Supersymmetric Liouville theory: A statistical mechanical approach

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The statistical mechanical system associated with the two-dimensional supersymmetric Liouville theory is obtained through an infrared-finite perturbation expansion. Considering the system confined in a finite volume and in the presence of a uniform neutralizing background, we show that the grand-partition function of this system describes a one-component gas, in which the Boltzmann factor is weighted by an integration over the Grassmann variables. This weight function introduces the dimensional reduction phenomenon. After performing the thermodynamic limit, the resulting supersymmetric quantum theory is translationally invariant.

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I. INTRODUCTION

The two-dimensional Liouville field theory (LT) attracts considerable interest due to the important role played by it in the quantum theory of bosonic strings [1]. For space-time dimensions $D < 10$, the supersymmetric (SUSY) version of the Liouville theory plays a similar role in the quantum theory of fermionic strings [2]. The effective theory of free Fermi strings, which describes the sum over random surfaces with fermionic structure, is described by the SUSY LT. The computation of scattering amplitudes for fermionic strings is then reduced to the computation of correlation functions of the two-dimensional SUSY Liouville field theory.

General results, some of them exact [3,5–10], have been established in the literature of statistical mechanical systems associated with two-dimensional quantum field theories, like the standard sine-Gordon (SG), SUSY SG, and Liouville theories (LT's).

The standard sine-Gordon theory has been known for a long time to be equivalent to the two-dimensional Coulomb gas of point particles in the grand-canonical ensemble [3]. Using perturbation expansion, the equation of state of this statistical mechanical system is obtained and predicts the existence of a Kosterlitz-Thouless (KT) phase transition [4] at a critical temperature $\beta_c^2 = 8\pi$.

The supersymmetric generalization of the two-dimensional sine-Gordon theory was considered in Refs. [5–7]. In Ref. [7] it was shown that supersymmetry introduces the dimensional reduction phenomenon. The equation of state of the corresponding statistical mechanical system shows that the supersymmetry preserves the KT phase transition at the new critical temperature $\beta_c^2 = 4\pi$.

The statistical mechanical system associated with the two-dimensional Liouville theory was considered in Refs. [8,9]. The equation of state is obtained exactly [8], showing the absence of a KT phase transition, and corresponds to the equation of state of a two-dimensional ideal gas.

The main purpose of this paper is to study the statistical mechanical system associated with the supersymmetric generalization of the Liouville theory (SUSY LT).

Analogously to what happens in the LT [8], in its supersymmetric version the monotonicity of the potential piece associated with the primary boson field is responsible for the infrared instability (non-neutrality) of the corresponding statistical system. In order to ensure translational invariance of the fundamental bosonic sector, as well as to prevent the infrared instability problem, a constant neutralizing background must be introduced. A subsidiary condition on the auxiliary field must be also introduced such that, in the thermodynamic limit, the supercharge annihilates the vacuum and supersymmetry is not destroyed in the resulting quantum theory.

This paper is organized as follows. In Sec. II we obtain the grand-partition function describing the infrared stable (screened) statistical system associated with the SUSY LT. The exact equation of state is obtained. In Sec. III we show the existence of a translationally invariant ground state which implies also that the superfield satisfies a "free field weak condition." We summarize and discuss our results in Sec. IV.

II. VACUUM FUNCTIONAL AND EQUATION OF STATE

In superfield notation, the two-dimensional SUSY LT is defined by the action[1]

$$\mathcal{A} = \int d^2z\, d^2\theta \left\{ \frac{1}{2}\mathcal{D}_+\Phi(z,\theta)\mathcal{D}_-\Phi(z,\theta) - \alpha_0\, e^{\beta\,\Phi(z,\theta)} \right\}. \tag{2.1}$$

[1]The conventions used are the following. The Hermitian γ^μ matrices are $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A Majorana spinor is defined by $\psi = \begin{pmatrix} \chi^* \\ \chi \end{pmatrix}$, such that $\psi_c = \gamma^0\,\psi^* = \psi$.

In terms of component fields, the real scalar superfield is given by

$$\Phi(z,\theta) = \varphi(z) + \bar{\Theta}\,\psi(z) + \frac{1}{2}\bar{\Theta}\,\Theta\,\mathcal{F}(z) \ , \qquad (2.2)$$

where φ is a scalar field, ψ is a two-component Majorana fermion, and $\mathcal{F}$ is a subsidiary scalar field. The superspace Grassmann coordinates are given in terms of the Majorana spinor $\Theta = (\theta, \theta^*)^T$. The covariant derivatives are

$$\mathcal{D}_- = \partial_{\theta^*} - \theta^* \partial_- \ ,$$

$$\mathcal{D}_+ = \partial_\theta + \theta\,\partial_+ \ , \qquad (2.3)$$

with $\partial_\pm = \partial_0 \pm i\partial_1$.

The component field representation is obtained by expanding the functionals of the superfield in a Taylor series with respect to the Grassmann variables. Using (2.2) and (2.3), expanding the superpotential in (2.1), we obtain, after integration over the Grassmann variables,

$$\mathcal{A} = \int d^2z \left[\frac{1}{2}\partial_\mu\varphi(z)\,\partial^\mu\varphi(z) + \bar{\psi}(z)\gamma_\mu\partial^\mu\psi(z) - \frac{1}{2}\mathcal{F}^2(z) - \alpha_0\beta\mathcal{F}(z)\,e^{\beta\,\varphi(z)} - \frac{1}{2}\alpha_0\,\beta\,\bar{\psi}(z)\psi(z)e^{\beta\,\varphi(z)}\right] . \qquad (2.4)$$

Taking the minimum of the action with respect to $\mathcal{F}$ we obtain

$$\mathcal{F}(z) = \alpha_0\,\beta\,e^{\beta\,\varphi(z)} \ . \qquad (2.5)$$

Thus the subsidiary field can be eliminated from the action (2.4) and we get

$$\mathcal{A} = \int d^2z \left[\frac{1}{2}\partial_\mu\varphi(z)\partial^\mu\varphi(z) + \frac{1}{2}\bar{\psi}(z)\gamma_\mu\partial^\mu\psi(z) - \frac{1}{2}\alpha_0\,\beta^2\,e^{2\beta\,\varphi(z)} - \frac{1}{2}\alpha_0\,\beta^2\,\bar{\psi}(z)\psi(z)\,e^{\beta\,\varphi(z)}\right] . \qquad (2.6)$$

The above action is not invariant under translation of the primary bosonic φ field. As we will see, the monotonicity of the potential associated with φ is responsible for the infrared instability (non-neutrality) of the corresponding statistical system. This is more conveniently performed using the superfield notation.

To begin with, let us consider the modified action

$$\mathcal{A}_\mathcal{V} = \int d^2z\,d^2\theta \left\{\frac{1}{2}\mathcal{D}_+\Phi(z,\theta)\,\mathcal{D}_-\Phi(z,\theta) - \alpha_0\,e^{\beta\,[\,\Phi(z,\theta)-\Phi_v\,(\theta)\,]}\right\} \ , \qquad (2.7)$$

where

$$\Phi_v(\theta) = \frac{1}{\mathcal{V}}\int d^2z\,d^2\omega\,(\omega\,\omega^* \ - \ \theta\,\theta^*)\,\Phi(z,\omega) \ . \qquad (2.8)$$

The volume is $\mathcal{V} = \int d^2z\,d^2\theta\,\theta\,\theta^* = \pi\,\mathcal{R}^2$. The action (2.7) is invariant under translation of φ and in the limit $\mathcal{R} \to \infty$ reduces to the original action (2.1).

The first term in expression (2.8) introduces the uniform neutralizing background via

$$\varphi_v = \frac{1}{\mathcal{V}}\int d^2z\,\varphi(z) \ , \qquad (2.9)$$

which ensures the infrared stability of the associated statistical system. The second term in (2.8) introduces a subsidiary condition on the auxiliary field $\mathcal{F}$, such that the correct supersymmetry properties are implemented in the resulting quantum theory.

Let us consider the Euclidean vacuum functional written in terms of the action (2.7):

$$\mathcal{Z} = \mathcal{Z}_0^{-1}\int[d\Phi]\exp\left\{-\int d^2z\,d^2\theta\left(\frac{1}{2}\Phi(z,\theta)\mathcal{D}_+\mathcal{D}_-\,\Phi(z,\theta) + \alpha_0\,e^{\beta[\,\Phi(z,\theta)\,-\,\Phi_v\,(\theta)\,]}\right)\right\} \ , \qquad (2.10)$$

where $\mathcal{Z}_0$ is the free ($\alpha_0 = 0$) vacuum functional. The statistical mechanical description is obtained by making the gas expansion [3]

$$\exp\left\{-\alpha_0\int d^2z\,d^2\theta\,e^{\beta\,[\Phi(z,\theta)-\Phi_v\,(\theta)\,]}\right\} = \sum_{n=0}^{\infty}\frac{(-\alpha_0)^n}{n!}\int\prod_{i=1}^{n} d^2z_i\,d^2\theta_i\exp\left(\beta\sum_{j=1}^{n}[\,\Phi(z_j,\theta_j) - \Phi_v\,(\theta_j)]\right) . \qquad (2.11)$$

Inserting the expansion above into (2.10), we obtain

$$\mathcal{Z} = \sum_{n=0}^{\infty}\frac{(-\alpha_0)^n}{n!}\int_{|z_i|<\mathcal{R}}\prod_{i=1}^{n} d^2z_i\,d^2\theta_i\,\Gamma_0^{SL}(z_1,\theta_1;\ldots;z_n,\theta_n) \ , \qquad (2.12)$$

where the free correlation function Γ_0^{SL} is given by

$$\Gamma_0^{SL}(z_1,\theta_1;\ldots;z_n,\theta_n) = \left\langle 0 \left| \prod_{i=1}^{n} e^{\beta\,[\,\Phi(z_i,\theta_i)\,-\,\Phi_v(\theta_i)\,]} \right| 0 \right\rangle_0 \quad , \tag{2.13}$$

and $\langle\ldots\rangle_0$ denotes the average with respect to the free theory. The free n-point function (2.13) can be written as a quadratic functional integral over the superfield. Using (2.8), we can write

$$\begin{aligned}\Gamma_0^{SL}(z_1,\theta_1;\ldots;z_n,\theta_n) &= \left\langle 0 \left| \exp\left\{ \int d^2z\, d^2\theta\, \mathcal{J}(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n)\,\Phi(z,\theta) \right\} \right| 0 \right\rangle_0 \\ &= \mathcal{Z}_0^{-1}\int [d\Phi] \exp\left\{ -\int d^2z\, d^2\theta \left(\frac{1}{2}\Phi(z,\theta)\mathcal{D}_+\mathcal{D}_-\Phi(z,\theta) - \mathcal{J}(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n)\,\Phi(z,\theta) \right) \right\},\end{aligned} \tag{2.14}$$

with

$$\mathcal{J}_{(SL)}(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n) = \rho(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n) + f(\theta_1,\ldots,\theta_n) \tag{2.15}$$

and

$$\rho(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n) = \beta \sum_{i=1}^{n} \left[\delta^{(2)}(z-z_i)\,\delta^{(2)}(\theta-\theta_i) - \frac{1}{\mathcal{V}}\,\delta^{(2)}(\theta) \right] \; , \tag{2.16}$$

$$f(\theta_1,\ldots,\theta_n) = \frac{\beta}{\mathcal{V}} \sum_{i=1}^{n} \delta^{(2)}(\theta_i) \; . \tag{2.17}$$

The distribution ρ given by expression (2.16) corresponds to a neutral "charge" density in superspace:

$$\int d^2z\, d^2\theta\, \rho(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n) = 0 \; . \tag{2.18}$$

The neutral charge density ρ_L associated with the LT is obtained from (2.15) via

$$\mathcal{J}_{(L)}(z,z_1,\ldots,z_n) = \int d^2\theta\; \mathcal{J}_{(SL)}(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n) \; . \tag{2.19}$$

Performing the quadratic functional integration in (2.14), we obtain

$$\begin{aligned}\Gamma_0^{SL}(z_1,\theta_1;\ldots;z_n,\theta_n) = \exp\Big\{ \frac{1}{2} \int & d^2z\, d^2\theta\, d^2z'\, d^2\theta' \\ &\times \mathcal{J}_{(SL)}(z,\theta;z_1,\theta_1;\ldots;z_n,\theta_n)\,\Delta(z,\theta;z',\theta')\,\mathcal{J}_{(SL)}(z',\theta';z_1,\theta_1;\ldots;z_n,\theta_n) \Big\}.\end{aligned} \tag{2.20}$$

The infrared and ultraviolet regularized Euclidian massless superpropagator is given by [5,7]

$$\Delta(z_i,\theta_i;z_j,\theta_j) = -\frac{1}{4\pi}\ln \mu_0^2\left(R_{ij}^2 + \epsilon^2\right) \;, \tag{2.21}$$

where the distance between two points (z_i,θ_i) and (z_j,θ_j) in superspace is given by [6,7]

$$R_\mu^{ij} = z_\mu^i - z_\mu^j + \bar{\Theta}^i \gamma_\mu \Theta^j \; . \tag{2.22}$$

Expanding the superpropagator in terms of the Grassmann variables, we can write

$$\Delta(z,\theta;z',\theta') = -\frac{1}{4\pi}\ln\mu_0^2 + D_\epsilon(z) + \mathcal{G}_\epsilon(z,\theta;z',\theta') \;, \tag{2.23}$$

where

$$D_\epsilon(z) = -\frac{1}{4\pi}\ln(\,|\,z\,|^2 + \epsilon^2\,) \;, \tag{2.23a}$$

$$\mathcal{G}_\epsilon(z,\theta;z',\theta') = -\frac{1}{4\pi}\left[\theta^*\theta'^* d_{z,z'^*} - \theta\,\theta' d_{z,z'}\right] - \delta^{(2)}(\theta)\,\delta^{(2)}(\theta')\,\delta^{(2)}_\epsilon(z-z') \ , \tag{2.23b}$$

$$d_{z,z'} = \frac{(z_0 - z'_0) + i(z_1 - z'_1)}{|\,z - z'\,|^2 + \epsilon^2} \ , \tag{2.23c}$$

$$\delta^{(2)}_\epsilon(z-z') = \frac{1}{\pi}\frac{\epsilon^2}{[\,|\,z-z'|^2 + \epsilon^2\,]^2} \ . \tag{2.23d}$$

Inserting (2.23) into (2.20), the free correlation function is then given by

$$\Gamma_0^{SL}(z_1,\theta_1;\ldots;z_n,\theta_n) = \lim_{\mu_0,\epsilon\to 0}\left[\mu_0^2\right]^{-\frac{n^2\beta^2}{8\pi}}\left[\mu_0^2\right]^{\frac{n^2\beta^2}{8\pi}}\left[e^{\frac{3}{2}}\,\epsilon^2\right]^{-\frac{n\beta^2}{8\pi}}\left[\mathcal{R}^2\right]^{\frac{n\beta^2}{8\pi}}$$
$$\times\Gamma_0^L(z_1,\ldots,z_n)\,\mathcal{W}(z_1,\theta_1;\ldots;z_n,\theta_n)\,, \tag{2.24}$$

where

$$\Gamma_0^L(z_1,\ldots,z_n) = \exp\left\{-\frac{\beta^2}{4\pi}\sum_{i<j}^{n}\ln\left[e^{\frac{3}{2}}\frac{(\,|\,z_i - z_j\,|^2 + \epsilon^2)}{\mathcal{R}^2}\right] - n\frac{\beta^2}{4\pi}\sum_{i=1}^{n}\frac{|\,z_i\,|^2}{\mathcal{R}^2}\right\} \tag{2.25}$$

and

$$\mathcal{W}(z_1,\theta_1;\ldots;z_n,\theta_n) = \exp\left\{\frac{\beta^2}{2\pi}\sum_{i<j}^{n}\theta_i\theta_j d_{i,j}\right\}\exp\left\{-\frac{\beta^2}{2\pi}\sum_{i<j}^{n}\theta_i^*\theta_j^*\,d^*_{i,j}\right\}$$
$$\times\exp\left\{-\beta^2\sum_{i<j}^{n}\delta^{(2)}(\theta_i)\delta^{(2)}(\theta_j)\left[\delta^{(2)}_\epsilon(z_i - z_j) - \frac{2}{\mathcal{V}}\int d^2z\,\delta^{(2)}_\epsilon(z - z_j)\right.\right.$$
$$\left.\left.+\frac{1}{\mathcal{V}^2}\int d^2z\,d^2z'\delta^{(2)}_\epsilon(z-z')\right]\right\}. \tag{2.26}$$

The free correlation function Γ_0^L given by Eq. (2.25) corresponds to the Boltzmann factor of the screened two-dimensional one-component plasma [10] associated with the LT [8,9]. The neutrality condition (2.18) ensures that the μ_0-dependent terms cancel and the n-point function (2.24) is free of infrared instabilities.

The first term in the exponent of (2.25) represents the interaction energy of the equally charged gas particles and the second one the interaction energy between gas particles and uniform background. The divergent ϵ and $\mathcal{R}$-dependent terms appearing in (2.24) represent, respectively, the particle-particle and background-background self-energies, and must be eliminated by a renormalization prescription. In order to eliminate these unphysical contributions, we introduce the renormalized coupling constant [8,11]

$$\alpha = \alpha_0\left(\frac{e^{\frac{3}{2}}\epsilon^2}{\mathcal{R}^2}\right)^{-\frac{\beta^2}{8\pi}} . \tag{2.27}$$

Using the scaling $z_i = \mathcal{R}\hat{z}_i$, the Euclidean vacuum functional (2.12) can be written as

$$\mathcal{Z} = \lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}\left(\frac{\mathcal{V}}{\pi}\right)^n\int_{|\hat{z}_i|<1}\prod_{i=1}^{n}d^2\,\hat{z}_i\,\Gamma_0^L(\hat{z}_i,\ldots,\hat{z}_n)\prod_{j=1}^{n}d^2\,\theta_j\,\mathcal{W}_\mathcal{V}(\hat{z}_1,\theta_1;\ldots;\hat{z}_n,\theta_n) \ . \tag{2.28}$$

In terms of the new variables, the contribution of Grassmann variables is given by

$$\mathcal{W}_\mathcal{V}(\hat{z}_1,\theta_1;\ldots;\hat{z}_n,\theta_n) = \exp\left\{\frac{\beta^2}{2\pi}\left(\frac{\pi}{\mathcal{V}}\right)^{\frac{1}{2}}\sum_{i<j}^{n}\theta_i\theta_j\hat{d}_{i,j}\right\}\exp\left\{-\frac{\beta^2}{2\pi}\left(\frac{\pi}{\mathcal{V}}\right)^{\frac{1}{2}}\sum_{i<j}^{n}\theta_i^*\theta_j^*\hat{d}^*_{i,j}\right\}$$
$$\times\exp\left\{-\beta^2\frac{\pi}{\mathcal{V}}\sum_{i<j}^{n}\delta^{(2)}(\theta_i)\delta^{(2)}(\theta_j)\left[\delta^{(2)}_{\hat\epsilon}(\hat{z}_i - \hat{z}_j) - 2\int d^2\hat{z}\,\delta^{(2)}_{\hat\epsilon}(\hat{z} - \hat{z}_i)\right.\right.$$
$$\left.\left.+\int d^2\hat{z}\,d^2\hat{z}'\delta^{(2)}_{\hat\epsilon}(\hat{z}-\hat{z}')\right]\right\}. \tag{2.29}$$

The $(\frac{\mathcal{V}}{\pi})^n$ factor in Eq. (2.28) is due to the scaling in the integration elements in Eq. (2.12).

Identifying the renormalized coupling constant α with the fugacity and β^2 with the inverse temperature, the vacuum functional (2.28) can be regarded as the grand partition function of the "screened" one-component gas [8,9] in which the Boltzmann factor is weighted by a function

$$\mathcal{W}_{\mathcal{V}}(\hat{z}_1,\ldots,\hat{z}_n)=\int\prod_{i=1}^{n}d^2\theta_i\,\mathcal{W}_{\mathcal{V}}(\hat{z}_1,\theta_1;\ldots;\hat{z}_n,\theta_n)\ . \tag{2.30}$$

In order to display the effective volume dependence of the grand-canonical partition function (2.28), we must expand the weight function (2.29) in terms of the Grassmann variables. The only terms which give a nonzero contribution to the integrals over the Grassmann variables in Eq. (2.30) are those with n even and are proportional to $\mathcal{V}^{-\frac{n}{2}}$. Taking this into account, the volume dependence can be factorized in the weight function (2.30), and we can write

$$\mathcal{W}_{\mathcal{V}}(\hat{z}_1,\ldots,\hat{z}_{2n})=\left(\frac{\mathcal{V}}{\pi}\right)^{-\frac{(2n)}{2}}\mathcal{W}(\hat{z}_1,\ldots,\hat{z}_{2n})\ , \tag{2.30a}$$

where $\mathcal{W}$ is independent of volume. The grand-partition function (2.28) can be rewritten as

$$\mathcal{Z}=\sum_{n=0}^{\infty}\frac{(\alpha)^{2n}}{(2n)!}\left(\frac{\mathcal{V}}{\pi}\right)^{\frac{(2n)}{2}}\mathcal{S}^{(n|n)}\ , \tag{2.31}$$

where $\mathcal{S}^{(n|n)}$ is independent of volume and is given by

$$\mathcal{S}^{(n|n)}=\int_{|\hat{z}_i|<1}\prod_{i=1}^{2n}d^2\hat{z}_i\,\Gamma_0^L(\hat{z}_1,\ldots,\hat{z}_{2n})\times\mathcal{W}(\hat{z}_1,\ldots,\hat{z}_{2n})\ . \tag{2.32}$$

Thus, as a consequence of the Grassmann algebra introduced by the weight function (2.30), only configurations with n even contribute to the partition function. As we shall see, this restriction imposed by the weight function implies the dimensional reduction phenomenon.

Next, we shall obtain the equation of state governing the statistical mechanical system associated with the SUSY LT. To this end we introduce the grand-canonical potential

$$\Omega=-\mathcal{K}\mathcal{T}\ln\mathcal{Z}\ , \tag{2.33}$$

such that the pressure is given by

$$\mathcal{P}=-\left(\frac{\partial\Omega}{\partial\mathcal{V}}\right)_{\alpha,\beta}=\mathcal{K}\mathcal{T}\mathcal{Z}^{-1}\left(\frac{\partial\mathcal{Z}}{\partial\mathcal{V}}\right)_{\alpha,\beta}\ . \tag{2.34}$$

The partition function $\tilde{\mathcal{Z}}$ describing the statistical mechanical system associated with the LT [8] can be obtained directly from Eq. (2.28). Getting out the weight function (2.30) that carries the Grassmann variable dependence, we obtain, from (2.28),

$$\tilde{\mathcal{Z}}=\lim_{\epsilon\to0}\sum_{n=0}^{\infty}\frac{(-\alpha)^{2n}}{(2n)!}\left(\frac{\mathcal{V}}{\pi}\right)^{2n}\int_{|\hat{z}_i|<1}\prod_{i=1}^{2n}d^2\hat{z}_i\times\Gamma_0^L(\hat{z}_i,\ldots,\hat{z}_n)\ , \tag{2.35}$$

where the Boltzmann factor Γ_0^L is given by Eq. (2.25).

In the case of LT, inserting (2.35) into (2.34) we obtain the equation of state [8]

$$\mathcal{P}\mathcal{V}=\langle\mathcal{N}\rangle\mathcal{K}\mathcal{T}\ , \tag{2.36}$$

where $\mathcal{N}=n$ is the total number of particles and the average number of particles is given by

$$\langle\mathcal{N}\rangle=\tilde{\mathcal{Z}}^{-1}\sum_{n=0}^{\infty}\frac{(\alpha)^n}{n!}\mathcal{N}\left(\frac{\mathcal{V}}{\pi}\right)^n\tilde{\mathcal{S}}^{(n|n)}\ , \tag{2.37}$$

with

$$\tilde{\mathcal{S}}^{(n|n)}=\int_{|\hat{z}_i|<1}\prod_{i=1}^{n}d^2\hat{z}_i\,\Gamma_0^L(\hat{z}_1,\ldots,\hat{z}_n)\ . \tag{2.38}$$

Contrary to what occurs in sine-Gordon theory [3], the equation of state (2.36) indicates the absence of a critical temperature predicting that the system does not undergo a Kosterlitz-Thouless phase transition.

For SUSY LT, inserting (2.31) into (2.34), we obtain the equation of state

$$\mathcal{P}\mathcal{V}=\frac{1}{2}\langle\mathcal{N}\rangle\mathcal{K}\mathcal{T}\ , \tag{2.39}$$

with $\mathcal{N}=2n$ and the average number of particles is given by

$$\langle\mathcal{N}\rangle=\mathcal{Z}^{-1}\sum_{n=0}^{\infty}\frac{(\alpha)^{2n}}{(2n)!}\mathcal{N}\left(\frac{\mathcal{V}}{\pi}\right)^{2n}\mathcal{S}^{(n|n)}\ . \tag{2.40}$$

Comparing Eq. (2.39) with Eq. (2.36) we see that SUSY LT exhibits the dimensional reduction phenomenon without a Kosterlitz-Thouless phase transition.

III. TRANSLATIONAL INVARIANCE

The equation of motion for the operator-valued superfield is given by

$$\mathcal{D}_+\mathcal{D}_-\,\Phi(x,\theta)+\alpha_0\beta\left\{:e^{\beta[\Phi(x,\theta)-\Phi_v(\theta)]}:-\frac{\delta^{(2)}(\theta)}{\mathcal{V}}\int d^2z\,d^2\omega\ :e^{\beta[\Phi(z,\omega)-\Phi_v(\theta)]}:\right\}-\frac{\alpha_0\beta}{\mathcal{V}}\int d^2z\,d^2\omega\,\delta^{(2)}(\omega)\ :e^{\beta[\Phi(z,\omega)-\Phi_v(\omega)]}:=0\ , \tag{3.1}$$

which implies the subsidiary condition

$$\mathcal{F}'(x) = \alpha_0 \beta \left\{ : e^{\beta[\varphi(x) - \varphi_v]} : - \frac{1}{\mathcal{V}} \int d^2 z \, : e^{\beta[\varphi(z) - \varphi_v]} : \right\} . \tag{3.2}$$

In order to show that in the thermodynamic limit the resulting quantum theory is invariant under translation, let us consider the vacuum expectation value

$$\langle 0 | : e^{\beta[\Phi(x,\theta) - \Phi_v(\theta)]} : | 0 \rangle = \mathcal{Z}^{-1} \sum_{n=0}^{\infty} \frac{(-\alpha_0)^n}{(n!)} \int \prod_{i=1}^{n} d^2 z_i \, d^2 \omega_i \times \left(e^{\frac{3}{2}} \frac{\epsilon^2}{\mathcal{R}^2} \right)^{\frac{\beta^2}{8\pi}} \left\langle 0 \left| \prod_{j=1}^{n} e^{\beta[\Phi(z_j,\omega_j) - \Phi_v(\omega_j)]} e^{\beta[\Phi(x,\theta) - \Phi_v(\theta)]} \right| 0 \right\rangle_0 , \tag{3.3}$$

in which the expansion (2.11) was introduced. In the above expression the normal-ordered exponential operators are defined by [8,11]

$$: e^{\beta[\Phi(x,\theta) - \Phi_v(\theta)]} : = \left(\frac{e^{\frac{3}{2}} \epsilon^2}{\mathcal{R}^2} \right)^{\frac{\beta^2}{8\pi}} e^{\beta[\Phi(x,\theta) - \Phi_v(\theta)]} . \tag{3.4}$$

Performing the quadratic functional integration on the free correlation function appearing in expression (3.3), we obtain

$$\langle 0 | e^{\beta[\Phi(x,\theta) - \Phi_v(\theta)]} | 0 \rangle = \mathcal{Z}^{-1} \mathcal{Z}_0 \sum_{n=0}^{\infty} \frac{(-\alpha_0)^n}{n!} \int \prod_{i=1}^{n} d^2 z_i \, d^2 \omega_i \exp\left\{ \frac{1}{2} \int d^2 z \, d^2 \omega \, d^2 z' \, d^2 \omega' \times \tilde{\mathcal{J}}_{(SL)}(z,\omega;x,\theta;z_1,\omega_1;\ldots;z_n,\omega_n) \, \Delta(z,\omega;z',\omega') \times \tilde{\mathcal{J}}_{(SL)}(z',\omega';x,\theta;z_1,\omega_1;\ldots;z_n,\omega_n) \right\} , \tag{3.5}$$

where the new distribution is given by

$$\tilde{\mathcal{J}}_{(SL)}(z,\omega;x,\theta;z_1,\omega_1;\ldots;z_n,\omega_n) = \tilde{\rho}(z,\omega;x,\theta;z_1,\omega_1;\ldots;z_n,\omega_n) + \tilde{f}(\theta;\omega_1,\ldots,\omega_n) , \tag{3.6}$$

with

$$\tilde{\rho}(z,\omega;x,\theta;z_1,\omega_1;\ldots;z_n,\omega_n) = \beta \sum_{i=1}^{n} \left\{ \delta^{(2)}(z - z_i) \delta^{(2)}(\omega - \omega_i) - \frac{\delta^{(2)}(\omega)}{\mathcal{V}} \right\} + \beta \left\{ \delta^{(2)}(z - x) \delta^{(2)}(\omega - \theta) - \frac{\delta^{(2)}(\omega)}{\mathcal{V}} \right\} , \tag{3.7}$$

$$\tilde{f}(\theta;\omega_1,\ldots,\omega_n) = \frac{\beta}{\mathcal{V}} \left\{ \sum_{i=1}^{n} \delta^{(2)}(\omega_i) - \delta^{(2)}(\theta) \right\} . \tag{3.8}$$

Computing the integrals in (3.5) and using the renormalized coupling constant (2.27), we get

$$\langle 0 | : e^{\beta[\Phi(x,\theta) - \Phi_v(\theta)]} : | 0 \rangle = \mathcal{Z}^{-1} \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} \int \prod_{i=1}^{n} d^2 z_i \, d^2 \omega_i \times \exp\left\{ -\frac{\beta^2}{4\pi} \sum_{i<j}^{n} \ln \left(e^{\frac{3}{2}} \frac{[|z_i - z_j|^2 + \epsilon^2]}{\mathcal{R}^2} \right) - \frac{\beta^2}{4\pi} \sum_{i=1}^{n} \ln \left(e^{\frac{3}{2}} \frac{[|z_i - x|^2 + \epsilon^2]}{\mathcal{R}^2} \right) \right\} \times \exp\left\{ -(n+1) \frac{\beta^2}{4\pi} \left(\sum_{i=1}^{n} \frac{|z_i|^2}{\mathcal{R}^2} + \frac{|x|^2}{\mathcal{R}^2} \right) \right\} \exp\left\{ \frac{\beta^2}{2\pi} \sum_{i<j}^{n} \omega_i \omega_j d_{z_i,z_j} \right\} \times \exp\left\{ -\frac{\beta^2}{2\pi} \sum_{i<j}^{n} \omega_i{}^* \omega_j{}^* d^*_{z_i,z_j} \right\}$$

$$\times \exp\Bigg\{-\beta^2 \sum_{i<j}^{n} \delta^{(2)}(\omega_i)\delta^{(2)}(\omega_j)\Bigg[\delta_\epsilon^{(2)}(z_i - z_j) - \frac{2}{\mathcal{V}}\int d^2z\delta_\epsilon^{(2)}(z - z_i)$$
$$+\frac{1}{\mathcal{V}^2}\int d^2zd^2z'\delta_\epsilon^{(2)}(z - z')\Bigg]\Bigg\} \exp\left\{\frac{\beta^2}{4\pi}\sum_{i=1}^{n}\omega_i\theta d_{z_i,x}\right\}\exp\left\{-\frac{\beta^2}{4\pi}\sum_{i=1}^{n}\omega_i^*\theta^* d_{z_i,x}^*\right\}$$
$$\times \exp\Bigg\{-\beta^2 \sum_{i=1}^{n} \delta^{(2)}(\omega_i)\delta^{(2)}(\theta)\Bigg[\delta_\epsilon^{(2)}(z_i - x) - \frac{2}{\mathcal{V}}\int d^2z\,\delta_\epsilon^{(2)}(z - z_i)$$
$$+\frac{1}{\mathcal{V}^2}\int d^2z\,d^2z'\,\delta_\epsilon^{(2)}(z - z')\Bigg]\Bigg\} \ . \tag{3.9}$$

Making the change of variables $z_i \to z_i - x$, we obtain that the vacuum expectation value given by (3.9) is independent of x in the thermodynamic limit. Expanding the exponentials in (3.9) which carry the dependence on the Grassmann variables, we find

$$\lim_{\mathcal{V}\to\infty}\langle 0| : e^{\beta[\Phi(x,\theta)-\Phi_v(\theta)]} : |0\rangle = A\,\delta^{(2)}(\theta) + B \ , \tag{3.10}$$

where A and B are constants corresponding, respectively, to the terms of the series (3.9) with odd and even powers of the coupling constant.

Taking the vacuum expectation value of the superfield equation of motion (3.1) and using (3.9), in the thermodynamic limit we obtain the "free field weak condition"

$$\langle 0|\,\mathcal{D}_+\mathcal{D}_-\,\Phi(x,\theta)\,|0\rangle = 0 \ . \tag{3.11}$$

In terms of primary fields, we obtain, from (3.11),

$$\langle 0|\,\gamma^\mu\partial_\mu\psi(x)\,|0\rangle = \alpha\beta^2\langle 0| : e^{\beta[\varphi(x)-\varphi_v]} : \psi(x)\,|0\rangle = 0 \ , \tag{3.12}$$

$$\langle 0|\,\Box\,\varphi(x)\,|0\rangle = \alpha\beta^2\left(\langle 0| : e^{\beta[\varphi(x)-\varphi_v]} : \mathcal{F}'(x)\,|0\rangle - \frac{1}{\mathcal{V}}\int d^2z\langle 0 : e^{\beta[\varphi(z)-\varphi_v]} : \mathcal{F}'(z)\,|0\rangle\right)$$
$$+\alpha\beta^2\left(\langle 0|\,\bar\psi(x)\psi(x) : e^{\beta[\varphi(x)-\varphi_v]}\,|0\rangle - \frac{1}{\mathcal{V}}\int d^2z\langle 0|\,\bar\psi(z)\psi(z) : e^{\beta[\varphi(z)-\varphi_v]} : |0\rangle\right) = 0 \ . \tag{3.13}$$

According to Eq. (3.13), in the thermodynamic limit we obtain $\langle 0|\varphi|0\rangle = \text{const}$, and the resulting quantum theory is translationally invariant.

As a consequence of Eq. (3.11) and the subsidiary condition (3.2), in the thermodynamic limit the subsidiary field acquires a zero vacuum expectation value

$$\lim_{\mathcal{V}\to\infty}\langle 0|\,\mathcal{F}'(x)\,|0\rangle = \alpha\,\beta^2\left\{\langle 0| : e^{\beta[\varphi(x)-\varphi_v]} : |0\rangle - \frac{1}{\mathcal{V}}\int d^2z\langle 0| : e^{\beta[\varphi(z)-\varphi_v]} : |0\rangle\right\} = 0 \ , \tag{3.14}$$

which ensures the correct supersymmetry properties. The infinitesimal supersymmetry transformations are

$$\delta\varphi = \bar\zeta\psi \ , \tag{3.15a}$$

$$\delta\psi = (\mathcal{F} - i\gamma^\mu\partial_\mu\varphi)\zeta \ , \tag{3.15b}$$

$$\delta\mathcal{F} = -i\bar\zeta\gamma^\mu\partial_\mu\psi \ , \tag{3.15c}$$

where ζ is an anticommuting parameter. Denoting the supercharge by $\mathcal{Q}$, we can write, from (3.15c) and (3.14)

$$\langle 0\,|\,[\bar\zeta\mathcal{Q},\psi]\,|\,0\rangle = \lim_{\mathcal{V}\to\infty}\zeta\langle 0\,|\,\mathcal{F}'\,|\,0\rangle = 0 \ , \tag{3.16}$$

since $\langle 0|\gamma_\mu\partial^\mu\varphi|0\rangle = 0$. Thus, the supercharge $\mathcal{Q}$ annihilates the vacuum and supersymmetry is not destroyed in the thermodynamic limit.

IV. CONCLUSIONS AND OUTLOOK

The standard sine-Gordon theory has been known for a long time to be equivalent to the two-dimensional Coulomb gas of point particles in the grand-canonical ensemble. Using the same methods applied in this paper, the equation of state of this statistical mechanical system is obtained as

$$\mathcal{P}\mathcal{V} = \left(1 - \frac{\beta^2}{8\pi}\right)\langle\mathcal{N}\rangle\mathcal{K}\mathcal{T} \ . \tag{4.1}$$

Although this equation of state exhibits a simple form, it is not a trivial equation since all interactions are contained in the infinite series (perturbative expansion) enclosed in the $\langle \mathcal{N} \rangle$ term. The negative β-dependent factor characterizes the existence of a Kosterlitz-Thouless (KT) phase transition at a critical temperature corresponding to $\beta_c^2 = 8\pi$.

The supersymmetric generalization of the two-dimensional sine-Gordon theory was considered in Refs. [5–7]. In the latter paper, one of the main results is that supersymmetry introduces a dimensional reduction phenomenon in the corresponding equation of state, which is given by

$$\mathcal{P}\mathcal{V} = \left(\frac{1}{2} - \frac{\beta^2}{8\pi}\right)\langle \mathcal{N} \rangle \mathcal{K}\mathcal{T} \ . \tag{4.2}$$

The existence of a KT phase transition is preserved in the supersymmetric extension.

Let us resume what happens in Liouville theory and SUSY LT.

The quantum statistical mechanical system associated with the two-dimensional Liouville theory was considered in Refs. [8,9]. The equation of state is obtained exactly and is given by

$$\mathcal{P}\mathcal{V} = \langle \mathcal{N} \rangle \mathcal{K}\mathcal{T}. \tag{4.3}$$

In this case there is no KT phase transition.

Considering the supersymmetric system in presence of a uniform neutralizing background and confined to a finite volume, we showed that the corresponding grand-partition function is infrared finite and describes a one-component gas, in which the Boltzmann factor is weighted by an integration over the superspace Grassmann coordinates. This weight function introduces the dimensional reduction phenomenon and the statistical system obeys the equation of state

$$\mathcal{P}\mathcal{V} = \frac{1}{2}\langle \mathcal{N} \rangle \mathcal{K}\mathcal{T} \ . \tag{4.3a}$$

After performing the thermodynamic limit, the resulting supersymmetric quantum theory is translationally invariant.

Nevertheless, the grand-partition function (2.31) contains ultraviolet singularities, since in the limit $\epsilon \to 0$, the weighted Boltzmann factor (2.32) becomes singular in regions where two or more charges become close to each other.

A detailed nonperturbative analysis of the ultraviolet singularities in the statistical mechanical system associated with the sine-Gordon and SUSY sine-Gordon theories was performed in Refs. [7,12]. In the range $2\pi \leq \beta^2 < 4\pi$ the ultraviolet divergences in the SUSY sine-Gordon theory factorize and may be eliminated by an infinite subtractive renormalization in the corresponding Lagrangian.

The partition function describing the one-component gas associated with LT exhibits a short distance behavior more divergent than that of the Coulomb gas associated with sine-Gordon theory. However in SUSY LT, due to the integration over the Grassmann coordinates, the ultraviolet divergence structure resembles the sine-Gordon ones. In this way, it seems to be interesting to apply the method developed in [7,12] to SUSY LT, in order to investigate whether the ultraviolet divergences factorize in this case and, eventually, can be eliminated by a renormalization prescription.

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[1] A. Polyakov, Phys. Lett. **103B**, 207 (1981); J. L. Gervais and A. Neveu, Nucl. Phys. **B209**, 125 (1982).
[2] A. Polyakov, Phys. Lett. **103B**, 211 (1981); A. Polyakov, *ibid.* **82B**, 247 (1979).
[3] J. Fröhlich, Commun. Math. Phys. **47**, 2331 (1976); S. Samuel, Phys. Rev. D **18**, 1916 (1978); S. T. Chui and P. A. Lee, Phys. Rev. Lett. **35**, 35 (1975); E. C. Marino, Nucl. Phys. **B251** [FS13], 227 (1985).
[4] J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).
[5] Y. Y. Goldschmidt, Nucl. Phys. **B270** [FS16], 29 (1986).
[6] M. M. Doria and Y. Lifa, Nucl. Phys. **B270** [FS16], 519 (1986).
[7] A. Lima-Santos and E. C. Marino, Nucl. Phys. **B336**, 547 (1990).
[8] E. C. Marino and G. C. Marques, Phys. Lett. **161B**, 146 (1985).
[9] L. V. Belvedere, J. Phys. A **24**, 4549 (1991).
[10] C. Deutsch and M. Lavaud, Phys. Rev. A **9**, 2598 (1974); C. Deutsch, H. E. Dewitt, and Y. Furutani, *ibid.* **20**, 2631 (1979); C. Deutsch, J. Math. Phys. **17**, 1404 (1976); B. Jancovici, Phys. Rev. Lett. **46**, 386 (1981); D. Nicolaides, Phys. Lett. **103A**, 277 (1984).
[11] S. Coleman, Phys. Rev. D **11**, 2088 (1974).
[12] A. Lima-Santos and E. C. Marino, J. Stat. Phys. **55**, 157 (1989).