Semilocal self-dual Chern-Simons solitons

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We consider a nonrelativistic Chern-Simons theory of planar matter fields interacting with the Chern-Simons gauge field in a $SU(N)_{\text{global}}\times U(1)_{\text{local}}$ invariant fashion. We find that this model admits static zero-energy self-dual soliton solutions. We also present a set of exact soliton solutions. The exact time-dependent solutions are also obtained, when this model is considered in the background of an external uniform magnetic field.

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Soliton solutions in Chern-Simons (CS) gauge theories have received considerable attention over the past few years due to their possible relevance to the planar condensed matter systems. It is known that the Abelian Higgs model with a CS term admits finite energy charged vortex solutions [1]. Further, the pure CS Higgs theory admits static self-dual soliton solutions with a ϕ^6-type scalar potential [2]. Moreover, in the nonrelativistic limit of this theory [3], the charge density solves the Liouville equation at the self-dual limit, all of whose solutions are well known. When this nonrelativistic model is modified by including an external magnetic field [4] or a harmonic force [5], exact time-dependent soliton solutions can be obtained. The self-dual nonrelativistic case for the non-Abelian gauge group has also been considered [6], which provides a unified dynamical framework for a variety of two-dimensional nonlinear equations [7].

In this Brief Report, we consider a nonrelativistic CS theory with a gauge group as in the case of semilocal Nielsen-Olesen strings [8] or semilocal charged vortices [9,10]. In particular, we consider the Jackiw-Pi (JP) model [3] but with the gauge group enlarged to $SU(N)_{\text{global}}\times U(1)_{\text{local}}$. We find that this model admits static zero-energy self-dual soliton solutions. Interestingly enough, we are also able to find a set of exact soliton solutions. These solitons are characterized by the magnetic flux $\Phi = -\frac{2\pi}{e}\frac{\kappa}{|\kappa|}(N+1)|\,n\,|$, the charge $Q = -\frac{\kappa}{e}\Phi$, and the angular momentum $J = Q$, where n is the winding number and κ and e are two-dimensional constants to be discussed below. We also present exact time-dependent solutions of the model in the presence of an external uniform magnetic field.

Consider the nonrelativistic Lagrangian

$$\mathcal{L} = i\Psi^\dagger\left(\partial_t + ieA^0\right)\Psi - \frac{1}{2m}|\,(\partial_i + ieA_i)\Psi\,|^2 + \frac{g}{2}(\Psi^\dagger\Psi)^2 + \frac{\kappa}{4}\epsilon^{\mu\nu\alpha}A_\mu F_{\nu\alpha}, \quad (1)$$

where Ψ is N component scalar field, i.e., $\Psi^\dagger = (\psi_0^*, \psi_1^*, \ldots, \psi_{N-1}^*)$ (here $*$ denotes complex conjugation). The Lagrangian (1) is invariant under a $SU(N)_{\text{global}}\times U(1)_{\text{local}}$ transformations. For $N = 1$, the Lagrangian (1) essentially describes the JP model. The $N = 2$ case was previously discussed and some exact solutions were obtained in Ref. [11]. Note that the scalar-field self-interaction may be attractive or repulsive according as g is positive or negative, respectively. However, as we will see shortly, the self-interaction is always attractive for zero-energy self-dual soliton solutions as in the case of JP model.

The equations of motion which follow from (1) are

$$\frac{\kappa}{2}\epsilon^{\nu\alpha\beta}F_{\alpha\beta} = eJ^\nu, \quad (2)$$

$$i\partial_t\Psi = -\frac{1}{2m}D_iD_i\Psi + eA^0\Psi - g|\,\Psi\,|^2\Psi, \quad (3)$$

where the conserved matter current J^ν is given by

$$J^\nu = (\rho, J^i) = \left[\Psi^\dagger\Psi, \frac{i}{2m}[\Psi^\dagger(D^i\Psi) - (D^i\Psi)^\dagger\Psi]\right]. \quad (4)$$

The zero component of (2), i.e., Gauss' law, implies that the solution with charge Q also carries magnetic flux $\Phi = -\frac{e}{\kappa}Q$. Equation (3) is a (2+1)-dimensional gauged nonlinear Schrödinger equation where the gauge-field variables can be expressed solely in terms of the matter-field variables with the help of Eq. (2).

The energy for the Lagrangian (1) is

$$E = \int d^2x\left[\frac{1}{2m}(D_i\Psi)^\dagger D_i\Psi - \frac{g}{2}(\Psi^\dagger\Psi)^2\right], \quad (5)$$

which can be rewritten using the Bogomol'nyi [12] trick as

$$E = \int d^2x\left[\frac{1}{2m}|\,(D_1 \pm iD_2)\Psi\,|^2 - \left(\frac{g}{2} \pm \frac{e^2}{2m\kappa}\right)(\Psi^\dagger\Psi)^2\right], \quad (6)$$

where a surface term has been dropped since it vanishes for the well-behaved field variables. Now note that for the choice of g as $g = \mp\frac{e^2}{m\kappa}$, the energy satisfies the bound

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$E \geq 0$. The bound is saturated when the following first order self-dual equations are satisfied:

$$(D_1 \pm iD_2)\Psi = 0 \ . \tag{7}$$

It should be noted that the Eq. (7) is identical to the corresponding equation of Ref. [3] except that Ψ now is a N component scalar field.

In order to solve Eq. (7), we write down the gauge potential A_i in the Coulomb gauge,

$$A_i = -\frac{1}{e}\epsilon_{ij}\partial_j\chi \ . \tag{8}$$

Now it is trivial to check that Eq. (7) can be rewritten as

$$(\partial_1 \pm i\partial_2)e^{\mp\chi}\psi_j = 0 \ . \tag{9}$$

Thus we have the general solution to Eq. (9) in the form [13]

$$\psi_j = e^{\pm\chi}f_j(z) \ , \tag{10}$$

where $z = x + iy$ and $f_j(z)$'s are arbitrary analytic functions. With the help of Gauss' law and Eqs. (8) and (10), the decoupled equation for the χ is

$$\nabla^2\chi = \frac{e^2}{\kappa}e^{\pm 2\chi}\sum_{j=0}^{N-1} \mid f_j(z) \mid^2, \tag{11}$$

where $\nabla^2 = 4\partial_z\partial_{\bar{z}}$. Note that Eq. (11) reduces to the Liouville equation in case the summation on the right hand side is equal to some real constant. However, we are interested here in more general solutions. Let us first discuss the simplest case of $N = 2$, in which case Eq. (11) takes the form

$$\nabla^2\chi = \frac{e^2}{\kappa}e^{\pm 2\chi}\left(|f_0(z)|^2 + \mid f_1(z) \mid^2\right). \tag{12}$$

The above equation can be solved exactly provided we assume a particular form for $f_1(z)$ in terms of $f_0(z)$. In particular, we choose

$$\mid f_1(z) \mid^2 = \mid f_0(z) \mid^2 \left|\int f_0(z)dz\right|^2 . \tag{13}$$

The solution of Eq. (12) is

$$\chi = \mp\frac{3}{2}\ln\left[\left(\frac{\alpha}{6}\right)^{\frac{1}{3}}\left(1 + \left|\int f_0(z)dz\right|^2\right)\right], \tag{14}$$

where the sign of κ in (12) must be opposite to that of $\pm$ and $\alpha = \frac{e^2}{|\kappa|}$. Note that for this choice of sign $g = \mp\frac{e^2}{m\kappa}$ is always positive, and hence the scalar-field self-interaction is attractive. If one now chooses $f_0(z) = nc_0z^{n-1}(z = re^{i\theta}, \mid n \mid \geq 1)$, then one obtains the rotationally symmetric solutions

$$\psi_j(r) = \frac{\sqrt{6}\mid n \mid}{\sqrt{\alpha}r}\left(\frac{r}{r_0}\right)^{\frac{\delta}{2}}\left[\left(\frac{r}{r_0}\right)^n + \left(\frac{r_0}{r}\right)^n\right]^{-\frac{3}{2}} e^{i(|n|j+|n|-1)\theta}, \quad j = 0, 1, \tag{15}$$

where $\delta = \mid n \mid (2j - 1)$.

As far as we are aware of, for arbitrary N no exact analytic solution of Eq. (11) is known and at present we do not know how to solve Eq. (11) exactly except in a few specific cases. For example, Eq. (11) can be solved exactly if one assumes

$$\mid f_j(z) \mid^2 = {}^{N-1}C_j \ \mid f_0(z) \mid^2 \left|\int f_0(z)dz\right|^{2j}, \tag{16}$$

where ${}^{N-1}C_j = \frac{(N-1)!}{j!(N-1-j)!}$. After substituting (16) into (11), we have

$$\nabla^2\chi = \frac{e^2}{\kappa}e^{\pm 2\chi}\mid f_0(z) \mid^2\left(1 + \left|\int f_0(z)dz\right|^2\right)^{N-1} . \tag{17}$$

As in the $N = 2$ case we fix the convention that the sign of κ must be opposite to that of $\pm$. Now one can check that

$$\chi = \mp\frac{N+1}{2}\ln\left[a\left(1 + \left|\int f_0(z)dz\right|^2\right)\right] \tag{18}$$

solves Eq. (17), where $a = [\frac{\alpha}{2(N+1)}]^{\frac{1}{N+1}}$. The ψ_j's are thus given by

$$\psi_j = \sqrt{{}^{N-1}C_j}\sqrt{\frac{2(N+1)}{\alpha}}\frac{f_0(z)\left[\int f_0(z)dz\right]^j}{\left[1 + \mid \int f_0(z)dz \mid^2\right]^{\frac{N+1}{2}}} . \tag{19}$$

It should be mentioned at this point that the familiar Liouville solution can be embedded into the $SU(N)_{global}\times U(1)_{local}$ invariant theory for any N by choosing all the $f_j(z)$'s to be equal. Further, for any $N' < N$, the solutions as given by (18) and (19) can be embedded into the higher N theory.

The radially symmetric solutions for ψ_j's can be obtained from Eq. (19) by putting $f(z) = nc_0z^{n-1}$ ($z = re^{i\theta}$). We find

$$\psi_j(r) = \sqrt{{}^{N-1}C_j}\sqrt{\frac{2(N+1)}{\alpha}}\frac{\mid n \mid}{r}\left(\frac{r}{r_0}\right)^{\frac{\delta}{2}}\left[\left(\frac{r}{r_0}\right)^n + \left(\frac{r_0}{r}\right)^n\right]^{-\frac{(N+1)}{2}} e^{i(|n|j+|n|-1)\theta}, \tag{20}$$

where $\delta = \mid n \mid (2j+1-N)$. Note that the single valuedness of ψ_j's demands that $\mid n \mid$ necessarily be an integer. All the ρ_j's ($\rho_j = \psi_j^\dagger \psi_j$) vanish at asymptotic infinity as $r^{-2(|n|N-|n|j+1)}$, implying that the rate of falloff is higher for lower values of j, reaching a maximum at $j = 0$. Near the origin ρ_j's behave as $r^{2\{|n|(j+1)-1\}}$ so that all the ρ_j's are nonsingular except ρ_0, which is nonsingular only when $\mid n \mid \geq 1$. We shall therefore restrict ourselves to $\mid n \mid \geq 1$, throughout this paper.

The gauge potential A_2 is given by

$$A_2 = \mp \frac{\mid n \mid (N+1)}{er} \left[1 + \left(\frac{r_0}{r}\right)^{2|n|}\right]^{-1} . \tag{21}$$

Near the origin the gauge potential goes to zero as $r^{2|n|}$ and interestingly enough it is independent of N. However, at the asymptotic infinity, $erA_2 \sim -\mid n \mid (N+1)$, keeping track of the global group structure. In fact, the profile of the gauge potential as given in (21) is same to the corresponding $N = 1$ case except for an overall multiplication factor of $\frac{1}{2}(N+1)$. As a consequence the magnetic field B also has the same profile as in the case of JP model except for a overall multiplication factor, i.e.,

$$B = \pm \frac{2n^2(N+1)}{er^2} \left(\frac{r_0}{r}\right)^{2|n|} \left[1 + \left(\frac{r_0}{r}\right)^{2|n|}\right]^{-2}, \tag{22}$$

so that the magnetic flux is modified to $\Phi = -\frac{2\pi}{e}\frac{\kappa}{|\kappa|} \mid n \mid (N+1)$. Note that the flux Φ is N dependent and $\mid \frac{e}{2\pi}\Phi \mid$ is quantized in terms of $(N+1)\mid n \mid$. So the flux quantum increases as one considers the higher values of N, i.e., enlarges the global symmetry. Also the flux quantum is even for odd N, while it can be both even and odd for even N. The charge Q and the angular momentum J are also quantized in this case as they are related to the magnetic flux by $Q = J = -\frac{\kappa}{e}\Phi = \frac{2\pi}{\alpha}\mid n \mid (N+1)$.

So far we have discussed only a set of specific solutions of Eqs. (11), which are expressed in terms of one unknown function $f_0(z)$. However, one would like to know the more general solutions of (11). Though we do not know the most general solutions of the Eqs. (11), one can obtain a set of exact solutions in case $N = \frac{1}{2}N'(N'-1)$ (where $N' \geq 3$, i.e., N=3, 6, 10, 15,...) in terms of N' unknown functions. For example, when $N = N' = 3$ the solution is given by

$$\chi = \mp ln \left[\frac{\sqrt{\alpha}}{2}(\mid \phi_1(z) \mid^2 + \mid \phi_2(z) \mid^2 + \mid \phi_3(z) \mid^2)\right], \tag{23}$$

where $f_j(z)$'s are chosen as

$$f_i(z) = \frac{1}{2}\epsilon_{ijk} W_{jk}(z),$$
$$W_{ij}(z) = \phi_j(z)\partial_z\phi_i(z) - \phi_i(z)\partial_z\phi_j(z), \quad i,j,k = 1,2,3. \tag{24}$$

The restriction on the solution (23) is that the analytic functions $\phi_i(z)$'s have no common zeros and are arbitrary otherwise. Hence in this case no rotationally symmetric solution is possible. However, if one assumes any one of the three $\phi_j(z)$'s is equal to unity, then it is possible to have rotationally symmetric solution analogous to Eq. (19). Notice from Eqs. (11), (23), and (24) that there is a freedom in choosing $f_i(z)$'s in terms of $W_{ik}(z)$ as the requirement to have exact solution is

$$\frac{1}{2}\sum_{i,j=1}^{3} \mid W_{ij}(z) \mid^2 = \sum_{i=1}^{3} \mid f_i(z) \mid^2 . \tag{25}$$

The particular choice in Eq. (24) is for notational convenience. Similar solutions can also be written down for other values of N (6, 10, 15, ...) using the identity

$$\nabla^2 \ln \left(\sum_{j=1}^{N'} \mid \phi_j(z) \mid^2\right) = \frac{1}{2}\left[\sum_{i,j=1}^{N'} \mid W_{ij}(z) \mid^2\right] \left[\sum_{m=1}^{N'} \mid \phi_m(z) \mid^2\right]^{-2}, \tag{26}$$

where $W_{ij}(z)$ is defined as in Eq. (24), but now with $i,j = 1,2,\ldots,N'$. Note that the first sum on the right side of Eq. (26) contains $N = \frac{1}{2}N'(N'-1)$ number of terms of the form $\mid W_{ij}(z) \mid^2$.

Let us now discuss time-dependent solutions of Eq. (1) in the case when it is considered in the background of a uniform magnetic field. To this end notice that the action (1) is invariant under dilation,

$$\mathbf{x} \to \mathbf{x}' = \Omega^{-1}\mathbf{x}, \quad t \to t' = \Omega^{-2}t, \quad \Psi \to \Psi' = \Omega\Psi(t,\mathbf{x}), \quad A_k \to A'_k = \Omega A_k, \quad A_0 \to A'_0 = \Omega^2 A_0, \tag{27}$$

where Ω is a constant. However, when the action (1) is considered in the background of an external magnetic field $\mathcal{B}$, only the Hamiltonian remains a conserved quantity. This fact was utilized in Ref. [4] to construct time-dependent solutions for the JP model (N=1) in the presence of $\mathcal{B}$ by starting from the static soliton solutions with $\mathcal{B} = 0$. We find that the same conclusions are also valid for arbitrary N. In particular, the Lagrangian (1) in the presence of an external uniform magnetic field can be written as

$$\tilde{\mathcal{L}} = i\tilde{\Psi}^\dagger \left(\tilde{\partial}_t + ie\tilde{A}^0\right)\tilde{\Psi} - \frac{1}{2m}\mid (\tilde{\partial}_i + ie\tilde{A}_i - ea_i)\tilde{\Psi} \mid^2 + \frac{g}{2}(\tilde{\Psi}^\dagger\tilde{\Psi})^2 + \frac{\kappa}{4}\epsilon^{\mu\nu\alpha}\tilde{A}_\mu \tilde{F}_{\nu\alpha}, \tag{28}$$

where $a_i = -\frac{B}{2}\epsilon_{ij}x_j$ and $\tilde{\partial}_\mu = \frac{\partial}{\partial \tilde{x}^\mu}$. One can easily check that under the transformations (with $w = \frac{eB}{m}$)

$$t = \frac{2}{w}\tan(\frac{w\tilde{t}}{2}), \quad \mathbf{x} = \begin{bmatrix} 1 & \tan(\frac{w\tilde{t}}{2}) \\ -\tan(\frac{w\tilde{t}}{2}) & 1 \end{bmatrix} \tilde{\mathbf{x}},$$

$$\psi_j(t,\mathbf{x}) = \cos\left(\frac{w\tilde{t}}{2}\right) \exp\left[i\frac{mw}{4}\tilde{r}^2\tan\left(\frac{w\tilde{t}}{2}\right)\right] \tilde{\psi}_j(\tilde{t},\tilde{\mathbf{x}}),$$

$$A_\mu(t,\mathbf{x}) = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\tilde{A}_\nu(\tilde{t},\tilde{\mathbf{x}}), \tag{29}$$

$\tilde{S} = \int \tilde{\mathcal{L}} d^3\tilde{x}$ is transformed into the action without the magnetic field $S = \int \mathcal{L} d^3x$ where $\mathcal{L}$ is as given by (1). Thus Eq. (29) is not a symmetry transformation of the action $\tilde{S}$. However, it relates the soliton solutions of (1) to that of the (28). Using the exact rotationally symmetric soliton solutions of the Lagrangian (1) as given in Eqs. (20) and (21), it is now straightforward to write down the time-dependent soliton solutions for the field variables $\tilde{\psi}_j$, $\tilde{A}^0$, and $\tilde{A}_i$ with the help of Eq. (29). The whole analysis goes through even when the solitons of (1) are considered in the background of a harmonic force [5,14].

Finally, the following comments are in order.

(i) For $N = 2$, the Lagrangian studied here can be obtained by taking the nonrelativistic limit of a relativistic semilocal theory considered in Ref. [9]. In fact, the Lagrangian of [9] but with the $SU(N)_{global} \times U(1)_{local}$ symmetry reduces to Eq. (1) in the nonrelativistic limit.

(ii) It is known that the relativistic $N = 2$ theory admits semilocal topological as well nontopological soliton solutions [9] (actually this is also true for arbitrary N). However, as we have shown, in the nonrelativistic limit only semilocal nontopological vortices are admissible. Can one extend the model (1) and also obtain semilocal topological solitons? We have checked that if the term ev_0A_0 is added to the Lagrangian (1) and the potential $-\frac{g}{2}(\Psi^\dagger\Psi)^2$ is modified to $\frac{g}{2}(\Psi^\dagger\Psi - v_0)^2$ where v_0 is a constant, then semilocal self-dual topological soliton solutions can be obtained for $g = \pm\frac{e^2}{m\kappa}$ by following the discussion of Ref. [15]. These self-dual solutions are characterized by nonzero energy $E = \frac{ev_0}{2m}\Phi$ unlike the semilocal nontopological solitons. Further, one finds that the decoupled equations for the matter fields are

$$\nabla^2 \ln\rho_j = \pm\frac{2e^2}{\kappa}\left(\sum_{l=0}^{N-1} \rho_l - v_0\right), \quad j = 0,1,2,\ldots,N-1. \tag{30}$$

For the special case of $N = 2$ [and when the constant $\pm\frac{2e^2}{\kappa}$ on the right side of (30) is positive] these equations are identical to those obtained in [16,17], and hence their analysis about the solutions [17] as well as the stability [16] goes through in this case.

(iii) Recently, Knecht *et al.* [18] have done the Painlevè analysis of the JP model and have shown that the model is not integrable, although it naturally admits integrable reductions which are the familiar Liouville and $1+1$ nonlinear Schrödinger equations. It would be interesting to repeat the same exercise for the $SU(N)_{global} \times U(1)_{local}$ case.

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