Gluonic and leptonic decays of heavy quarkonia and the determination of $\alpha_s(m_c)$ and $\alpha_s(m_b)$

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The determination of the QCD running coupling constants $\alpha_s(m_c)$ and $\alpha_s(m_b)$ is studied with heavy quarkonia $c\bar{c}$ and $b\bar{b}$ decays. The decay rates of $V \to 3g$ and $V \to e^+e^-$ for $V = J/\psi$ and Υ are given in terms of the Bethe-Salpeter amplitudes. To the first-order relativistic correction of on-shell quarks, for the leptonic decay, we have $\Gamma(V \to e^+e^-) = (4\pi\alpha^2 e_Q^2/m_Q^2)|\int d^3q[1 - (2\vec{q}^2/3m_Q^2)]\psi_{\rm Sch}(\vec{q})|^2$, which agrees with the NRQCD result, while for the gluonic decay we find $\Gamma(V \to 3g) = [40(\pi^2 - 9)\alpha_s^3(m_Q)/81m_Q^2]|\int d^3q[1 - (2.95\vec{q}^2/m_Q^2)]\psi_{\rm Sch}(\vec{q})|^2$. Here $\psi_{\rm Sch}(\vec{q})$ is the $Q\bar{Q}$ bound-state wave function in momentum space, and m_Q is the heavy quark mass. This result clearly shows that the relativistic correction (due to the $\vec{q}^2/m_Q^2$ term in the decay widths) suppresses the gluonic decay more severely than the leptonic decay. We then estimate these decay widths by further including the first-order QCD radiative corrections (given in the $\overline{\rm MS}$ scheme and at the heavy quark mass scale) on the basis of the factorization assumption, and using the meson wave functions which are obtained with a QCD-inspired interquark potential. Using the experimental values of the ratio $R_g \equiv \Gamma(V \longrightarrow 3g)/\Gamma(V \longrightarrow e^+e^-) \approx 10$, 32 for $V = J/\psi$, Υ, respectively, and the calculated widths, we find $\alpha_s(m_c) = 0.26$–0.29 and $\alpha_s(m_b) = 0.19$–0.21 at $m_c = 1.5$ GeV and $m_b = 4.9$ GeV. These values for the QCD running coupling constant are substantially enhanced, as compared with the ones obtained without relativistic corrections, and are potentially consistent with the QCD scale parameter $\Lambda^{(4)}_{\overline{\rm MS}} \approx 200$ MeV. We emphasize, however, that our numerical results of the running coupling constant mainly serve as an improved estimate rather than a precise determination for which the existing theoretical uncertainties due to higher order relativistic corrections and the scheme dependence of the radiative corrections should be further clarified, and a first principles estimate of the nonperturbative bound-state effects should be further studied.

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I. INTRODUCTION

Determining the QCD running coupling constant α_s at different energy scales is very important in the verification of the fundamental theory of strong interactions. Among others the heavy quarkonia decays may provide very useful information for α_s at the heavy quark mass scale. Decay rates of heavy quarkonia in the nonrelativistic limit with QCD radiative corrections have been studied (see, e.g., Refs. [1–3]). However, the decay rates of many processes are subject to substantial relativistic corrections. In particular, the rate of $J/\psi \to 3g$ and accordingly the determination of $\alpha_s(m_c)$ through this process depend rather crucially on the relativistic corrections. Presently the ratios of gluonic to leptonic widths of J/ψ and Υ have been precisely measured in experiment. However, if using the nonrelativistic expressions for decay widths which, in accordance with the factorization assumption for the long-distance part and the short-distance part of the decay amplitudes, are proportional to $|R_S(0)|^2$, where $|R_S(0)|$ is the radial wave function at the origin, and comparing them with the corresponding experimental values, one would get [2,3] $\alpha_s(m_c) = 0.19$ and $\alpha_s(m_b) = 0.17$ which are defined in the modified minimal subtraction $\overline{\rm MS}$ scheme. They are substantially smaller than the expected values determined from other experimental results or the QCD scale parameter (for a review on quantum chromodynamics, see Ref. [4]).

Theoretically, the difficulty probably is mainly due to the large relativistic effects on the decay widths. It is known that there are at least two important sources of relativistic effects for these processes, one is from the kinematical corrections to decay amplitudes, the other is from the bound-state wave-function corrections which are concerned with the nonperturbative dynamical effect of quark-antiquark interactions. In Ref. [5] the first-order relativistic corrections were considered based on a phenomenological model. But only the relativistic correction originated from kinematics was discussed with only $|R_S(0)|$ involved and no explicit methods with dynamical considerations were given to calculate the decay widths. As a result of lack of estimates for relativistic corrections to the decay widths in the determination of $\alpha_s(m_c)$ and $\alpha_s(m_b)$, either a conjectural parametrization for the v^2/c^2 term is made to get these coupling constants enhanced [3], or the v^2/c^2 term is arbitrarily neglected

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but a large effective mass for the gluons is introduced [6]. Apparently, a better estimate rather than arbitrary guesses for the relativistic corrections is indeed needed, though this is certainly a very difficult task without a deep understanding of quark confinement.

In this paper, as an attempt to tackle this problem, we will use the Bethe-Salpeter (BS) formalism [7] to derive the decay amplitudes and to calculate the decay widths of $V \to e^+e^-$ and $V \to 3g$, where V is a vector heavy quarkonium state. In this approach, the meson is considered as a bound state consists of a pair of constituent quark and antiquark (i.e., higher Fock states such as $|Q\bar{Q}g\rangle$ and $|Q\bar{Q}gg\rangle$ are neglected, which may be justified to the first-order relativistic corrections of S wave heavy quarkonium decays) and described by the BS wave function which satisfies the BS equation. A phenomenological QCD-inspired interquark potential will be used to solve for the wave functions and to calculate the decay matrix elements. These may allow us to give the expressions which take into account both relativistic and QCD radiative corrections to next-to-leading order. We may then estimate both kinematical and dynamical relativistic corrections to the decay widths, and determine the QCD coupling constant from the calculated and observed ratios of gluonic to leptonic decay widths. The remainder of this paper is organized as follows. In Sec. II we study the relativistic corrections to $V \to e^+e^-$ and $V \to 3g$ widths. In Sec. III we discuss the bound-state wave functions and determine the coupling constants $\alpha_s(m_c)$ and $\alpha_s(m_b)$ defined in the $\overline{\rm MS}$ scheme by comparing the experimental values of the gluonic and leptonic widths of J/ψ and Υ with their calculated values. A summary and discussion will be given in the last section, where comparison with other theoretical methods will be discussed and theoretical uncertainties in our calculation such as higher-order relativistic corrections and the renormalization scheme dependence in QCD radiative corrections will be emphasized.

II. RELATIVISTIC CORRECTIONS TO $V \to e^+e^-$ AND $V \to 3g$

It is argued [1,2] that the heavy quarkonium $Q\bar{Q}$ annihilation into gluons, photons, or lepton pairs can occur only at small distances of order $\frac{1}{m_Q}$, where m_Q is the heavy quark mass, while the quark confinement becomes effective at long distances of the order of meson radius $r_{Q\bar{Q}} \sim \frac{1}{m_Q(v/c)}$, where v is the relative velocity of the Q and $\bar{Q}$. For $\frac{v}{c} \ll 1$, these two length scales are well separated, and hence it is expected that the decay rates can be factorized into a short-distance part, which can be calculated using perturbation theory, and a long-distance part, which depends on the nonperturbative dynamics of the bound state. In fact, recently this factorization has been justified more rigorously based on the nonrelativistic QCD (NRQCD) effective theory (see Refs. [11,12]).

Since the short-distance part is governed by the heavy quark mass scale, it may be treated as an unbound $Q\bar{Q}$ pair annihilation (i.e., quarks are on their mass shells) and calculated perturbatively as an expansion in powers of $\alpha_s(m_Q)$. These QCD radiative corrections have been calculated in Refs. [1,2] (see also [3]), where the coupling constant $\alpha_s(m_Q)$ is usually defined in the $\overline{\rm MS}$ scheme (the modified minimal subtraction scheme) at the heavy quark mass scale.

The long-distance part of the decay rates is usually assumed to be proportional to $|R(0)|^2$ (for S-wave $Q\bar{Q}$ states), where $R(0)$ is the radial wave function at the origin of the bound state. However, this is only the lowest order result in $\frac{v^2}{c^2}$. If higher-order corrections in $\frac{v^2}{c^2}$ are considered, other quantities [e.g., $\nabla^2 R(0)$] related to the bound-state wave function must be involved. In certain cases these relativistic effects can be quite substantial and have to be taken into account.

In the following we will estimate these relativistic corrections in the Bethe-Salpeter formalism, and then give expressions for the decay rates with both relativistic corrections and QCD radiative corrections on the basis of the factorization assumption.

A. The $V \to e^+e^-$ decay

We first consider the leptonic decay for $V = J/\psi$ and Υ (see also [8]). This process proceeds via the $Q\bar{Q}$ annihilation. Define the Bethe-Salpeter wave function, in general, for a $Q_1\bar{Q}_2$ bound state $|P\rangle$ with overall mass M and momentum $P = (\sqrt{\vec{P}^2 + M^2},\ \vec{P})$

$$\chi(x_1, x_2) = \langle 0|T\psi_1(x_1)\bar{\psi}_2(x_2)|P\rangle, \tag{1}$$

where T represents the time-order product, and transform it into momentum space

$$\chi_P(q) = e^{-iP\cdot X}\int d^4x\, e^{-iq\cdot x}\chi(x_1, x_2). \tag{2}$$

Here q_1 (m_1) is the quark momentum (mass), q_2 (m_2) the antiquark momentum (mass), and q the relative momentum,

$$X = \eta_1 x_1 + \eta_2 x_2, \quad x = x_1 - x_2,$$
$$P = q_1 + q_2, \quad q = \eta_2 q_1 - \eta_1 q_2, \tag{3}$$

where $\eta_i = \frac{m_i}{(m_1+m_2)}$ $(i = 1, 2)$.

In this formalism the quarkonium annihilation matrix elements can be written as

$$\langle 0 \mid \overline{Q}IQ \mid P\rangle = \int d^4q\, {\rm Tr}\,[I(q,P)\chi_P(q)]\,, \tag{4}$$

where $I(q,P)$ is the interaction vertex of the $Q\bar{Q}$ with other fields (e.g., the photons or gluons) which, in general, may also depend on the variable q^0. If $I(q,P)$ is independent of q^0 (e.g., if quarks are on their mass-shells in the annihilation), Eq. (4) can be written as

$$\langle 0 \mid \overline{Q}IQ \mid P\rangle = \int d^3q\, {\rm Tr}[I(\vec{q}, P)\Phi_P(\vec{q})], \tag{5}$$

where

$$\Phi_P(\vec{q}) = \int dq^0\chi_P(q) \tag{6}$$

is the three-dimensional BS wave function of the $Q\bar{Q}$ meson. Note that in the approximation that the quarks are on mass-shells the decay amplitude (5) is greatly simplified and only the three-dimensional BS wave function is needed (but this does not necessarily require the interquark interaction to be instantaneous). In the BS formalism in the meson rest frame, where $\vec{q}_1 = -\vec{q}_2 = \vec{q}$, $P = (M, 0)$, we have

$$\Phi_P^{0^-}(\vec{q}) = \Lambda_+^1(\vec{q})\gamma^0(1+\gamma^0)\gamma_5\gamma^0\Lambda_-^2(-\vec{q})\varphi(\vec{q}),$$
$$\Phi_P^{1^-}(\vec{q}) = \Lambda_+^1(\vec{q})\gamma^0(1+\gamma^0)\not{e}\gamma^0\Lambda_-^2(-\vec{q})f(\vec{q}), \quad (7)$$

where $\Phi_P^{0^-}(\vec{q})$, and $\Phi_P^{1^-}(\vec{q})$ represent the three-dimensional wave functions of 0^- and 1^- mesons, respectively, $\not{e} = e_\mu\gamma^\mu$, e_μ is the polarization vector of 1^- meson, φ and f are scalar functions which can be obtained by solving the BS equation for 0^- and 1^- mesons, and $\Lambda_+(\Lambda_-)$ are the positive (negative) energy projector operators

$$\Lambda_+^1(\vec{q}) = \Lambda_+(\vec{q}_1) = \frac{1}{2E}(E + \gamma^0\vec{\gamma}\cdot\vec{q}_1 + m\gamma^0),$$
$$\Lambda_-^2(-\vec{q}) = \Lambda_-(\vec{q}_2) = \frac{1}{2E}(E - \gamma^0\vec{\gamma}\cdot\vec{q}_2 - m\gamma^0),$$
$$E = \sqrt{\vec{q}^2 + m^2}. \quad (8)$$

For the process $V \to e^+e^-$ with the electron (positron) momentum $k_1(k_2)$ and helicity $r_1(r_2)$,

$$I(q, P) = -ie\gamma_\mu, \quad (9)$$

which is independent of q^0, the amplitude can be written as

$$T = e^2 e_Q \langle 0 \mid \bar{Q}\gamma_\mu Q \mid V\rangle \bar{u}_{r_1}(k_1)\gamma^\mu v_{r_2}(k_2)\frac{1}{M^2}, \quad (10)$$

where M is the meson mass and e_Q is the electric charge of the quark Q ($Q = c, b$). Define the decay constant f_V by

$$f_V M e_\mu \equiv \langle 0 \mid \bar{Q}\gamma_\mu Q \mid V\rangle = \int d^3q \,\mathrm{Tr}[\gamma_\mu \Phi_P(\vec{q})], \quad (11)$$

where e_μ is the polarization vector of V meson. Then with (7) we can easily find

$$f_V = \frac{2\sqrt{3}}{M}\int d^3q \left(\frac{m+E}{E} - \frac{\vec{q}^{\,2}}{3E^2}\right) f(\vec{q}), \quad (12)$$

where $E = \sqrt{\vec{q}^{\,2} + m^2}$, and $\sqrt{3}$ is the color factor. Summing over the polarizations of the final state and averaging over that of the initial state, it is easy to get the decay width

$$\Gamma(V \to e^+e^-) = \tfrac{4}{3}\pi\alpha^2 e_Q^2 f_V^2/M. \quad (13)$$

Including further the first-order QCD radiative correction and assuming that the short-distance radiative correction and the long-distance relativistic correction can be factorized, we then get the following expression for the decay width with both relativistic and QCD radiative corrections:

$$\Gamma(V \to e^+e^-) = \frac{4}{3}\pi\alpha^2 e_Q^2 \frac{f_V^2}{M}\left(1 - \frac{16\alpha_s(m_Q)}{3\pi}\right). \quad (14)$$

Expanding f_V in terms of $\vec{q}^{\,2}/m^2$, to the first order we get

$$f_V = -\frac{4\sqrt{3}}{M}\int d\vec{q}\, f(\vec{q})\left(1 - \frac{5}{12}\frac{\vec{q}^{\,2}}{m^2}\right). \quad (15)$$

Equation (14) with (15) can also be written in the on-shell approximation [see (57) and (59) below] where quarks are assumed to be on the mass shell and hence the meson mass M is replaced by $M = 2E = 2\sqrt{\vec{q}^{\,2} + m^2}$ in the integration. In the numerical calculation of the decay width, in order to maintain the physical phase space of the decay process we will use (14) with (15) or (12) with the meson mass M taken to be its observed value (see the last section for further discussions).

B. The $V \to 3g$ decay

We next study the hadronic decay for $V = J/\psi$ and Υ. We consider a $J^{PC} = 1^{--}$ $Q\bar{Q}$ bound state decaying into three gluons. The decay width is given by

$$\Gamma = \frac{1}{2M}\int d\phi Z, \quad (16)$$

where the integration $\int d\phi$ is over the final-state phase space. Z is defined by

$$Z = \sum |T|^2, \quad (17)$$

where $\sum$ represents summing over the polarizations of the final state and averaging over that of the initial state. The decay matrix element T is

$$T = -ig_s^3\,\mathrm{Tr}(T_aT_bT_c)\int d^4q\,\mathrm{Tr}\chi_P(q)I(k_1, k_2, k_3; q_1, q_2), \quad (18)$$

where

$$I(k_1, k_2, k_3; q_1, q_2) = \frac{\not{\epsilon}_3(\not{k}3 - \not{q}_2 + m)\not{\epsilon}_2(\not{q}_1 - \not{k}_1 + m)\not{\epsilon}_1}{[(k_3 - q_2)^2 - m^2][(q_1 - k_1)^2 - m^2]} + \text{all permutations of } 1,2,3. \quad (19)$$

Here k_i and ϵ_i (i=1,2,3) represent the momenta and polarizations of the three gluons; T_a,T_b,T_c are the color SU(3) matrices, and a, b, c are the color indices of the three gluons; q_1 and q_2 are the momenta of quark and antiquark, and their time components satisfy $q_1^0 + q_2^0 = M$. As usual, we assume [1,5]

$$q^0 = q_1^0 - q_2^0 = 0, \quad q_1^0 = q_2^0 = M/2. \quad (20)$$

Thus $I(k_1, k_2, k_3, q_1, q_2)$ is independent of q^0, and T becomes

$$T = -ig_s^3\,\mathrm{Tr}(T_aT_bT_c)\int d^3q\,\mathrm{Tr}\{\Phi_P^{1^-}(\vec{q})I(k_1, k_2, k_3; \vec{q})\}. \quad (21)$$

Substituting (7) and (19) into (21) and (17), we get the expression for Z. In the extremely nonrelativistic limit the dependence on $\vec{q}$ of $\Lambda_+(\vec{q})$, $\Lambda_-(\vec{q})$, and $I(k_1,k_2,k_3;\vec{q})$ in Z is neglected and we obtain the usual nonrelativistic formula for $V\to 3g$. In order to go beyond the leading order result we expand the matrix element T in terms of $\vec{q}^{\,2}/m^2$.

We first consider a special case in which the quarks are off-shell by taking $q_1^0=q_2^0=M/2=m$. After performing the trace of γ matrix, in this case we get, to the first order of v^2/c^2,

$$Z=\frac{960g_s{}^6}{81}\int d^3q\,d^3q'\frac{1}{M^4}\times\left[A_0(x_1,x_2,x_3)\left(1+\frac{\vec{q}^{\,2}+\vec{q}^{\,\prime 2}}{6m^2}\right)+A_1(x_1,x_2,x_3)\frac{\vec{q}^{\,2}+\vec{q}^{\,\prime 2}}{3m^2}\right]f(\vec{q})f(\vec{q}^{\,\prime}),\tag{22}$$

where $f(\vec{q})$ is the scalar wave function of vector meson which comes from (7), and where

$$A_0(x_1,x_2,x_3)=\frac{16}{x_1^2}+\frac{16}{x_2^2}+\frac{32}{x_1x_2}-\frac{32}{x_1^2x_2}-\frac{32}{x_1x_2^2}+\frac{16}{x_1^2x_2^2}+\text{ (two other permutations)},$$

$$A_1(x_1,x_2,x_3)=-\frac{16}{x_1^2}-\frac{16}{x_2^2}-\frac{41}{x_1x_2}+\frac{12}{x_1^2x_2}+\frac{12}{x_1x_2^2}-\frac{40}{x_1^2x_2^2}-\frac{8}{x_1^3}-\frac{8}{x_2^3}-\frac{8}{x_1^3x_2}-\frac{8}{x_1x_2^3}+\frac{48}{x_1^3x_2^2}+\frac{48}{x_1^2x_2^3}-\frac{32}{x_1^3x_2^3}+\text{ (two other permutations)},$$

$$x_i=\frac{2\omega_i}{M},\quad i=1,2,3,\tag{23}$$

where ω_i represents the gluonic energy. The final-state phase space is given by

$$d\phi=\frac{1}{3!}(2\pi)^4\delta^4\left(P-\sum_{i=1}^{3}k_i\right)\prod_{i=1}^{3}\frac{d^3k_i}{(2\pi)^3 2\omega_i}\ .\tag{24}$$

After performing the integration, we find the decay width of $V\to 3g$ with relativistic corrections to be

$$\Gamma(V\to 3g)=\frac{640(\pi^2-9)\alpha_s^3(m_Q)}{81M^3}g_V^2,\tag{25}$$

where to the first-order relativistic correction g_V is

$$g_V=\int d^3q\left[1-\left(\frac{(41\pi^2/3-48)}{32(\pi^2-9)}-1/6\right)\frac{\vec{q}^{\,2}}{m^2}\right]f(\vec{q})=\int d^3q\left(1-2.96\frac{\vec{q}^{\,2}}{m^2}\right)f(\vec{q}).\tag{26}$$

This integral will diverge when the wave function f is determined by the interquark potential which is Coulombic at short distances where one gluon-exchange dominates. We could introduce a cutoff scale Λ of order $m(v/c)$ to regularize the integral. However, examining the interaction vertex I given in (19) we can easily see that g_V should be a convergent quantity if all higher-order corrections are taken into consideration, because in (19) I becomes inversely proportional to $\vec{q}^{\,2}$ as $\vec{q}^{\,2}\to\infty$. In the absence of calculations for higher-order relativistic corrections which may involve higher Fock states we will take the following expression for the regulation of g_V:

$$g_V\approx\int d^3q\left(1+2.96\frac{\vec{q}^{\,2}}{m^2}\right)^{-1}f(\vec{q}),\tag{27}$$

which gives the same result as (26) to the first-order relativistic correction and differs from (26) only by higher-order terms which are difficult to estimate because to that order other higher-order Fock states such as $|Q\bar{Q}g\rangle$ may be involved as well. Similarly, for the leptonic decay the first-order expression of decay constant f_V in (15) may also be regularized in the same manner, which differs from the "full" expression (12) only by higher-order terms, and the difference between (12) and the regularized value is found to be small.

In connection with (20), a more relevant treatment is to take the on-shell condition, which assumes the quark and antiquark to be on the mass shell

$$q_1^0=q_2^0=M/2=E=\sqrt{m^2+\vec{q}^{\,2}}.\tag{28}$$

The advantage of this assumption is that gauge invariance is maintained for the on-shell quarks but at the price of treating the quark and antiquark just as free particles in a bound state. An apparent problem in this scheme is that with a fixed value of the meson mass M (e.g., its observed value) if the quark mass takes a fixed value then $\vec{q}^{\,2}$ will be fixed but not weighted by the wave function as in the usual bound-state description. In order to connect the decay process, which occurs at short distances, where quarks are approximately on shell, with the bound-state wave function, which is mainly determined by the long-distance confinement force, we have to make a compromise between the on-shell condition and the bound-state description. We will use (28) and expand the matrix element (21) and (17) in terms of $\frac{\vec{q}^{\,2}}{m^2}$ and allow $\vec{q}^{\,2}$ (so the meson mass accordingly) to vary in accordance with the bound-state wave function $f(\vec{q})$ which is to be determined by the long-distance dynamics, or phenomenologically by some dynamical models. With this treatment, we get another expression for Z:

$$Z=\frac{960g_s{}^6}{81}\int d^3q\,d^3q'\frac{1}{M^4}\times\left[A_0(x_1,x_2,x_3)\left(1-\frac{5\vec{q}^{\,2}+5\vec{q}^{\,\prime 2}}{6m^2}\right)+A_1(x_1,x_2,x_3)\frac{\vec{q}^{\,2}+\vec{q}^{\,\prime 2}}{3m^2}\right]f(\vec{q})f(\vec{q}^{\,\prime}),\tag{29}$$

where

$$A_0(x_1, x_2, x_3) = \frac{16}{x_1^2} + \frac{16}{x_2^2} + \frac{32}{x_1x_2} - \frac{32}{x_1^2x_2} - \frac{32}{x_1x_2^2} + \frac{16}{x_1^2x_2^2} + \text{(two other permutations)},$$

$$A_1(x_1, x_2, x_3) = \frac{20}{x_1^2} + \frac{20}{x_2^2} + \frac{28}{x_1x_2} - \frac{56}{x_1^3x_2} - \frac{56}{x_1x_2^3} - \frac{88}{x_1^2x_2^2} + \frac{72}{x_1^3x_2^2} + \frac{72}{x_1^2x_2^3} + \frac{16}{x_1^3} + \frac{16}{x_2^3} - \frac{32}{x_1^3x_2^3} + \text{(two other permutations)}. \tag{30}$$

Then to the first-order relativistic correction we get

$$g_V = \int d^3q\ f(\vec{q}) \left[1 - \left(\frac{36 - \frac{7}{3}\pi^2}{8(\pi^2 - 9)} + \frac{5}{6}\right)\frac{\vec{q}^{\,2}}{m^2}\right] \approx \int d^3q\ f(\vec{q}) \left(1 + 2.70\frac{\vec{q}^{\,2}}{m^2}\right)^{-1}, \tag{31}$$

where g_V has been regulated in the same manner as (27).

Including further the first-order QCD radiative correction and assuming again that the radiative and relativistic corrections can be factorized, we then get the following expression for the decay width with both relativistic and QCD radiative corrections

$$\Gamma(V \to 3g) = \frac{640(\pi^2 - 9)\alpha_s^3(m_Q)}{81M^3}\left(1 - C\frac{\alpha_s(m_Q)}{\pi}\right) g_V^2, \tag{32}$$

where $C = 3.7(4.9)$ for $Q = c(b)$ with $m_{c(b)} = 1.5(4.9)$ GeV [2,3]. In the extremely nonrelativistic limit, the $\vec{q}^{\,2}/m^2$ term is zero, and the scalar wave function $f(q)$ satisfies the relation [see (55) below]

$$\int d^3q\ f(\vec{q}) = \sqrt{\frac{M}{4}}\ \psi(0), \tag{33}$$

where $\psi(0)$ is the nonrelativistic Schrödinger wave function at the origin in coordinate space, we then get the well-known zero-order result [2,3,5]

$$\Gamma(V \to 3g) = \frac{160(\pi^2 - 9)\alpha_s^3(m_Q)}{81M^2}|\psi(0)|^2. \tag{34}$$

With the two expressions (26) and (31), obtained in the two different treatments concerning the on-shell condition, we find that they both give very close results. This may largely reduce the uncertainties in our calculations associated with the on-shell or off-shell descriptions for the quarks which decay at short distances (therefore approximately on shell) and are bound together at large distances (therefore off shell).

Comparing (26) and (31) with (15), we see that the suppression due to relativistic correction for $V \to 3g$ is much more severe than for $V \to e^+e^-$. This result then rules out the conjecture that the relativistic correction to $J/\psi \to 3g$ may be negligibly small [6].

III. DETERMINATION OF $\alpha_S(M_Q)$ AND BOUND-STATE WAVE FUNCTIONS

To calculate the widths of leptonic and gluonic decays, we have to know the wave functions $f(\vec{q})$ for $c\bar{c}$ and $b\bar{b}$ states, which are determined mainly by the long-distance interquark dynamics. In the absence of a deep understanding for quark confinement at present, we will follow a phenomenological approach by using QCD inspired interquark potentials including both spin-independent and spin-dependent potentials, which are supported by both lattice QCD calculations and heavy quark phenomenology, as the interaction kernel in the BS equation. We begin with the bound-state BS equation [7] in momentum space

$$(\not{q}_1 - m_1)\chi_P(q)(\not{q}_2 + m_2) = \frac{i}{2\pi}\int d^4k G(P, q-k)\chi_P(k), \tag{35}$$

where q_1 and q_2 represent the momenta of quark and antiquark, respectively, $G(P, q-k)$ is the interaction kernel which dominates the interquark dynamics. In solving Eq. (35), we will employ the instantaneous approximation since for heavy quarks the interaction is dominated by instantaneous potentials. This is also because, at present we do not know how to reliably include the retardation effects of quark confinement beyond the static confinement potential. Once we know them we could add them as higher-order terms to the instantaneous kernel. Meanwhile, we will neglect negative energy projectors in the quark propagators which are of even higher orders. We then get the reduced Salpeter equation [7] for the three-dimensional BS wave function $\Phi_P(\vec{q})$ defined in (6)

$$\Phi_P(\vec{q}) = \frac{1}{P^0 - E_1 - E_2}\Lambda_+^1\gamma^0 \times \int d^3k\, G(P, \vec{q} - \vec{k})\Phi_P(\vec{k})\gamma^0\Lambda_-^2, \tag{36}$$

where $G(P, \vec{q} - \vec{k})$ represents the instantaneous potential.

We employ the following interquark potentials including a long-ranged confinement potential (Lorentz scalar) and a short-ranged one-gluon exchange potential (Lorentz vector) [8]:

$$V(r) = V_S(r) + \gamma_\mu \otimes \gamma^\mu V_V(r),$$
$$V_S(r) = \lambda r\frac{(1 - e^{-\alpha r})}{\alpha r},$$
$$V_V(r) = -\frac{4}{3}\frac{\alpha_s(r)}{r}e^{-\alpha r}, \tag{37}$$

where the introduction of the factor $e^{-\alpha r}$ is to regulate the infrared divergence and also to incorporate the color screening effects of the dynamical light quark pairs on the $Q\bar{Q}$ linear confinement potential [9]. In momentum space the potentials become [8]

$$G(\vec{p}) = G_S(\vec{p}) + \gamma_\mu \otimes \gamma^\mu G_V(\vec{p}),$$
$$G_S(\vec{p}) = -\frac{\lambda}{\alpha}\delta^3(\vec{p}) + \frac{\lambda}{\pi^2}\frac{1}{(\vec{p}^{\,2}+\alpha^2)^2},$$
$$G_V(\vec{p}) = -\frac{2}{3\pi^2}\frac{\alpha_s(\vec{p})}{\vec{p}^{\,2}+\alpha^2}, \tag{38}$$

where $\alpha_s(\vec{p})$ is the quark-gluon running coupling constant and is assumed to become a constant of $O(1)$ as $\vec{p}^{\,2} \to 0$:

$$\alpha_s(\vec{p}) = \frac{12\pi}{27}\frac{1}{\ln(a+\vec{p}^{\,2}/\Lambda_{\rm QCD}^2)}. \tag{39}$$

The constants λ, α, a, and $\Lambda_{\rm QCD}$ are the parameters that characterize the potential.

Substituting (7) and (38) into Eq. (36), one derives the equation for the 1^- meson wave function $f(\vec{q})$ in the meson rest frame

$$\begin{aligned}
Mf_1(\vec{q}) = {} & (E_{q1}+E_{q2})f_1(\vec{q}) \\
& -\frac{E_{q1}+m_1+E_{q2}+m_2}{4E_{q1}E_{q2}[3(E_{q1}+m_1)(E_{q2}+m_2)+\vec{q}^{\,2}]}\Bigg((E_{q1}E_{q2}+m_1m_2+\vec{q}^{\,2}) \\
& \times\int d^3k[G_S(\vec{q}-\vec{k})-2G_V(\vec{q}-\vec{k})]\frac{3(E_{k1}+m_1)(E_{k2}+m_2)+\vec{k}^2}{E_{k1}+E_{k2}+m_1+m_2}f_1(\vec{k}) \\
& -2\vec{q}^{\,2}\int d^3k[G_S(\vec{q}-\vec{k})-2G_V(\vec{q}-\vec{k})]\frac{E_{k2}m_1+E_{k1}m_2}{m_1+m_2}f_1(\vec{k}) \\
& +(E_{q1}m_2+E_{q2}m_1)\int d^3k\,G_S(\vec{q}-\vec{k})\frac{3(E_{k1}+m_1)(E_{k2}+m_2)-\vec{k}^2}{E_{k1}+E_{k2}+m_1+m_2}f_1(\vec{k}) \\
& -(m_1+m_2)\int d^3k[G_S(\vec{q}-\vec{k})+4G_V(\vec{q}-\vec{k})](\vec{q}\cdot\vec{k})f_1(\vec{k}) \\
& -2(E_{q1}-E_{q2})\int d^3k[G_S(\vec{q}-\vec{k})-2G_V(\vec{q}-\vec{k})](\vec{q}\cdot\vec{k})\frac{E_{k1}+m_1}{E_{k1}+m_1+E_{k2}+m_2}f_1(\vec{k}) \\
& +\int d^3k[4G_S(\vec{q}-\vec{k})-8G_V(\vec{q}-\vec{k})](\vec{q}\cdot\vec{k})^2\frac{f_1(\vec{k})}{E_{k1}+m_1+E_{k2}+m_2} \\
& -2(m_1-m_2)\int d^3k\,G_S(\vec{q}-\vec{k})\frac{E_{k1}-E_{k2}}{m_1+m_2}(\vec{q}\cdot\vec{k})f_1(\vec{k}) \\
& +(E_{q1}+3E_{q2})\int d^3k\,G_S(\vec{q}-\vec{k})(\vec{q}\cdot\vec{k})f_1(\vec{k}) \\
& -(6E_{q1}+2E_{q2})\int d^3k\,G_V(\vec{q}-\vec{k})\vec{q}\cdot\vec{k}f_1(\vec{k})\Bigg),
\end{aligned} \tag{40}$$

where $E_{qi} = \sqrt{\vec{q}^{\,2}+m_i^2}$, $E_{ki} = \sqrt{\vec{k}^2+m_i^2}$ $(i = 1,2)$, and

$$f_1(\vec{q}) = \frac{E_{q1}+m_1+E_{q2}+m_2}{4E_{q1}E_{q2}}f(\vec{q}). \tag{41}$$

The normalization condition $\int d^3q\,{\rm Tr}\{\Phi^\dagger(\vec{q})\Phi(\vec{q})\} = (2\pi)^{-3}2M$ for the BS wave function leads to [8]

$$\int d^3q\frac{(m_1+E_{q1})(m_2+E_{q2})}{8E_{q1}E_{q2}}|f(\vec{q})|^2 = \frac{M}{(4\pi)^3}. \tag{42}$$

Equation (40) looks complicated since higher-order terms in v^2/c^2 are all involved. To leading order in the nonrelativistic limit, Eq. (40) is just the ordinary nonrelativistic Schrödinger equation with simply a spin-independent linear plus Coulomb potential. To the first order of v^2/c^2, Eq. (40) becomes the well-known Breit equation with both spin-independent and spin-dependent potentials from vector (one-gluon) exchange and scalar (confinement) exchange.

For the heavy quarkonium $c\bar{c}$ and $b\bar{b}$ systems, $m_1 = m_2 = m$, Eq. (40) can be greatly simplified. By solving Eq. (40) we can find the wave functions for the 1^- mesons (see, e.g., Ref. [8]).

Substituting the obtained BS wave functions into (12) and (14), we then get

$$\Gamma(J/\psi \to e^+e^-) = 5.6\ {\rm keV}, \tag{43}$$

where we have used

$$m_c = 1.5\ {\rm GeV},\quad \lambda = 0.23\ {\rm GeV}^2,\quad \Lambda_{QCD} = 0.18\ {\rm GeV},$$
$$\alpha = 0.06\ {\rm GeV},\quad a = e = 2.7183. \tag{44}$$

With these parameters the $2S-1S$ spacing and $J/\psi-\eta_c$ splitting are required to fit the data. Our result is in agreement with the experimental value of $\Gamma(J/\psi \to e^+e^-) = 5.36 \pm 0.29$ keV [4]. Here in the above calculations the value of $\alpha_s(m_c)$ in the QCD radiative correction factor in (14) is chosen to be 0.29 [3], which is also consistent with our determination from the ratio of $B(J/\psi \to 3g)$ to $B(J/\psi \to e^+e^-)$ (see below).

Using the experimental data [4]

$$R_g \equiv \frac{\Gamma(J/\psi \to 3g)}{\Gamma(J/\psi \to e^+e^-)} \approx 10, \tag{45}$$

and the calculated widths from (14), (31), and (32) we find

$$\alpha_s(m_c) = 0.29, \tag{46}$$

as compared with the value without relativistic corrections (but with QCD radiative corrections)

$$\alpha_s^0(m_c) = 0.19. \tag{47}$$

Clearly, it is the strong suppression due to the relativistic correction to $J/\psi \to 3g$ that substantially enhances the value of $\alpha_s(m_c)$. [Note that a slightly larger value for $\alpha_s(m_c)$ could be obtained if using (27) rather than (31).]

In comparision we have chosen two other groups of parameters and solved the BS equation for $c\bar{c}$ states by requiring again both their $2S-1S$ spacing and $J/\psi - \eta_c$ splitting fitting the data. By the same procedure we get two values of $\alpha_s(m_c)$ corresponding to the obtained two new wave functions of J/ψ. With $m_c = 1.4$ GeV, $\lambda = 0.24$ GeV^2, and other parameters unchanged (the heavy quarkonia mass spectra are not sensitive to a and α for $\alpha \leq 0.06$ GeV) we get $\alpha_s(m_c) = 0.29$. With $m_c = 1.6$ GeV, $\lambda = 0.22$ GeV^2, and other parameters unchanged we get $\alpha_s(m_c) = 0.28$.

Meanwhile, we have also solved the nonrelativistic Schrödinger equation for the scalar wave function f in the nonrelativistic limit by using the same potentials and parameters as (38) and (44), but neglecting all first-order and higher-order spin-independent and spin-dependent terms in Eq. (40), we obtain

$$\alpha_s(m_c) = 0.28. \tag{48}$$

Note that in this limit the spin symmetry between the 1^- and 0^- mesons is restored and all the relativistic correction from the dynamical source is eliminated, and that with relativistic corrections the BS wave function falls more slowly in momentum space than the nonrelativistic Schrödinger wave function and consequently gives a slightly larger α_s [see (46)] than the Schrödinger wave function [see (48)].

In order to see further the sensitivity of the value of $\alpha_s(m_c)$ to the wave functions, we have also naively tried the Gaussian function

$$f(q) = N \exp\left(-\frac{4\vec{q}^{\,2}}{3q_0^2}\right), \tag{49}$$

where N is the normalzation factor, and q_0^2 is the mean value of the momentum squared of the quark inside the meson, which may be roughly estimated by using the scaling law $q_0^2 = mC/2$ ($C = 0.73$ GeV) found for heavy quarkonia (see, e.g., Ref. [10]). Then with $m_c = 1.5$ GeV we find that while the gluonic and leptonic decay widths both become smaller (owing to the fact that the Gaussian wave function has a smaller value at the origin in coordinate space than the wave function determined by the linear plus Coulomb potential), the ratio of the two widths does not change very much and gives $\alpha_s(m_c) = 0.26$.

The results (46) and (48) indicate that the determination of $\alpha_s(m_c)$ is not sensitive to the dynamical relativistic corrections to the wave functions, and the result with (49) may further indicate that it is even not very sensitive to the form of wave functions (within a reasonable range of choice for the wave functions), and the most important effect comes from the kinematic corrections.

For the $b\bar{b}$ system, with a similar calculation for the decay rates of $\Upsilon \to 3g$ and $\Upsilon \to e^+e^-$ with both relativistic and QCD radiative corrections taken into account, and using the observed value of the ratio [4]

$$R_g \equiv \frac{\Gamma(\Upsilon \to 3g)}{\Gamma(\Upsilon \to e^+e^-)} \approx 32, \tag{50}$$

we find

$$\alpha_s(m_b) = 0.19 - 0.20, \tag{51}$$

where 0.20 is obtained with the BS wave function while 0.19 with the Gaussian wave function, and the nonrelativistic Schrödinger wave function gives a value between 0.19 and 0.20 (all for $m_b = 4.9$ GeV). If without relativistic corrections (but with QCD radiative corrections), we would get $\alpha_s^0(m_b) = 0.17$. Moreover, with both relativistic and QCD radiative corrections, and $\alpha_s(m_b) = 0.20$, we get

$$\Gamma(\Upsilon \to e^+e^-) = (0.5 - 1.3)\ \text{keV}, \tag{52}$$

where 1.3 keV is obtained with the BS wave function, which is in agreement with data [4], while 0.5 keV is obtained with the Gaussian wave function, which indicates that the Gaussian wave function significantly underestimates the wave function at the origin and then leads to a smaller leptonic width. The Schrödinger wave function gives 1.1 keV for the leptonic width.

We may then conclude that by estimating the relativistic corrections to the gluonic and leptonic decays of heavy quarkonia, we find that the relativistic effects substantially suppress the $V \to 3g$ decays, and consequently the determined values of the QCD running coupling constant at the heavy quark mass scale can get enhanced, and can be potentially consistent with other theoretical and experimental results. In particular, for the Υ with estimated small value of $v^2/c^2 = 0.08$–0.09 the higher order relativistic corrections should be small and therefore the result (51) for the b quark should be more reliable than (46) and (48) for the c quark. From (51) and (52) we see again that although the values of leptonic and hadronic widths themselves are sensitive to the wave functions, the value of $\alpha_s(m_b)$ is not, and only varies in a very narrow range. Then with (51) $\alpha_s(\mu) = 0.19$–0.20 at $\mu = m_b = 4.9$ GeV and with the relation between $\alpha_s(\mu)$ and $\Lambda_{\overline{\rm MS}}^{(n_f)}$ to two-loop accuracy,

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln \frac{\mu^2}{\Lambda_{\overline{\rm MS}}^2}} \left[1 - \frac{\beta_1 \ln\ln \frac{\mu^2}{\Lambda_{\overline{\rm MS}}^2}}{\beta_0^2 \ln \frac{\mu^2}{\Lambda_{\overline{\rm MS}}^2}}\right], \tag{53}$$

where $\beta_0 = 11 - \frac{2}{3}n_f$, $\beta_1 = 102 - \frac{38}{3}n_f$, and n_f is the number of light quark flavors, we find $\Lambda^{(4)}_{\overline{\rm MS}} = 220 - 260$ MeV. Here experimental errors are not included.

IV. SUMMARY AND DISCUSSION

In this paper we calculated the first-order relativistic corrections to $V \to e^+e^-$ and $V \to 3g$ for the vector heavy quarkonia $V = J/\psi$ and Υ, based on the BS formalism for the decay amplitudes and bound-state wave functions. We then combine the relativistic corrections with the known QCD radiative corrections to give the decay widths to the next to leading order. We derived the coefficients of the v^2/c^2 term in the decay rates with two different treatments regarding the on-shell condition of bound quarks, and obtained very similar values for the coefficients. These results may largely reduce the uncertainty in our calculations concerning the on-shell condition in these decay processes. To maintain gauge invariance in the decay amplitudes, the on-shell quarks are certainly better than off-shell quarks, and this is also justified by the fact that the annihilation can take place only at short distances. The large negative value of the coefficient in the $\vec{q}^{\,2}/m^2$ term [see (31)] in the $V \to 3g$ width may encourage us to conclude that the relativistic effects suppress $V \to 3g$ decays much more severely than $V \to e^+e^-$ decays, therefore can make the coupling constants $\alpha_s(m_c)$ and $\alpha_s(m_b)$ substantially enhanced, as compared with the values obtained without relativistic corrections.

For a more accurate estimate of relativistic corrections, the higher orders, e.g., $(v^2/c^2)^2$ terms should be taken into account, but this is difficult in our approach within the $|Q\bar{Q}\rangle$ sector, since to the higher orders the effects of dynamical gluons, e.g., the $|Q\bar{Q}g\rangle$ and $|Q\bar{Q}gg\rangle$ Fock states may be involved. In connection with the uncertainty of higher-order corrections, we have regularized the singularity associated with the derivative of the wave functions at the origin in a simple manner, as shown in (26), (27), and (31). It is evident that (27) and (31) are valid to the first order of v^2/c^2 but are uncertain to higher-order corrections, which are not only concerned with higher-order terms in (18) and (19), but also with the contribution of, e.g., the $|Q\bar{Q}g\rangle$ component in the meson wave function. Clearly, the lack of estimate of higher-order relativistic corrections is one of the main theoretical uncertainties in our approach, especially for the $c\bar{c}$ system. Fortunately this should not be too troublesome for the Υ, since its v^2/c^2 is only about 0.09, small enough to make the higher order v^2/c^2 corrections unimportant. Consequently, the result for α_s should be more reliable for bottomonium than for charmonium.

We have solved the BS equation for the bound-state wave functions with QCD inspired interquark potentials (linear confinement potential plus one gluon exchange potential) as the BS kernel. With some popular parameters for the potentials we obtained the wave functions and used them to calculate the gluonic and leptonic decay widths and their ratios and then determine the strong coupling constants. By comparing the BS wave functions with nonrelativistic Schrödinger wave functions and Gaussian-type wave functions, we find that the coupling constants are not very sensitive to the dynamical relativistic effects on the wave functions (note that this is partly because the coupling constant is approximately proportional to the cube root of the ratio of gluonic and leptonic widths). We may then conclude that the relativistic effects on the ratio of the widths and the strong coupling constant mainly originate from the kinematic part of decay amplitudes. This may allow us to obtain a reasonable range for the values of the coupling constant, despite of the uncertainty in the estimate of the dynamical relativistic effects. Indeed, it is difficult to control the systematic accuracy within the potential model. In particular, the spin-independent relativistic correction to the confinement potential and the retardation correction connected with confinement are far from being thoroughly understood. This also causes an uncertainty in the estimate of decay widths and values of the strong coupling constant. Nevertheless, from (51) and (52) it can be seen that three very different wave functions (i.e., the BS wave function, the Schrödinger wave function, and the Gaussian wave function) lead to very different leptonic widths (and gluonic widths accordingly) but give very close values for $\alpha_s(m_b)$. This might indicate that at least in the case of $b\bar{b}$ the uncertainty in the bound state wave functions would not cause a large uncertainty in $\alpha_s(m_b)$.

Despite the theoretical uncertainties mentioned above, our results for the first-order relativistically corrected gluonic and leptonic widths [in particular (31) and (15), respectively] are useful for an improved estimate of the strong coupling constant as compared with that obtained without relativistic corrections. Using the experimental values of the ratio $R_g \equiv \frac{\Gamma(V\to 3g)}{\Gamma(V\to e^+e^-)} \approx 10,\ 32$ for $V = J/\psi,\ \Upsilon$, respectively, and the calculated widths with both relativistic and QCD radiative corrections to the first order, we found $\alpha_s(m_c) = 0.26 - 0.29$ and $\alpha_s(m_b) = 0.19 - 0.20$ in the $\overline{\rm MS}$ scheme. The numerical uncertainties due to the potential parameters and the different choices of wave functions within our approach have been included in above values (experimental errors are not included). These values for the QCD running coupling constant are potentially consistent with the QCD scale parameter $\Lambda^{(4)}_{\overline{\rm MS}} \approx 200$ MeV. In particular, with $\alpha_s(m_b) = 0.19$–0.20, which is a more reliable estimate than for charmonium, we get $\Lambda^{(4)}_{\overline{\rm MS}} = 220$–$260$ MeV.

Recently, there has been significant progress in the study of heavy quarkonium decays based on a more fundamental approach of the nonrelativistic QCD (NRQCD) effective theory (see [11,12]). The factorization theorem was further discussed, and some important issues (e.g., the infrared divergences in the P-wave state decay rates) were clarified in this study. It will be interesting to compare their results with ours in connection with the gluonic and leptonic decay widths and the determination of the strong coupling constant at the heavy quark mass scales from these decays.

In fact, our (14) combined with (15) can be written in terms of the standard Schrödinger wave function (with relativistic corrections) $\psi_{\rm Sch}$ as

$$\Gamma(V \to e^+e^-) = \frac{16\pi\alpha^2 e_Q^2}{M^2}\left(1 - \frac{16\alpha_s(m_Q)}{3\pi}\right) \times \left|\int d^3q \left(1 - \frac{\vec{q}^{\,2}}{6m^2}\right)\psi_{\rm Sch}(\vec{q})\right|^2, \quad (54)$$

where $\psi_{\rm Sch}(\vec{q})$ is related to $f(\vec{q})$ through the normalization condition (42) which leads to

$$\psi_{\rm Sch}(\vec{q}) = \frac{1}{\sqrt{M}}\left(\frac{m+E}{E}\right) f(\vec{q}),$$
$$(2\pi)^3 \int d^3q\ \psi^*_{\rm Sch}(\vec{q})\psi_{\rm Sch}(\vec{q}) = 1. \quad (55)$$

Our (32) combined with (31) can also be written as

$$\Gamma(V \to 3g) = \frac{160(\pi^2-9)\alpha_s^3(m_Q)}{81M^2}\left(1 - C\frac{\alpha_s(m_Q)}{\pi}\right) \times \left|\int d^3q \left[1 - \left(2.70 - \frac{1}{4}\right)\frac{\vec{q}^{\,2}}{m^2}\right]\psi_{\rm Sch}(\vec{q})\right|^2. \quad (56)$$

Here in (54) and (56), which are essentially the formulas used for our calculations, the meson mass M, which originates from the phase space of the decay process, is taken to be the observed physical value. If we further use the on-shell condition (28): $M = 2E = 2\sqrt{m^2+\vec{q}^{\,2}}$, to replace the observed value of the meson mass M then (54) and (56) will become

$$\Gamma(V \to e^+e^-) = \frac{4\pi\alpha^2 e_Q^2}{m^2}\left(1 - \frac{16\alpha_s(m_Q)}{3\pi}\right)\left|\int d^3q\left(1 - \frac{2\vec{q}^{\,2}}{3m^2}\right)\psi_{\rm Sch}(\vec{q})\right|^2, \quad (57)$$

$$\Gamma(V \to 3g) = \frac{40(\pi^2-9)\alpha_s^3(m_Q)}{81m^2}\left(1 - C\frac{\alpha_s(m_Q)}{\pi}\right)\left|\int d^3q\left[1 - \left(2.70 + \frac{1}{4}\right)\frac{\vec{q}^{\,2}}{m^2}\right]\psi_{\rm Sch}(\vec{q})\right|^2. \quad (58)$$

It is easy to see that to the first order of $\frac{v^2}{c^2}$, in coordinate space (57) and (58) can be expressed as

$$\Gamma(V \to e^+e^-) = \frac{\alpha^2 e_Q^2}{m^2}\left(1 - \frac{16\alpha_s(m_Q)}{3\pi}\right)\left(|R(0)|^2 + \frac{4}{3m^2}{\rm Re}[R^*(0)\nabla^2 R(0)]\right), \quad (59)$$

$$\Gamma(V \to 3g) = \frac{10(\pi^2-9)\alpha^3(m_Q)}{81m^2}\left(1 - C\frac{\alpha_s(m_Q)}{\pi}\right)\left(|R(0)|^2 + \frac{5.90}{m^2}{\rm Re}[\ R^*(0)\nabla^2 R(0)]\right), \quad (60)$$

where $R(0)$ is the Schrödinger radial wave function at the origin of the V ($V = J/\psi, \Upsilon$) meson. Equation (59) is exactly the same as that given in Ref. [11] with the NRQCD effective theory, if we identify our bound-state wave functions with their regularized operator matrix elements, i.e.:

$$R(0) = \sqrt{\frac{2\pi}{3}}\epsilon\cdot\langle 0|\chi^\dagger\sigma\psi|V\rangle, \quad (61)$$

$$\nabla^2 R(0) = -\sqrt{\frac{2\pi}{3}}\epsilon\cdot\langle 0|\chi^\dagger\sigma\left(\frac{-i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|V\rangle \times[1 + O(v^2/c^2)]. \quad (62)$$

In our calculations we have used (54) and (56) (with also a regularization procedure) in which the observed physical value for the meson mass M is taken to maintain the physical phase space. Of course, we may also use (57) and (58) to determine $\alpha_s(m_Q)$. Since $\alpha_s(m_Q)$ is essentially determined by the ratio of the gluonic width to the leptonic width, use of (57) and (58) will not lead to much of a difference from the use of (54) and (56). E.g., instead of (51), use of (57) and (58) with the same potential parameters as before gives $\alpha_s(m_b) = 0.20 - 0.21$ for $m_b = 4.9$ GeV, while the obtained gluonic and leptonic widths themselves become slightly smaller.

Our results (54)–(60) differ from Ref. [5] in two respects. First, to the first order of v^2/c^2 we have two independent nonperturbative quantities $R(0)$ and $\nabla^2 R(0)$, whereas in Ref. [5] (and other earlier treatments on the factorization of quarkonium decays) only one quantity, i.e., $R(0)$ is considered. Second, these two quantities can be calculated in our approach based on a QCD-inspired quark potential model by solving the Bethe-Salpeter equation. Although this is not a first principle theory and it is difficult to control the systematic uncertainty in the potential model calculation, it may provide a rather useful estimate of these quantities, since not only the zeroth order spin-independent potential but also the first-order spin-dependent potential, i.e., the Breit-Fermi Hamiltonian, which stems from one gluon exchange and has a good theoretical and phenomenological basis, are considered in the calculation, and different wave functions are also chosen to estimate the uncertainties in the calculation. Nevertheless, for more reliable estimates we hope that these nonperturbative quantities of the bound states can be eventually calculated from more fundamental theoretical methods, e.g., the lattice QCD simulations.

We have used the QCD radiative corrections given in the $\overline{\rm MS}$ scheme [1–3]. However, as pointed out in Ref. [13] that the renormalization scheme dependence of perturbative QCD will plague attempts to make high precision tests of the theory, and the inclusive hadronic width of vector quarkonium does have a serious scheme ambiguity. In the so-called effective charge scheme [being essentially the $\overline{\text{MS}}$ scheme but with a specific scale $\mu = Q^*$ at which all the dependence on the light flavor numbers is absorbed into the leading term of $\alpha_s(\mu)$] proposed in Ref. [13] the ratio of gluonic and leptonic widths of J/ψ or Υ can even be negative. This may indicate that perturbative QCD is risky for describing these decays. The scheme ambiguity problem in QCD has been discussed in Ref. [14] where various schemes are extensively compared and it is argued that there is no known "correct" answer to the question which scheme, among all the infinite number of possible schemes, should be chosen in order to best compare the truncated finite order QCD predictions with experimental results. The renormalization scheme problem has also been discussed in Ref. [15]. Presumably, while the effective charge scheme in Ref. [13] is physically motivated, the lack of a clean resolution of the scheme ambiguity at present might allow one to try other schemes. In fact the $\overline{\text{MS}}$ scheme (with a scale around the heavy quark mass) has been widely used in the studies concerning the determination of QCD coupling constant from the inclusive hadronic widths (and the leptonic widths) of vector heavy quarkonia (see, e.g., Refs. [3,4,14,15] and references therein). Nevertheless, this does not mean that the use of it is really justified and one should keep in mind that there is possibly a large uncertainty due to the scheme ambiguity in the first order radiative corrections to the inclusive hadronic widths, and hopefully the explicit calculation of higher order radiative corrections will further clarify this ambiguity.

Recently there has been significant progress in lattice QCD toward the determination of QCD coupling constant α_s and other fundamental parameters from heavy quarkonium spectra and decay rates [16–18]. Lattice simulations of the 1S-1P or 1S-2S energy spacings of heavy quarkonia, compared with their observed values, can determine the inverse lattice spacing, a^{-1}. This information can be converted to the determination of α_s in the $\overline{\text{MS}}$ scheme when both nonperturbative simulations and perturbative expansions for some short-distance quantities are used. This approach gives a well-defined meaning of α_s and other fundamental parameters from first principle calculations. The average over Fermilab [16] and NRQCD [17] results (especially for the new unquenched method) for α_s seems close to our estimate $\Lambda^{(4)}_{\overline{\text{MS}}} = 220$–$260$ MeV.

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