaus are observed from $n \approx 14$.

Notice the strong suppression of the background term in the improved operators. For $c \approx 1$ the reduction is about a factor 8 when performing one improving step, and about a factor 30 by two improving steps. For a larger number of improving steps, the suppression is expected to be larger.

The suppression of the background term in Eq. (4) together with the relevant increase of Z should drastically

change the relative weights of the contributions to χ_L in the relevant region for Monte Carlo simulations. By a standard Monte Carlo simulation at $\beta=2.6$ on a 16^4 lattice, we measured $\chi_L^{(i)}$ for $i=1,2$ and $c=0.6,0.8,1.0$. We performed 15 000 sweeps using an overrelaxed algorithm; this sample size is already sufficient to show the better properties of the operators $q_L^{(i)}(x)$. Data for $\chi_L^{(i)}$ are given in Table III. For comparison we also calculated $a^4\chi$ by cooling [16].

For the standard operator we found $\chi_L = 2.21(11) \times 10^{-5}$, which, due to the large corresponding background term $M = 2.10(5) \times 10^{-5}$, does not allow one to determine $a^4\chi$ at this value of β . Instead the improved operators $q_L^{(i)}(x)$ provide, using Eq. (4), reliable estimates of $a^4\chi$ having about 10% of uncertainty, which are consistent with each other and are also consistent with the determination from cooling: $\chi_{\text{cool}} = 1.3(2) \times 10^{-5}$, although the latter seems to be systematically lower. This fact may be explained taking into account that $Q_L = \sum_x q_L(x)$, which is used to estimate the topological charge of cooled configurations, underestimates the topological charge content (for the lattice size we are working with), as we found out explicitly when we constructed an instanton configuration on the lattice.

The determinations of Z and M should not be subject to relevant finite-size scaling effects (as explicitly checked in Ref. [10]), since they have their origin in short-ranged fluctuations. Thus finite-size corrections to our estimates of Z and M should be negligible. Larger finite-size effects are expected on the topological modes, as can be argued from numerical studies available in the literature. For this reason the measurement of χ_L , which receives contributions also from topological modes, was performed on a larger lattice. We should say that we did not perform a complete analysis of the finite-size corrections to χ_L , since our purpose was just to show the better behavior of the improved operators

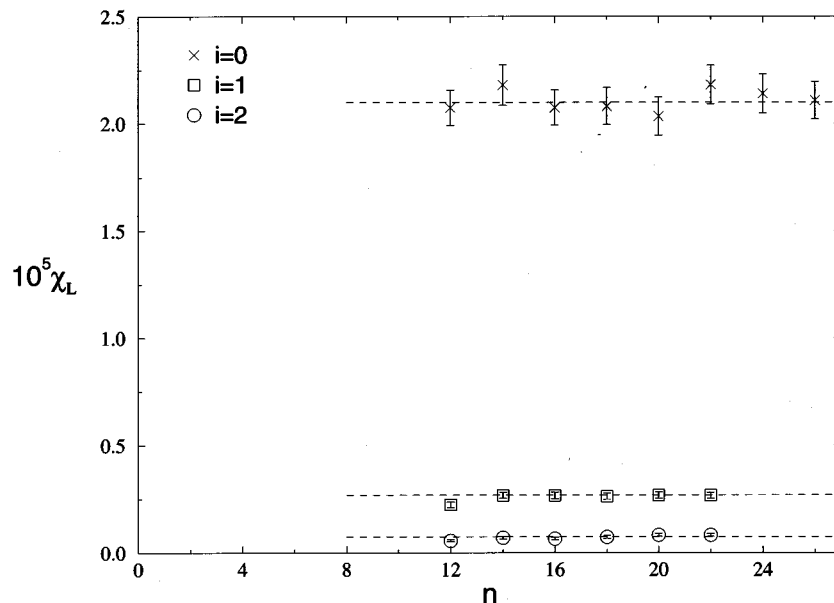


FIG. 4. $\chi_{L,n}$ versus n in the heating procedure of the flat configuration. Data for the standard operator and improved operators for $i=1,2$ at $c=1.0$ are shown.

TABLE III. We present $\chi_L^{(i)}(\beta=2.6)$ for $i=1,2$ and various values of c , as obtained by a standard Monte Carlo simulation on a 16^4 lattice (15 000 sweeps using an overrelaxed algorithm). We also give the corresponding values of $a^4\chi$ as obtained from Eq. (4). Data for $a^4\chi$ must be compared with the cooling result: $\chi_{\text{cool}}=1.3(2)\times 10^{-5}$.

	i	$c=0.6$	$c=0.8$	$c=1.0$
χ_L	1	$1.02(7)\times 10^{-5}$	$0.89(7)\times 10^{-5}$	$0.74(7)\times 10^{-5}$
	2	$0.93(7)\times 10^{-5}$	$0.92(7)\times 10^{-5}$	$0.71(6)\times 10^{-5}$
$a^4\chi=(\chi_L-M)/Z^2$	1	$1.5(3)\times 10^{-5}$	$1.6(3)\times 10^{-5}$	$1.6(3)\times 10^{-5}$
	2	$1.5(2)\times 10^{-5}$	$1.4(2)\times 10^{-5}$	$1.4(2)\times 10^{-5}$

$q_L^{(i)}(x)$ and not the determination of χ for the SU(2) gauge model. So we limited ourselves to a numerical study not requiring a supercomputer.

If the improvement for SU(3) is similar to that achieved for SU(2), using the optimal operator for $i=2$ at $g_0^2=1$ the unphysical term in Eq. (4) is expected to become a small part of the total signal, allowing a precise determination of χ by the field theoretical method.

V. CONCLUSIONS

We have analyzed the properties of a class of improved lattice topological charge density operators constructed by a smearinglike procedure. Such operators look promising for the lattice calculation of the on-shell proton matrix element of the topological charge density operator in full QCD, which is related to the so-called proton spin content. Indeed their use should overcome the difficulty due to the large noise observed in preliminary quenched studies [11,5], and

they have a multiplicative renormalization much closer to one.

Improved operators are also expected to provide a much better determination of the topological susceptibility by the field theoretical method in the SU(3) gauge theory, by strongly reducing the unphysical background term while enhancing the term containing χ with larger values of the multiplicative renormalization. This should allow a precise and independent check of the alternative cooling method determinations (see, e.g., Refs. [16,17]), whose systematic errors are not completely controlled. Furthermore the improved operators may also open the road to a more reliable lattice investigation of the behavior of the topological susceptibility at the deconfinement transition, where cooling does not give satisfactory results [18].

The smearinglike procedure (6) may be used to improve any local operator involving link variables, and a renormalization study would again be called for in all cases. We hope to return to this issue in the future.

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