

Action for $N=4$ supersymmetric self-dual Yang-Mills theory

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The $N = 4$ supersymmetric self-dual Yang-Mills theory in four-dimensional space with the signature $(2, 2)$ is formulated in harmonic superspace. The on-shell constraints of the theory are reformulated in the equivalent form of vanishing curvature conditions for three gauge connections (one harmonic and two space-time). The constraints are then obtained as variational equations from a superspace action of the Chern-Simons-type. The action is manifestly $SO(2, 2)$ invariant. It can be viewed as the Lorentz-covariant form of the light-cone superfield action proposed by Siegel.

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I. INTRODUCTION

The interest in self-dual theories in a four-dimensional space-time with the signature $(2, 2)$ has risen considerably after the observation of Ooguri and Vafa [1] that the string with a local $N = 2$ world-sheet supersymmetry has only one state describing self-dual Yang-Mills theory (open string) and self-dual gravity (closed string). Soon afterwards Parkes [2] proposed a field-theory action,¹ allegedly corresponding to the amplitudes of this string. This action uses a special Lorentz-noncovariant gauge for Yang-Mills theory of the type first considered by Yang [4], in which only one degree of freedom is left (as opposed to the three degrees of freedom in the covariant self-duality condition). In addition to the lack of covariance, this action contradicts standard dimensional counting, as noted in [5, 6] (it requires a dimensionful coupling constant which is not natural in a four-dimensional Yang-Mills theory).

In [7] Siegel put forward the idea that the $N = 4$ string, when properly quantized, is in fact equivalent to the $N = 2$ one. He further argued that the corresponding field theory is actually the $N = 4$ supersymmetric self-dual Yang-Mills (SSDYM) theory in the open case or $N = 8$ self-dual supergravity in the closed case. In [6] he also presented a Green-Schwarz-type formulation of that string.²

In general, the multiplet of N -extended SSDYM theory contains helicities from $+1$ to $1 - N/2$. So, it includes all the helicities from $+1$ to -1 in the maximal case $N = 4$ only. As a consequence, in the latter case the degrees of freedom appear in Lagrangian pairs and one is able to write down a Lorentz-covariant action for

the theory. So far this action has been presented either in component (i.e., not manifestly supersymmetric) [5, 9] or light-cone superspace (i.e., not manifestly Lorentz-covariant) form [10, 5]. Our aim in this paper will be to write down the $N = 4$ SSDYM action in a form which is both manifestly Lorentz-invariant and supersymmetric. In some sense it is a covariantization of the light-cone action of Siegel [10, 5], obtained with the help of harmonic variables for one of the $SL(2, R)$ factors of the Lorentz group $SO(2, 2) \sim SL(2, R)_L \times SL(2, R)_R$. Harmonic superspace [11] has proven to be the adequate tool for manifestly supersymmetric formulations of many supersymmetric theories. The $N = 4$ SSDYM theory is just another example in this series. It should be mentioned that some time ago a different variational principle reproducing the self-duality condition on (nonsupersymmetric) Yang-Mills fields has been proposed in [12]. It used $SU(2)$ harmonics and involved a nonpropagating Lagrange multiplier. However, according to the analysis in [13], it does not describe any scattering and thus cannot be considered as a conventional field-theory action.

Self-dual Yang-Mills theory and supergravity have been studied from a different point of view in a series of papers by Devchand and Ogievetsky [14–16]. There the accent was on parametrizing all the solutions of such self-dual equations and eventually constructing some of those solutions. In a sense, they considered a kind of a twistor transform of the self-dual theories based on the harmonic superspace formalism. In this paper we shall use a similar formalism, but our main purpose will be to write down an *action* for the $N = 4$ SSDYM theory rather than to look for solutions to its field equations.

In Sec. II we recall some basic facts about harmonic superspace. We apply the formal rules developed for the case of $SU(2)/U(1)$ harmonics [11], ignoring possible subtleties due to the noncompactness of the coset $SL(2, R)/GL(1, R)$ in the case of Lorentz harmonics under consideration. In Sec. III we use harmonic superspace to rewrite the constraints of N -extended SSDYM in the form of integrability conditions. It then becomes possible to formulate the theory in terms of three gauge connections depending on one-fourth of the original number of

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¹A similar action for self-dual Yang-Mills theory had earlier appeared in a different context in [3].

²Some comments on the validity of the latter have recently been made in [8].

Grassmann variables. This is in fact the covariantization of the light-cone superspace used by Siegel (see also [15]). In it the self-duality equations have the form of zero-curvature conditions for the three gauge connections (the harmonic connection and the two harmonic projections of the space-time connection). This immediately suggests to write down an action of the *Chern-Simons type*. The $GL(1, R)$ weight (closely related to the physical dimension of the fields) of the Chern-Simons form only matches that of the superspace measure in the maximal case of $N = 4$ SSDYM. So, this action only makes sense for $N = 4$, although the same constraints and the same Chern-Simons form can be written down for any value $0 \leq N \leq 4$.

II. HARMONIC SUPERSPACE WITH SIGNATURE (2, 2)

The space with signature (2, 2) can be parametrized by coordinates $x^{\alpha\alpha'}$, where α and α' are spinor indices of $SL(2, R)_L$ and $SL(2, R)_R$ [the Lorentz group is $SO(2, 2) \sim SL(2, R)_L \times SL(2, R)_R$]. The N -extended superspace has the coordinates

$$x^{\alpha\alpha'}, \theta_a^\alpha, \theta^{\alpha'a}, \quad (1)$$

where a are co- or contravariant indices of the automorphism group $GL(N, R)$. In it one can realize N -extended supersymmetry as

$$\delta x^{\alpha\alpha'} = -\frac{1}{2}(\epsilon_a^\alpha \theta^{\alpha'a} + \epsilon^{\alpha'a} \theta_a^\alpha), \quad \delta \theta_a^\alpha = \epsilon_a^\alpha, \quad \delta \theta^{\alpha'a} = \epsilon^{\alpha'a}. \quad (2)$$

The corresponding algebra of supercovariant derivatives is

$$\begin{aligned} \{D_\alpha^a, D_\beta^b\} &= 0, \\ \{D_{\alpha'a}, D_{\beta'b}\} &= 0, \\ \{D_\alpha^a, D_{\beta'b}\} &= \delta_b^a \partial_{\alpha\beta}. \end{aligned} \quad (3)$$

We choose to "harmonize" one-half of the Lorentz group, e.g., the factor $SL(2, R)_R$. To this end we introduce real harmonic variables $u^{\pm\alpha'}$ defined as two $SL(2, R)_R$ spinors forming an $SL(2, R)_R$ matrix:

$$u^{\pm\alpha'} \in SL(2, R)_R : u^{+\alpha'} u_{\alpha'}^- = 1 \quad (4)$$

[raising and lowering the $SL(2, R)_R$ spinor indices is done with the ϵ tensor]. The index \pm refers to the weight of these variables with respect to transformations of $GL(1, R)_R \subset SL(2, R)_R$. Thus, the harmonic variables defined in this way should describe the noncompact coset $SL(2, R)_R/GL(1, R)_R$. However, we are going to apply to them the formal rules of harmonic calculus on the compact coset $SU(2)/U(1)$ [11]. In a certain sense, this corresponds to making a Wick rotation from signature (2, 2) to (4, 0). It is beyond the scope of this paper to give a rigorous justification of this approach. Nevertheless,

the formal rules will allow us to write down a superspace action which has the correct component content.

Here we give a short summary of the rules of harmonic calculus which we are going to use. Harmonic functions are defined by their harmonic expansion

$$f^{(q)}(u) = \sum_{n=0}^{\infty} f^{\alpha_1 \dots \alpha_{2n+q}} u_{(\alpha_1}^+ \dots u_{\alpha_{n+q}}^+ u_{\alpha_{n+q+1}}^- \dots u_{\alpha_{2n+q}}^-). \quad (5)$$

By definition, they are homogeneous under the action of $GL(1, R)_R$, i.e., they carry a certain weight q [in (5) $q \geq 0$]. From (5) it is clear that the harmonic functions are collections of infinitely many irreducible representations of $SL(2, R)_R$ (multispinors).

The principal differential operator compatible with the defining constraint (4) is the harmonic derivative

$$\partial^{++} = u^{+\alpha'} \frac{\partial}{\partial u^{-\alpha'}} : \partial^{++} u^{+\alpha'} = 0, \quad \partial^{++} u^{-\alpha'} = u^{+\alpha'}. \quad (6)$$

The other harmonic derivative on the two-dimensional coset $SL(2, R)_R/GL(1, R)_R$ is ∂^{--} ($\partial^{--} u^{-\alpha'} = 0$, $\partial^{--} u^{+\alpha'} = u^{-\alpha'}$), but we shall never make use of it.³

A direct consequence of the above definitions is the following lemma:

$$\begin{aligned} \partial^{++} f^{(q)}(u) \\ = 0 \Rightarrow \begin{cases} f^{(q)}(u) = 0, & q < 0 \\ f^{(q)}(u) = f^{\alpha_1 \dots \alpha_q} u_{(\alpha_1}^+ \dots u_{\alpha_q)}^+, & q \geq 0. \end{cases} \end{aligned} \quad (7)$$

Finally, harmonic integration amounts to projecting out the singlet part of a weightless integrand, according to the formal rule

$$\int du f^{(q)}(u) = \begin{cases} 0, & q \neq 0 \\ f_{\text{singlet}}, & q = 0. \end{cases} \quad (8)$$

This integration rule is designed to give a Lorentz-invariant result. It is compatible with integration by parts for the harmonic derivative ∂^{++} .

With the help of the above harmonic variables we can define Lorentz-covariant $GL(1, R)_R$ projections of the supercovariant derivatives from (3):

$$D_a^+ = u^{+\alpha'} D_{\alpha'a}, \quad \partial_\alpha^+ = u^{+\alpha'} \partial_{\alpha\alpha'}. \quad (9)$$

Together with the harmonic derivative ∂^{++} they form an algebra equivalent to the original one (3):

³There exists yet another derivative compatible with (4), ∂^0 ($\partial^0 u_{\alpha'}^\pm = \pm u_{\alpha'}^\pm$). As follows from (5), it just counts the $GL(1, R)_R$ weight of the harmonic functions, $\partial^0 f^{(q)}(u) = q f^{(q)}(u)$.

$$\{D_\alpha^a, D_\beta^b\} = [D_\alpha^a, \partial_\beta^+] = 0, \quad \{D_\alpha^a, D_b^+\} = \delta_b^a \partial_\alpha^+, \quad (10)$$

$$\{D_a^+, D_b^+\} = [D_a^+, \partial_\beta^+] = [\partial_\alpha^+, \partial_\beta^+] = 0, \quad (11)$$

$$[\partial^{++}, D_\alpha^a] = [\partial^{++}, D_a^+] = [\partial^{++}, \partial_\alpha^+] = 0. \quad (12)$$

To see the equivalence it is sufficient to apply the lemma (7) to the commutation relations (12) and thus restore the unprojected derivatives from (9). When removing the harmonics $u^{+\alpha'}$, $u^{+\beta'}$ from a relation such as, e.g., $\{D_a^+, D_b^+\} = 0$ we could, in principle, obtain terms proportional to $\epsilon_{\alpha'\beta'}$ in the right-hand side. However, the Lorentz index structure and the dimensions of the available superspace operators do not allow this (except for possible central charge terms, which we do not consider) and we reconstruct the original algebra (3).

The structure of the algebra (10), (11) suggests several new realizations of the N -extended supersymmetry algebra in subspaces of the harmonic superspace involving only part of the Grassmann variables θ . One of them is the chiral superspace which does not contain the variables θ_a^α . It is characterized by the coordinate shift

$$\text{Chiral basis: } x^{\alpha\alpha'} \rightarrow x^{\alpha\alpha'} - \frac{1}{2} \theta_a^\alpha \theta^{\alpha'a}. \quad (13)$$

In addition, we shall regard the following $GL(1, R)_R$ harmonic⁴ projections as independent variables:

$$x^{\pm\alpha} = u_{\alpha'}^\pm x^{\alpha\alpha'}, \quad \theta^{\pm a} = u_{\alpha'}^\pm \theta^{\alpha\alpha'}.$$

In this basis the covariant derivatives from (10)–(12) become

$$\begin{aligned} D_\alpha^a &= \partial_\alpha^a, \\ D_a^+ &= \partial_a^+ + \theta_a^\alpha \partial_\alpha^+, \\ D^{++} &= \partial^{++} + \theta^{+a} \partial_a^+ + x^{+\alpha} \partial_\alpha^+. \end{aligned} \quad (14)$$

Here and in what follows we use the notation

$$\partial_\alpha^a = \frac{\partial}{\partial \theta_\alpha^a}, \quad \partial_a^\pm = \frac{\partial}{\partial \theta^{\mp a}}, \quad \partial_\alpha^\pm = \frac{\partial}{\partial x^{\mp \alpha}}.$$

Note the appearance of vielbein terms in the harmonic derivative D^{++} in (14). So, in this basis the *chiral* superfields defined by the constraint

$$D_\alpha^a \Phi = 0 \quad \Rightarrow \quad \Phi = \Phi(x^{\pm\alpha}, \theta^{\pm a}, u) \quad (15)$$

do not depend on θ_a^α . In the chiral subspace the supersymmetry transformations are

$$\delta x^{\pm\alpha} = -\epsilon_\alpha^{\pm a} \theta^{\pm a}, \quad \delta \theta^{\pm a} = u_{\alpha'}^\pm \epsilon^{\alpha'a}, \quad \delta u_{\alpha'}^\pm = 0. \quad (16)$$

Another possibility offered by the algebra (10), (11) is to eliminate the projections θ^{-a} from the supersymmetry

transformations. To this end one makes the shift $x^{-\alpha} \rightarrow x^{-\alpha} + (1/2)\theta_a^\alpha \theta^{-a}$, after which the spinor derivative D_a^+ becomes $D_a^+ = \partial/\partial \theta^{-a}$. Then the *analytic* superfields defined by

$$D_a^+ \Phi = 0 \quad \Rightarrow \quad \Phi = \Phi(x^{\pm\alpha}, \theta^{+a}, \theta_a^\alpha, u) \quad (17)$$

do not depend on θ^{-a} . In this analytic subspace supersymmetry acts as

$$\begin{aligned} \delta x^{+\alpha} &= -\frac{1}{2}(\epsilon_a^\alpha \theta^{+a} + \epsilon^{+a} \theta_a^\alpha), \\ \delta x^{-\alpha} &= -\epsilon^{-a} \theta_a^\alpha, \\ \delta \theta^{+a} &= u_{\alpha'}^+ \epsilon^{\alpha'a}, \\ \delta \theta_a^\alpha &= \epsilon_a^\alpha, \quad \delta u_{\alpha'}^\pm = 0. \end{aligned} \quad (18)$$

A peculiarity of the harmonic superspace under consideration is the existence [14] of an even smaller superspace containing only θ^{+a} . It is defined by imposing the chirality (15) and analyticity (17) constraints *simultaneously*:

$$D_\alpha^a \Phi = D_a^+ \Phi = \partial_\alpha^+ \Phi = 0 \quad \Rightarrow \quad \Phi = \Phi(x^{+\alpha}, \theta^{+a}, u). \quad (19)$$

Note that the third constraint is an inevitable corollary of the first two and of the anticommutation relations (10). In a suitable superspace basis supersymmetry is realized on $x^{+\alpha}$ and θ^{+a} only:

$$\delta x^{+\alpha} = -\epsilon_a^\alpha \theta^{+a}, \quad \delta \theta^{+a} = u_{\alpha'}^+ \epsilon^{\alpha'a}, \quad \delta u_{\alpha'}^\pm = 0. \quad (20)$$

Such superfields are automatically on shell, since

$$\partial_\alpha^+ \phi = 0 \quad \Rightarrow \quad \square \phi = 2\partial^{+a} \partial_a^- \Phi = 0. \quad (21)$$

III. SELF-DUAL SUPERSYMMETRIC YANG-MILLS THEORY

A. Superspace constraints

N -extended ($0 \leq N \leq 4$) supersymmetric Yang-Mills theory is described by the algebra of the gauge-covariantized superspace derivatives from (3):

$$\begin{aligned} \{\nabla_\alpha^a, \nabla_\beta^b\} &= \epsilon_{\alpha\beta} \tilde{\phi}^{ab}, \quad [\nabla_\alpha^a, \nabla_{\beta\beta'}] = \epsilon_{\alpha\beta} \tilde{\chi}_{\beta'}^a, \\ \{\nabla_\alpha^a, \nabla_{\beta'b}\} &= \delta_b^a \nabla_{\alpha\beta'}, \\ \{\nabla_{\alpha'a}, \nabla_{\beta'b}\} &= \epsilon_{\alpha'\beta'} \phi_{ab}, \quad [\nabla_{\alpha'a}, \nabla_{\beta\beta'}] = \epsilon_{\alpha'\beta'} \chi_{\beta a}, \\ [\nabla_{\alpha\alpha'}, \nabla_{\beta\beta'}] &= \epsilon_{\alpha'\beta'} F_{\alpha\beta} + \epsilon_{\alpha\beta} F_{\alpha'\beta'}, \end{aligned} \quad (22)$$

where $\tilde{\phi}^{ab} = -\tilde{\phi}^{ba}$, $\phi^{ab} = -\phi^{ba}$ (for $N = 1$ the scalars ϕ drop out). In the nonsupersymmetric case $N = 0$ only the last relation in (22) remains. For $N = 0, 1, 2$ the theory is off shell, whereas for $N = 3, 4$ it is on shell. In

⁴Of course, chiral superspace can be defined without harmonic variables [as follows from the algebra (3)]. However, the latter will be needed below for the purpose of writing down an action for the $N = 4$ SSDYM theory.

addition, in the case $N=4$ one should require the two sets of six scalars to be related,⁵ $\tilde{\phi}^{ab} = (1/2)\epsilon^{abcd}\phi_{ab}$.

Self-duality means that half of the field strengths vanish, e.g., all those appearing in (22) multiplied by $\epsilon_{\alpha\beta}$ [of course, the $N=4$ relation $\tilde{\phi}^{ab} = (1/2)\epsilon^{abcd}\phi_{ab}$ does not hold any longer]. Thus we obtain the constraints of SSDYM theory:

$$\begin{aligned} \{\nabla_{\alpha}^a, \nabla_{\beta}^b\} &= 0, \\ \{\nabla_{\alpha}^a, \nabla_{\beta'}^b\} &= \delta_b^a \nabla_{\alpha\beta'}, \\ [\nabla_{\alpha}^a, \nabla_{\beta\beta'}] &= 0, \end{aligned} \quad (23)$$

$$\begin{aligned} \{\nabla_{\alpha'a}, \nabla_{\beta'b}\} &= \epsilon_{\alpha'\beta'}\phi_{ab}, \\ [\nabla_{\alpha'a}, \nabla_{\beta\beta'}] &= \epsilon_{\alpha'\beta'}\chi_{\beta a}, \\ [\nabla_{\alpha\alpha'}, \nabla_{\beta\beta'}] &= \epsilon_{\alpha'\beta'}F_{\alpha\beta}. \end{aligned} \quad (24)$$

Since the self-duality condition $F_{\alpha'\beta'} = 0$ on the Yang-Mills field is a dynamical equation, the constraints (23), (24) now describe an on-shell theory for any value $0 \leq N \leq 4$. This supermultiplet contains helicities from $+1$ down to $1 - N/2$. Clearly, it only becomes self-conjugate, i.e., spans all the helicities from $+1$ to -1 in the maximal case $N=4$. As a consequence, in the latter case the degrees of freedom appear in Lagrangian pairs and one is able to write down an action for the theory. So far this action has been presented either in component (i.e., not manifestly supersymmetric) [5, 9] or light-cone superspace (i.e., not manifestly Lorentz-invariant) form [10, 5]. Our purpose in this paper will be to write down the $N=4$ SSDYM action in a form which is both manifestly Lorentz-invariant and supersymmetric. To this end we shall first relax (23), (24) in order to go off shell and then we shall find a variational principle from which (23), (24) will follow as field equations.

Our first step will be to obtain a set of (anti)-commutation relations completely free from curvatures with the help of the harmonic variables introduced⁶ in Sec. II. Defining the harmonic projections [cf. (9)]

$$\nabla_{\alpha}^+ = u^{+\alpha'} \nabla_{\alpha'\alpha}, \quad \nabla_{\alpha'}^+ = u^{+\alpha} \nabla_{\alpha\alpha'}, \quad (25)$$

we obtain, from (23), (24),

$$\{\nabla_{\alpha}^a, \nabla_{\beta}^b\} = 0, \quad (26)$$

$$\{\nabla_{\alpha}^a, \nabla_b^+\} = \delta_b^a \nabla_{\alpha}^+, \quad (27)$$

$$[\nabla_{\alpha}^a, \nabla_{\beta}^+] = 0, \quad (28)$$

$$\{\nabla_{\alpha}^+, \nabla_b^+\} = 0, \quad (29)$$

$$[\nabla_{\alpha}^+, \nabla_{\beta}^+] = 0, \quad (30)$$

$$[\nabla_{\alpha}^+, \nabla_{\beta}^+] = 0. \quad (31)$$

In fact, these constraints are equivalent to the initial set (23), (24). To see this one takes into account the linear harmonic dependence of the projected covariant derivatives (25) and then pulls out the harmonics u^+ from the relations (27)–(31). In doing so the terms proportional to $\epsilon_{\alpha'\beta'}$ appear on the right-hand side of Eqs. (24). The information contained in (25) can also be encoded in the form of commutation relations with the harmonic derivative ∂^{++} [cf. (12) and recall (7)]:

$$[\partial^{++}, \nabla_{\alpha}^a] = [\partial^{++}, \nabla_{\alpha}^+] = [\partial^{++}, \nabla_{\alpha}^+] = 0. \quad (32)$$

This means that we first assume that the gauge connections $A_{\alpha}^a, A_{\alpha}^+, A_{\alpha}^+$ are arbitrary functions of the harmonic variables $u_{\alpha'}^{\pm}$. Then the role of the constraints (32) is to reduce this dependence to a trivial one. In fact, we can go a step further and start from a framework in which not only the gauge connections but also the gauge group parameters have an arbitrary dependence of the harmonic variables. This implies that the harmonic derivative ∂^{++} is covariantized as well:

$$\nabla^{++} = \partial^{++} + A^{++}(x, \theta, u).$$

Then Eqs. (32) are replaced by covariant ones:

$$[\nabla^{++}, \nabla_{\alpha}^a] = 0, \quad (33)$$

$$[\nabla^{++}, \nabla_{\alpha}^+] = 0, \quad (34)$$

$$[\nabla^{++}, \nabla_{\alpha}^+] = 0. \quad (35)$$

In order to go back to the frame in which the harmonic dependence is trivial it is sufficient to eliminate the newly introduced harmonic connection $A^{++}(x, \theta, u)$ by a suitable harmonic-dependent gauge transformation:

$$A^{++}(x, \theta, u) = e^{-\Lambda(x, \theta, u)} \partial^{++} e^{\Lambda(x, \theta, u)}.$$

This is always possible, since there is only one such connection (no integrability conditions). Then we recover the original constraints (32), from which we deduce the trivial harmonic dependence of the remaining connections $A_{\alpha}^a, A_{\alpha}^+, A_{\alpha}^+$.

For our purposes it will be preferable to stay in the frame with nontrivial harmonic dependence of the gauge objects. Even so, the constraints (26)–(31) and (33)–(35) still allow us to choose alternative special gauge frames. One possibility typical for other harmonic gauge theories (see [11, 19, 20]) would be to use the zero-curvature constraint (29) and gauge away the connections A_{α}^+ (“analytic frame”). In such a frame the notion of an analytic (θ^- -independent) superfield (17) is preserved. However, we do not find it useful in the present context. Instead, we can choose a chiral gauge frame in which the connection A_{α}^a vanishes [its existence is guaranteed by the zero-curvature condition (26)]. Although this could be done even before introducing harmonic variables, the relevance of the latter will become clear shortly. So, using the chiral basis (13) for the spinor derivatives, we can trivialize the covariant derivative ∇_{α}^a :

$$\text{chiral gauge: } A_{\alpha}^a = 0 \rightarrow \nabla_{\alpha}^a = \partial_{\alpha}^a. \quad (36)$$

Note that in this new gauge frame (36) the gauge parame-

⁵This is the analogue of the reality condition on the scalars in $N=4$ SYM theory in the case of Minkowskian signature (1,3) [18]. In the case of signature (2,2) the scalars are real by definition.

⁶Our treatment of the SSDYM constraints is, up to a certain point, similar to that in [15].

ters are chiral (i.e., independent of θ_a^α) but still harmonic dependent, $\Lambda(x, \theta^{\pm a}, u)$. Further, from (28), (36), and (33) we find

$$\begin{aligned} \partial_\alpha^\alpha A_\beta^+ &= \partial_\alpha^\alpha A^{++} = 0 \Rightarrow A_\alpha^+ = A_\alpha^+(x, \theta^{\pm a}, u), \\ A^{++} &= A^{++}(x, \theta^{\pm a}, u). \end{aligned} \quad (37)$$

Equation (27) then has the general solution

$$A_\alpha^+ = a_\alpha^+(x, \theta^{\pm a}, u) + \theta_\alpha^\alpha A_\alpha^+(x, \theta^{\pm a}, u). \quad (38)$$

Substituting (38) into (29), using (14) and collecting the terms with 0, 1, and 2 θ_α^α , we obtain the constraints

$$\partial_\alpha^+ a_b^+ + \partial_b^+ a_\alpha^+ + \{a_\alpha^+, a_b^+\} = 0, \quad (39)$$

$$\partial_\alpha^+ A_\beta^+ - \partial_\beta^+ a_\alpha^+ + [a_\alpha^+, A_\beta^+] = 0, \quad (40)$$

$$\partial_\alpha^+ A_\beta^+ - \partial_\beta^+ A_\alpha^+ + [A_\alpha^+, A_\beta^+] = 0. \quad (41)$$

The first of them implies that the part $a_\alpha^+(x, \theta^{\pm a}, u)$ of A_α^+ (38) is pure gauge and can be gauged away by a suitable gauge transformation:

$$\delta A_\alpha^+ = \partial_\alpha^+ \Lambda + [A_\alpha^+, \Lambda] \quad (42)$$

with a chiral parameter Λ . From now on it will be convenient to work in the semianalytic gauge:

$$a_\alpha^+(x, \theta^{\pm a}, u) = 0. \quad (43)$$

Then from (40) we find

$$\partial_\alpha^+ A_\beta^+ = 0 \Rightarrow A_\alpha^+ = A_\alpha^+(x, \theta^{\pm a}, u). \quad (44)$$

The next step is to insert all the above results in Eq. (34). The θ_α^α -independent term gives

$$\partial_\alpha^+ A^{++} = 0 \Rightarrow A^{++} = A^{++}(x, \theta^{\pm a}, u) \quad (45)$$

and the term linear in θ_α^α yields the constraint

$$\partial_\alpha^+ A^{++} - D^{++} A_\alpha^+ + [A_\alpha^+, A^{++}] = 0. \quad (46)$$

Among the remaining constraints only that on the connections A_α^+ (41) is independent, Eqs. (30), (31), and (35) then follow.

Comparing the harmonic treatment of the SSDYM constraints given here with the more traditional approach to harmonic gauge theories in [11, 19, 20], we see that here we gauge away the spinor connection A_α^+ only partially (43) [the remaining part of it is related to the vector connection A_α^+ , see (38)]. At the same time, the other spinor connection A_α^α is fully gauged away [chiral gauge (36)]. This mixed chiral-semianalytic gauge explains why we needed to keep a nontrivial harmonic dependence when introducing the chiral gauge (36). In fact, we could go one more step further and fix a fully analytic gauge in which the entire spinor connection A_α^+ [or, equivalently, the vector connection A_α^+ , see (38)] is gauged away. This is permitted by the zero-curvature condition (41). In this case we would obtain a twistor transform of the on-shell SSDYM fields [see the discussion around Eq. (67)]. However, for the purpose of writing down an action we should keep the set of three gauge connections A_α^+ , A^{++} which are functions of only one-fourth of the Grassmann variables:

$$A_\alpha^+ = A_\alpha^+(x, \theta^+, u), \quad A^{++} = A^{++}(x, \theta^+, u). \quad (47)$$

These connections undergo gauge transformation

$$\begin{aligned} \delta A_\alpha^+ &= \partial_\alpha^+ \Lambda + [A_\alpha^+, \Lambda], \\ \delta A^{++} &= D^{++} \Lambda + [A^{++}, \Lambda], \\ \Lambda &= \Lambda(x, \theta^+, u), \end{aligned} \quad (48)$$

which are compatible with the chiral-semianalytic gauge (36), (43). The connections are put on shell by the three zero-curvature conditions:

$$\partial_\alpha^+ A^{++} - D^{++} A_\alpha^+ + [A_\alpha^+, A^{++}] = 0, \quad (49)$$

$$\partial_\alpha^+ A_\beta^+ - \partial_\beta^+ A_\alpha^+ + [A_\alpha^+, A_\beta^+] = 0. \quad (50)$$

All this represents an equivalent reformulation of the N -extended SSDYM theory and will serve as the basis for our action in the case $N = 4$. Before addressing the issue of the action, we would like to make a number of comments.

B. Supersymmetry transformations

The gauge connections (47) do not transform as superfields under the right-handed part (parameters ϵ_a^α) of the supersymmetry algebra (16). Indeed, A_α^+ in (38) is a supercovariant object but its component a_α^+ from the θ_α^α expansion is not:

$$\delta a_\alpha^+ = (\epsilon_b^\beta \theta^{+b} \partial_\beta^- + \epsilon_b^\beta \theta^{-b} \partial_\beta^+ - \epsilon^{+b} \partial_b^-) a_\alpha^+ - \epsilon_a^\alpha A_\alpha^+. \quad (51)$$

Here we have explicitly written out the supertranslation terms. Earlier we fixed the gauge (43) which is violated by the inhomogeneous term in (51). In order to correct this we have to accompany the supersymmetry transformation by a compensating gauge transformation (42) with the parameter

$$\Lambda = (\theta^{-a} \epsilon_a^\alpha) A_\alpha^+(x, \theta^+, u). \quad (52)$$

This gauge transformation affects the connections A_α^+ , A^{++} as well, so their supersymmetry transformation laws are modified:

$$\begin{aligned} \delta A_\alpha^+ &= (\epsilon_b^\beta \theta^{+b} \partial_\beta^- - \epsilon^{+b} \partial_b^-) A_\alpha^+ \\ &\quad + (\theta^{-b} \epsilon_b^\beta) (\partial_\alpha^+ A_\beta^+ - \partial_\beta^+ A_\alpha^+ + [A_\alpha^+, A_\beta^+]), \quad (53) \\ \delta A^{++} &= (\epsilon_b^\beta \theta^{+b} \partial_\beta^- - \epsilon^{+b} \partial_b^-) A^{++} \\ &\quad + (\theta^{-b} \epsilon_b^\beta) (D^{++} A_\beta^+ - \partial_\beta^+ A^{++} + [A^{++}, A_\beta^+]) \\ &\quad + (\theta^{+b} \epsilon_b^\beta) A_\beta^+. \quad (54) \end{aligned}$$

The terms containing θ^- are proportional to the constraints (49), (50), so they drop out. Remarkably, the connections $A_\alpha^+(x, \theta^+, u)$, $A^{++}(x, \theta^+, u)$ transform as if the supersymmetry transformations (16) did not involve θ^- at all:

$$\delta A_\alpha^+ = (\epsilon_b^\beta \theta^{+b} \partial_\beta^- - \epsilon^{+b} \partial_b^-) A_\alpha^+, \quad (55)$$

$$\delta A^{++} = (\epsilon_b^\beta \theta^{+b} \partial_\beta^- - \epsilon^{+b} \partial_b^-) A^{++} + (\theta^{+b} \epsilon_b^\beta) A_\beta^+. \quad (56)$$

It must be stressed that these objects should not be con-

fused with the chiral-analytic superfields defined by Eq. (19). The latter do not depend on $x^{-\alpha}$ and are thus automatically on shell. For our purposes it is essential that the superfields A_{α}^{+} , A^{++} can exist off shell too. Therefore we shall never require $\partial_{\alpha}^{+}A = 0$.

Thus, Eqs. (55) and (56) are the transformation laws of "semivariant" superfields. Another way to say this is to point out that commuting two such supersymmetry transformations one obtains the required translations in the direction $x^{-\alpha}$ only with the help of the constraint (49) and of a compensating gauge transformation (42) with parameter $\Lambda = (\epsilon_1^{-\alpha}\epsilon_{2\alpha} - \epsilon_2^{-\alpha}\epsilon_{1\alpha})A_{\alpha}^{+}$ [this follows from (52) as well]. From this point of view the harmonic derivative D^{++} (14) is not supercovariant either, i.e., it does not commute with the translation part of the transformations (55), (56):

$$[D^{++}, \epsilon_b^{\beta}\theta^{+\beta}\partial_{\beta}^{-} - \epsilon^{+\beta}\partial_{\beta}^{-}] = -\epsilon_b^{\beta}\theta^{+\beta}\partial_{\beta}^{+} + \epsilon^{+\beta}\partial_{\beta}^{+}. \quad (57)$$

The supersymmetry transformation rules (55), (56), in which θ^{-} never appears, were obtained using the on-shell constraints (49), (50). When writing down the $N=4$ action in Sec. IV we shall treat the connections $A_{\alpha}^{+}(x, \theta^{+}, u)$, $A^{++}(x, \theta^{+}, u)$ as *unconstrained* objects. Nevertheless, we shall apply the same supersymmetry rules to them. What is important in this context is to make sure that the left-hand sides of the constraints still form a supersymmetric set, i.e., that they transform into each other. Indeed, this is easy to check using the transformation laws (55), (56), (57).

C. Components

Now we would like to give a direct demonstration that the constraints (49), (50) do indeed describe N -extended SSDYM theory. To this end we shall exhibit the component content of the gauge superfields $A_{\alpha}^{+}(x, \theta^{+}, u)$, $A^{++}(x, \theta^{+}, u)$. Let us first consider the simplest case $N=1$. The harmonic connection has a very short Grassmann expansion:

$$A^{++} = a^{++}(x, u) + \theta^{+}\sigma^{+}(x, u). \quad (58)$$

The fields in (58) are harmonic, i.e., they contain infinitely many ordinary fields [recall (5)]. However, we still have the gauge transformations (48) with the parameter

$$\Lambda = \lambda(x, u) + \theta^{+}\rho^{-}(x, u). \quad (59)$$

Let us compare the harmonic expansions (5) of the bosonic components in (58) and (59):

$$\begin{aligned} a^{++}(x, u) &= u_{(\alpha}^{+}u_{\beta)}^{+}a^{\alpha\beta'}(x) \\ &\quad + u_{(\alpha}^{+}u_{\beta}^{+}u_{\gamma}^{+}u_{\delta)}^{-}a^{\alpha\beta'\gamma'\delta'}(x) + \dots, \\ \lambda(x, u) &= \lambda(x) + u_{(\alpha}^{+}u_{\beta)}^{-}\lambda^{\alpha\beta'}(x) \\ &\quad + u_{(\alpha}^{+}u_{\beta}^{+}u_{\gamma}^{-}u_{\delta)}^{-}\lambda^{\alpha\beta'\gamma'\delta'}(x) + \dots \end{aligned}$$

Clearly, the parameter $\lambda(x, u)$ contains enough components to completely gauge away the harmonic field $a^{++}(x, u)$ [note that the singlet part $\lambda(x)$ in $\lambda(x, u)$ is not used in the process; it remains nonfixed and plays the role of the ordinary gauge parameter]. Similarly, the parameter $\rho^{-}(x, u)$ can gauge away the entire field $\sigma^{+}(x, u)$. Thus, we arrive at $N=1$ Wess-Zumino gauge:

$$A^{++} = 0. \quad (60)$$

The other gauge connection has the expansion

$$A_{\alpha}^{+} = A_{\alpha}^{+}(x, u) + \theta^{+}\chi_{\alpha}(x, u). \quad (61)$$

The harmonic dependence in it can be eliminated by using the constraint (49). Substituting (60) and (61) into (49), we obtain $D^{++}A_{\alpha}^{+} = 0$, from which follow the harmonic equations

$$\begin{aligned} \partial^{++}A_{\alpha}^{+}(x, u) = 0 &\Rightarrow \mathcal{A}_{\alpha}^{+}(x, u) = u^{+\alpha'}\mathcal{A}_{\alpha\alpha'}(x), \\ \partial^{++}\chi_{\alpha}(x, u) = 0 &\Rightarrow \chi_{\alpha}(x, u) = \chi_{\alpha}(x). \end{aligned} \quad (62)$$

Then, inserting (62) into the remaining constraint (50), we obtain the self-duality equation for the Yang-Mills field $\mathcal{A}_{\alpha\alpha'}(x)$ and the Dirac equation for the chiral spinor $\chi_{\alpha}(x)$:

$$F^{\alpha'\beta'} = \partial^{\alpha(\alpha'}\mathcal{A}_{\alpha}^{\beta')} + \mathcal{A}^{\alpha(\alpha'}\mathcal{A}_{\alpha}^{\beta')} = 0, \quad \nabla^{\alpha\alpha'}\chi_{\alpha} = 0, \quad (63)$$

where $\nabla_{\alpha\alpha'} = \partial_{\alpha\alpha'} + [\mathcal{A}_{\alpha\alpha'}, \]$ denotes the usual Yang-Mills covariant derivative. This is precisely the content of the $N=1$ SSDYM multiplet.

It is not hard to find out the supersymmetry transformation laws of the component fields $\mathcal{A}_{\alpha\alpha'}$, χ_{α} . To this end we note that in order for the supersymmetry transformation (56) not to violate the Wess-Zumino gauge (60), we have to make a compensating gauge transformation (48) with parameter $\Lambda = (\theta^{+}\epsilon^{\alpha})\mathcal{A}_{\alpha}^{-}$ [where $\mathcal{A}_{\alpha}^{-} = u^{-\alpha'}\mathcal{A}_{\alpha\alpha'}(x)$]. Then the combination of (55) with this gauge transformation gives

$$\delta A_{\alpha}^{+} = -\epsilon^{+}\chi_{\alpha} - \theta^{+}\epsilon^{\beta}(\partial_{\beta}^{-}\mathcal{A}_{\alpha}^{+} + \partial_{\alpha}^{+}\mathcal{A}_{\beta}^{-} + [\mathcal{A}_{\alpha}^{-}, \mathcal{A}_{\beta}^{+}]).$$

This, together with the field equation for \mathcal{A} (63) leads to the transformation laws

$$\delta\mathcal{A}_{\alpha\alpha'} = -\epsilon_{\alpha'}\chi_{\alpha}, \quad \delta\chi_{\alpha} = -\frac{1}{2}\epsilon^{\beta}F_{\alpha\beta},$$

where $F_{\alpha\beta}$ is the self-dual part of the Yang-Mills curvature.

The maximal case $N=4$ follows the same pattern. There one can fix the $N=4$ WZ gauge:

$$\begin{aligned} A^{++} &= (\theta^{+})^{2ab}\phi_{ab}(x) \\ &\quad + (\theta^{+})^3u^{-\alpha'}\chi_{\alpha}^{\alpha}(x) \\ &\quad + (\theta^{+})^4u^{-\alpha'}u^{-\beta'}G_{\alpha'\beta'}(x), \end{aligned} \quad (64)$$

where

$$\begin{aligned}
(\theta^+)^{2ab} &= \frac{1}{2!} \theta^{+a} \theta^{+b}, \\
(\theta^+)^3_a &= \frac{1}{3!} \epsilon_{abcd} \theta^{+b} \theta^{+c} \theta^{+d}, \\
(\theta^+)^4 &= \frac{1}{4!} \epsilon_{abcd} \theta^{+a} \theta^{+b} \theta^{+c} \theta^{+d}.
\end{aligned}$$

The other gauge connection has the expansion

$$\begin{aligned}
A_\alpha^+ &= \mathcal{A}_\alpha^+(x, u) + \theta^{+a} \chi_{\alpha a}(x, u) + (\theta^+)^{2ab} B_{\alpha ab}^-(x, u) \\
&\quad + (\theta^+)^3_a \tau_\alpha^{-a}(x, u) + (\theta^+)^4 C_\alpha^{-3}(x, u). \quad (65)
\end{aligned}$$

With the help of the constraint (49) one eliminates the harmonic dependence in (65):

$$\begin{aligned}
\mathcal{A}_\alpha^+(x, u) &= u^{+\alpha'} \mathcal{A}_{\alpha\alpha'}(x), \quad \chi_{\alpha a}(x, u) = \chi_{\alpha a}(x), \\
B_{\alpha ab}^- &= u^{-\alpha'} \nabla_{\alpha\alpha'} \phi_{ab}(x), \\
\tau_\alpha^{-a}(x, u) &= \frac{1}{2} u^{-\alpha'} u^{-\beta'} \nabla_{\alpha\alpha'} \chi_{\beta'}^a(x), \\
C_\alpha^{-3}(x, u) &= \frac{1}{3} u^{-\alpha'} u^{-\beta'} u^{-\gamma'} \nabla_{\alpha\alpha'} G_{\beta'\gamma'}(x). \quad (66)
\end{aligned}$$

In addition, the constraint (50) implies that the physical fields ϕ_{ab} , $\chi_{\alpha a}$, $\chi_{\alpha'}^a$ satisfy their equations of motion. The fields $\mathcal{A}_{\alpha\alpha'}$ and $G_{\alpha'\beta'}$ describe a self-dual and an anti-self-dual gauge fields, respectively. Thus we find the complete content of the $N = 4$ SSDYM multiplet, as given in [10, 5, 9]. Following the $N = 1$ example, it is not hard to also derive the supersymmetry transformation laws from [5, 9].

Here we would like to comment on another possibility to fix the gauge. Above we showed that a nonsupersymmetric off-shell gauge of the Wess-Zumino type is necessary in order to obtain the standard components of the theory. However, on shell there exists an alternative, manifestly supersymmetric gauge. Indeed, the constraint (50) tells us that A_α^+ is *pure gauge on shell*. If we fix the supersymmetric on-shell gauge:

$$A_\alpha^+ = 0, \quad (67)$$

the remaining constraint (49) becomes simply

$$\partial_\alpha^+ A^{++} = 0. \quad (68)$$

It means that the components of A^{++} are harmonic fields $\phi(x^{+\alpha}, u)$ independent of $x^{-\alpha}$. As explained in (21), such fields are automatically on shell. In fact, what we encounter here are *twistor-type solutions* of massless equations of motion.⁷ [17]. Thus, we can say that the connection A^{++} now contains the set of twistor transforms of all the fields of the N -extended SSDYM multiplet. In other words, the solutions to the SSDYM equations are

encoded in a single on-shell superfield A^{++} . The situation here closely resembles the harmonic version of the twistor transform of the ordinary ($N = 0$) SDYM equations, where all the self-dual solutions (e.g., instantons) are parametrized by a single object $V^{++}(x_\alpha^+, u_\alpha^+)$ [19]. This line of study of the SSDYM system has been proposed and pursued in [14, 15]. In the present paper we are interested in an *action* for the $N = 4$ theory, therefore the on-shell gauge (67) will not be implemented.

D. Action for $N = 4$ SSDYM

Up to now the whole discussion applied equally well to all values $0 \leq N \leq 4$. The unique features of the case $N = 4$ only become important when one tries to write down an action. At the component level this is manifested in the fact that the $N = 4$ multiplet contains all the helicities from $+1$ (described by the self-dual field \mathcal{A}) down to -1 (the field G). It is precisely the field $G_{\alpha'\beta'}$ which can serve as a (propagating) Lagrange multiplier for the self-duality condition on $\mathcal{A}_{\alpha\alpha'}$. Similarly, the spinor fields $\chi_{\alpha'}^a$ and $\chi_{\alpha a}$ form a Lagrangian pair. The form of this action first given in [5, 9] is

$$\begin{aligned}
S &= \int d^4x \operatorname{Tr} \left\{ -\frac{1}{8} \nabla^{\alpha\alpha'} \phi^{ab} \nabla_{\alpha\alpha'} \phi_{ab} + \frac{1}{2} \chi_{\alpha'}^a \nabla^{\alpha\alpha'} \chi_{\alpha a} \right. \\
&\quad \left. + \frac{1}{6} G^{\alpha'\beta'} F_{\alpha'\beta'} - \phi^{ab} \chi_{\alpha'}^a \chi_{\alpha b} \right\}. \quad (69)
\end{aligned}$$

The purpose of the harmonic superspace formalism developed above was to write down an *action* for $N = 4$ SSDYM with manifest Lorentz invariance and supersymmetry. So far we have rewritten the on-shell constraints of the theory in the equivalent form (49), (50). Now we want to obtain these dynamical equations from a variational principle. Equations (49) and (50) have the form of vanishing curvature conditions. Note also the important fact that we have three gauge connections A^{++} , A_α^+ and, correspondingly, three curvatures made out of them. All this suggests to write down the *Chern-Simons form*

$$\begin{aligned}
L^{+4}(x, \theta^+, u) &= \operatorname{Tr} (A^{++} \partial^{+\alpha} A_\alpha^+ - \frac{1}{2} A^{+\alpha} D^{++} A_\alpha^+ \\
&\quad + A^{++} A^{+\alpha} A_\alpha^+). \quad (70)
\end{aligned}$$

Since the connections in (70) are not covariant superfields, we should find out how the Chern-Simons form transforms under supersymmetry. Using (55), (56), (57) it is easy to check that

$$\begin{aligned}
\delta L^{+4} &= (\epsilon_b^\beta \theta^{+b} \partial_\beta^- - \epsilon^{+b} \partial_b^-) L^{+4} \\
&\quad + \frac{1}{2} (\theta^{+b} \epsilon_b^\beta) \partial^{+\alpha} \operatorname{Tr} (A_\alpha^+ A_\beta^+), \quad (71)
\end{aligned}$$

i.e., it transforms into a total derivative with respect to the variables $x^{\pm\alpha}$ and θ^{+a} . Similarly, the gauge transformations (48) give

$$\delta L^{+4} = \partial^{+\alpha} \operatorname{Tr} (D^{++} \Lambda A_\alpha^+) - \frac{1}{2} D^{++} \operatorname{Tr} (\partial^{+\alpha} \Lambda A_\alpha^+), \quad (72)$$

⁷We should repeat here that our harmonic variables are defined on the noncompact coset $SL(2, R)_R / GL(1, R)_R$. Therefore the analogy with the standard twistor approach [17], based on the compact coset $S^2 \sim SU(2)/U(1)$, remains formal.

which is a total derivative with respect to the variables x_α^- and u_α^\pm . This allows us to write down the supersymmetric and gauge invariant action as an integral over the "1/4 superspace" $x^{\pm\alpha}, \theta^{+\alpha}, u_\alpha^\pm$:

$$S = \int d^4 x d u d^4 \theta^+ L^{+4}(x, \theta^+, u). \quad (73)$$

Obviously, the variation with respect to the superfields A^{++}, A_α^+ produces the desired field equations (49), (50). Let us now make sure that the action (73) has the correct physical dimension. Indeed, the gauge connections have dimensions $[A^{++}] = 0$ (the harmonic variables are dimensionless), $[A_\alpha^+] = 1$, so the dimension of the Lagrangian is $[L] = 2$. At the same time, the superspace measure has dimension $4[d x] + 4[d \theta] = -4 + 2 = -2$, thus exactly compensating that of the Lagrangian. Another property closely related to the physical dimension is the harmonic weight of the Lagrangian. By definition, the harmonic integral in (73) would only give a nonvanishing result if the integrand has zero weight [recall (8)]. This is indeed true, since the weight +4 of the Lagrangian is cancelled out by the weight of the Grassmann measure $d^4 \theta^+$. The last point clearly shows that an action of this type is only possible in the maximal case $N = 4$, although we could have written down the Chern-Simons form (70) for any $0 \leq N \leq 4$. The light-cone action of [2, 10, 5] can formally be written down for $0 \leq N < 4$ too, although then it requires a dimensional coupling constant, which is not natural for a Yang-Mills theory in four dimensions (see the discussion in [5, 6]).

Finally, we shall show that the component form of the action (73) is the same as (69). Inserting the Wess-Zumino gauge (64) for A^{++} and the expansion (65) of A_α^+ into (73) and doing the Grassmann integral, we obtain

$$S = \int d^4 x d u \operatorname{Tr} \left\{ \frac{1}{2} \phi^{ab} \nabla^{+\alpha} B_{\alpha ab}^- + \chi^{-\alpha} \nabla^{+\alpha} \chi_{\alpha a} + G^{--} F^{++} - C^{-3\alpha} \partial^{++} \mathcal{A}_\alpha^+ - \tau^{--\alpha a} \partial^{++} \chi_{\alpha a} - \frac{1}{4} B^{-\alpha ab} \partial^{++} B_{\alpha ab}^- - \phi^{ab} \chi_a^\alpha \chi_{\alpha b} \right\}, \quad (74)$$

where $\phi^{ab} = \frac{1}{2} \epsilon^{abcd} \phi_{cd}$, $\nabla^{+\alpha} = \partial^{+\alpha} + [A^{+\alpha}, \]$, $F^{++} = \partial^{+\alpha} \mathcal{A}_\alpha^+ + \mathcal{A}^{+\alpha} \mathcal{A}_\alpha^+$. The fields $B_{\alpha ab}^-(x, u)$, $C^{-3\alpha}(x, u)$, $\tau^{--\alpha a}(x, u)$ are clearly auxiliary. They give rise to the harmonic equations

$$\partial^{++} \mathcal{A}_\alpha^+ = 0, \quad \partial^{++} \chi_{\alpha a} = 0, \quad \partial^{++} B^{-\alpha ab} - \nabla^{+\alpha} \phi^{ab} = 0,$$

which allow us to eliminate the harmonic dependence of \mathcal{A}_α^+ and $\chi_{\alpha a}$ and to express $B^{-\alpha ab}$ in terms of ϕ^{ab} [see

(66)]. Afterwards the harmonic integral in (74) becomes trivial and we arrive at the action (69).

IV. CONCLUSIONS

In this paper we presented a harmonic superspace formulation of the N -extended supersymmetric self-dual Yang-Mills theory in a space with signature (2, 2). We were able to write down an action for the case $N = 4$ with manifest Lorentz invariance and supersymmetry. The most unusual feature is that the Lagrangian is a Chern-Simons form. In this the $N = 4$ SSDYM theory resembles the $N = 3$ SYM theory [signature (1, 3)] formulated in a harmonic superspace with harmonics parametrizing the coset $SU(3)/[U(1) \times U(1)]$ [20]. The main difference is that in the $N = 3$ SYM case the Chern-Simons form is made out of harmonic connections only, whereas in the $N = 4$ SSDYM case we used two space-time and one harmonic one. In both cases the manifestly supersymmetric formulation greatly facilitates the study of the quantum properties of the theory.

We remark that a similar formulation exists for the $N = 2$ free "self-dual" scalar multiplet defined in [5, 9]. It can be described by the *anticommuting* harmonic superfields $A_{\alpha i}(x, \theta^+, u)$, $A_i^+(x, \theta^+, u)$, where i is an index of, e.g., an internal symmetry group $SL(2, R)$. The action is very similar to (73):

$$S = \int d^4 x d u d^2 \theta^+ \left(A^{+i} \partial^{+\alpha} A_{\alpha i} - \frac{1}{2} A^{\alpha i} D^{++} A_{\alpha i} \right). \quad (75)$$

The most complicated case of a self-dual theory in the space with signature (2, 2) is $N = 8$ supergravity. As shown in [5], using a light-cone superspace it can be treated in the same fashion as the self-dual scalar and Yang-Mills theories. In a future publication we shall present a harmonic superspace formulation of $N = 8$ self-dual supergravity. It will allow us, in particular, to systematically derive all the supersymmetry transformation laws of the component fields (they were given in [5] only partially).

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