

# Connection between momentum cutoff and operator cutoff regularizations

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Operator cutoff regularization based on the original Schwinger proper-time formalism is examined. By constructing a regulating smearing function for the proper-time integration, we show how this regularization scheme simulates the usual momentum cutoff prescription yet preserves gauge symmetry even in the presence of the cutoff scales. The similarity between the operator cutoff regularization and the method of higher (covariant) derivatives is also observed. The invariant nature of the operator cutoff regularization makes it a promising tool for exploring the renormalization group flow of gauge theories in the spirit of the Wilson-Kadanoff blocking transformation.

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## I. INTRODUCTION

An essential step for identifying the physical contents of quantum field theory is the removal of ultraviolet (UV) divergences which are due to the presence of interactions. The procedure, known as renormalization, operates on the use of some regularization schemes to control the infinities followed by a redefinition of the parameters contained in the original Lagrangian in such a way that physical quantities are independent of the regularization choice. Although various methods are available, it is often desirable to choose one which respects all the symmetry properties present in the original theory. For example, when studying gauge theories such as QCD or QED, a momentum cutoff regulator would not be appropriate since it explicitly violates gauge invariance. One must therefore turn to gauge-invariant prescriptions such as dimensional regularization [1],  $\zeta$ -function regularization [2], the proper-time method [3], or the Pauli-Villars procedure [4]. On the other hand, because of the dimensionality dependence on the definition of  $\gamma_5$ , difficulties are encountered when applying dimensional regularization to chiral theories.

Despite the shortcoming of its gauge-noninvariant nature, the momentum cutoff has proven to be a useful regulator. In addition to being simple and more physical, one not only can immediately identify the divergent structures of the theory, but also derive readily the renormalization-group (RG) flow equations which in turn give predictions of how the theory behaves in different momentum regimes. The renowned RG formalism pioneered by Wilson and Kadanoff [5] is based on the use of this regulator. In addition, when probing physics in the infrared (IR), it is often advantageous to derive a low-energy effective theory by integrating out the irrelevant short-distance modes. The scale that separates the fast-fluctuating short-distance modes from the slowly varying

components appears naturally in the momentum cutoff regularization.

Momentum cutoff regularization can be formulated systematically by means of blocking transformation [6,7]. To illustrate the idea, consider the scalar field theory as an example. From the original field  $\phi(x)$  we first define a coarse-grained averaged blocked field  $\phi_k(x)$  for each given block of size  $k^{-d}$  in  $d$ -dimensional Euclidean space with a smearing function  $\rho_k^{(d)}(x)$  as

$$\begin{aligned}\Phi(x) &= \phi_k(x) = \int_y \rho_k^{(d)}(x-y)\phi(y), \\ \int_x &= \int d^d x.\end{aligned}\quad (1.1)$$

The role of  $\rho_k^{(d)}(x)$  is to provide an averaging of the fields within the block and retain the degrees of freedom that are relevant for studying the physics near the energy scale  $\sim k$ . Having defined  $\phi_k(x)$ , the corresponding blocked action  $\tilde{S}_k[\Phi]$  can be written as [8]

$$e^{-\tilde{S}_k[\Phi]} = \int D[\phi] \prod_x \delta(\phi_k(x) - \Phi(x)) e^{-S[\phi]}.\quad (1.2)$$

In the above, the infinite product of  $\delta$  functions strictly speaking only makes sense on the lattice where there are fewer blocked fields compared to the original fields. To render the procedure well defined, one may first replace this product by a Gaussian function [9]

$$\prod_x \delta(\phi_k(x) - \Phi(x)) \rightarrow \exp \left\{ -\omega \int d^d x [\phi_k(x) - \Phi(x)]^2 \right\},\quad (1.3)$$

where  $\omega$  is a large constant of dimension (mass)<sup>2</sup>. The physical limit corresponds to taking  $\omega \rightarrow \infty$ . It is also known that, if  $\rho_k^{(d)}(x)$  is a smooth Gaussian function, there will be no sharp boundary between the integrated and unintegrated modes [5]. On the other hand, the desired scale of separation is naturally set at  $p = k$  if one chooses

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$$\rho_k^{(d)}(x) = \int_{|p|<k} \frac{d^d p}{(2\pi)^d} e^{-ipx}, \quad (1.4)$$

or  $\rho_k^{(d)}(p) = \Theta(k - p)$ , i.e., a sharp step function in momentum space. Although such a sharp cutoff will in general produce nonlocal interactions in  $\tilde{S}_k[\Phi]$ , the RG flow equation based on the infinitesimal variation of  $k$  can still be formulated, and has been successfully carried out by Wegner and Houghton in [10]. When no confusion arises, we adopt the same general notation  $\rho_k^{(d)}$  for both coordinate and momentum space representations, and distinguish between them by the arguments they carry.

Choosing (1.4) as the smearing function, the Fourier modes can be decomposed into

$$\phi(p) = \begin{cases} \phi_{<}(p), & 0 \leq p \leq k, \\ \phi_{>}(p), & k < p < \Lambda, \end{cases} \quad (1.5)$$

where  $\phi_{<}$  and  $\phi_{>}$  are, respectively, the slow and the fast modes. This in turn implies

$$\phi_k(p) = \rho_k(p)\phi(p) = \phi_{<}(p), \quad (1.6)$$

which shows clearly how the fast modes are completely integrated over through blocking transformation. Since it is generally a hopeless task to compute  $\tilde{S}_k[\Phi]$  exactly, the complicated expression in (1.2) is frequently approximated by loop expansion. At the one-loop level, we have

$$\begin{aligned} \tilde{S}_k[\Phi] &= -\ln \int D[\phi_{<}]D[\phi_{>}] \prod_x \delta(\phi_k(x) - \Phi(x)) \exp\{-S[\phi_{<} + \phi_{>}]\} \\ &= -\ln \int D[\phi_{<}] \prod_p \delta(\phi_{<}(p) - \Phi(p)) \int D[\phi_{>}] \\ &\quad \times \exp\left\{-S[\phi_{<}] - \frac{1}{2} \int_p' \phi_{>}(p) K(\phi_{<}) \phi_{>}(-p) + \dots\right\} \\ &= -\ln \int D[\phi_{<}] \prod_p \delta(\phi_{<}(p) - \Phi(p)) \exp\{-S[\phi_{<}] - \frac{1}{2} \text{Tr}' \ln K(\phi_{<}) + \dots\} \\ &= S[\Phi] + \frac{1}{2} \text{Tr}' \ln K(\Phi) + \dots, \end{aligned} \quad (1.7)$$

where

$$\begin{aligned} K(\Phi) &= \left. \frac{\partial^2 S}{\partial \phi(x) \partial \phi(y)} \right|_{\Phi} \\ &= [-\partial^2 + V''(\Phi)] \delta^d(x - y), \end{aligned} \quad (1.8)$$

$$\begin{aligned} \int_p' &= \int_k^\Lambda \frac{d^d p}{(2\pi)^d} = S_d \int_k^\Lambda dp p^{d-1}, \\ S_d &= \frac{2}{(4\pi)^{d/2} \Gamma(d/2)}, \end{aligned} \quad (1.9)$$

and  $\text{Tr}'$  implies taking the trace over a restricted momentum range  $k \leq p \leq \Lambda$  as well as all possible internal indices. Without the prime notation, a complete momentum integration from zero to infinity is implied. Physically  $\tilde{S}_k[\Phi]$  can be interpreted as an effective action parametrized by the averaged field  $\Phi$  at the scale  $k$ , and it provides a smooth interpolation between the bare action  $S[\Phi]$  defined at  $k = \Lambda$  and the renormalized effective action  $\tilde{S}_{k=0}[\Phi]$  which generates one-particle-irreducible Feynman graphs. Thus, the RG flow pattern of the theory is readily obtained by studying the change of  $\tilde{S}_k[\Phi]$  in response to an infinitesimal change of the IR scale  $k$ .

With the advantages of choosing the sharp cutoff regulator (1.4), one then inquires how it can be possible to extend this scheme to other theories possessing addi-

tional symmetries. Such a formulation will have profound implications on gauge theories such as QCD, QED, supersymmetry, or quantum gravity. It may even offer new insights to the long-standing issue of quark confinement in the IR limit of strong interaction since the approach naturally yields an effective low-energy QCD Lagrangian upon integrating out systematically the short-distance modes [11]. Nonperturbative effects can be explored, too. There will be new higher-order interactions which are initially absent from the original Lagrangian, and they may be of great import or even dominate in the IR regime despite the suppression at high energy.

Unfortunately, deriving the RG equation based on the use of the momentum cutoff regulator is known to conflict with gauge symmetry. Yet the widely used dimensional regularization obscures the characteristics of the Wilson-Kadanoff RG albeit it is an invariant prescription. The first step toward applying the Wilson-Kadanoff RG to gauge theories is the implementation of both the UV and the IR cutoff scales without destroying gauge invariance. In [12], Oleszczuk demonstrated how this can be achieved via blocking transformation in a completely symmetry-preserving manner. The methodology of the "operator cutoff regularization" elegantly presented there relies on the construction of a smooth smearing function  $\rho(\Lambda^2 s)$ , where  $\Lambda$  is to be identified with the usual UV regulator, and  $s$  the proper-time variable carrying dimension  $(\text{mass})^{-2}$ . Embedding the smearing function into the  $s$  integration, one is led to the regularized parametrization

$\text{Tr}(\ln \mathcal{H} - \ln \mathcal{H}_0)|_{\text{OC}}$

$$= - \int_0^\infty \frac{ds}{s} \rho(\Lambda^2 s) \text{Tr}(e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s}), \quad (1.10)$$

where  $\mathcal{H}$  is an arbitrary fluctuation operator governing the quadratic fluctuations of the fields and  $\mathcal{H}_0$  its corresponding limit of vanishing background field. The subscript ‘‘OC’’ stands for operator cutoff. For bosonic theories,  $\mathcal{H}$  is a positive definite elliptic operator and  $\mathcal{H}^{-1}$  defines the propagator. With a suitable choice of  $\rho(\Lambda^2 s)$ , the conventional cutoff results may be recovered.

In the present work we follow closely the techniques outlined in [12] and generalize the operator cutoff regularization to arbitrary dimension  $d$  using the smearing function  $\rho_k^{(d)}(s, \Lambda)$ , where  $k$  will be shown to play the role of an effective IR cutoff for the theory. In this manner, any possible divergence originating from momentum integration, whether of UV or IR nature, will be turned into a singularity in  $s$  and subsequently regulated by  $\rho_k^{(d)}(s, \Lambda)$ . However, unlike the momentum cutoff approach, operator cutoff regularization is an invariant regularization since the proper-time variable  $s$  is independent of gauge transformation. Nonetheless, we emphasize that it is an invariant prescription provided that no cutoff scales are imposed on the momentum integral and the  $s$  integration is left as the last step. If the  $s$  integration is carried out first, divergences generated from taking the spacetime trace will manifest in the  $p$  integration and one may be forced to use some noninvariant regularization prescriptions. Even though the smearing procedure is now acting on the proper-time variable  $s$ , we retain the same general notation  $\rho_k^{(d)}$  here since the role of  $\rho_k^{(d)}(s, \Lambda)$  in  $s$  is similar to what  $\rho_k^{(d)}(x)$  [cf. (1.4)] does in the coordinate space. However, because of the difference in their origin, the functional forms of  $\rho_k^{(d)}(s, \Lambda)$  and  $\rho_k^{(d)}(x)$  are expected to be rather different. It is important to keep in mind that the form of  $\rho_k^{(d)}(s, \Lambda)$  is not unique at all; prescriptions such as the Pauli-Villars regulator and dimensional regularization can all be shown to fall under the generalized class of proper time by a suitable definition of smearing function [13]. Lucid discussions on the applications of proper-time regularization can also be found in [14].

What we shall demonstrate in this paper is that with a particular choice of  $\rho_k^{(d)}(s, \Lambda)$  operator cutoff regularization reproduces the usual one-loop blocked potential  $U_k(\Phi)$  which contains the IR cutoff scale  $k$ . When considering  $U_k(\Phi)$  in terms of Feynman diagrams, both operator cutoff and momentum cutoff regularizations yield the same results order by order in terms of coupling constant. The spirit of our operator cutoff formalism presented here is parallel to the idea of ‘‘invariant momentum space regularization’’ treated by Ball in [13]. However, when the full blocked action is considered, deviation between momentum cutoff and operator cutoff prescriptions occurs in the higher-order (covariant) derivative terms. We find that the effective blocked action regularized with the former contains gauge-noninvariant terms which are completely absent in the latter.

The organization of the paper is as follows: In Sec. II using scalar theory as an example we derive the generalized smearing function  $\rho_k^{(d)}(s, \Lambda)$  which provides the bridge for establishing the functional equivalence between the momentum cutoff and the operator cutoff regularizations at the level of one-loop blocked potential. Similarity between the operator cutoff and the Pauli-Villars regularizations is discussed. An attempt to equate the two regularization schemes beyond the leading-order blocked potential is made in Sec. III. It is found that at each order in the derivative expansion a new smearing function must be introduced in the operator cutoff formalism in order to give the same differential flow equations as that provided by the sharp cutoff. In general, to ensure equality between the two schemes to arbitrary order  $(\partial^2)^n$ , we need a total of  $n+1$  smearing functions; i.e.,  $\rho_k^{(d,m)}(s, \Lambda)$  with  $m = 0, 1, \dots, n$ . In Sec. IV we first reexamine the gauge-noninvariant nature of the momentum cutoff regularization by identifying explicitly the symmetry violating components and their corresponding proper-time counterparts. An invariant operator cutoff scheme is then proposed to eliminate the gauge-noninvariant sector and restore the symmetry. Similarity between the operator cutoff regularization and the method of higher covariant derivatives can also be inferred. Section V is reserved for summary and discussions.

## II. OPERATOR CUTOFF REGULARIZATION

Consider for simplicity the one-component scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi). \quad (2.1)$$

The one-loop contribution to the blocked action can be written as

$$\begin{aligned} \tilde{S}_k^{(1)}[\Phi] &= \frac{1}{2} \text{Tr}' \ln K(\Phi) \\ &= \frac{1}{2} \int_x \int_p' \ln \left( \frac{p^2 + V''(\Phi)}{p^2 + V''(0)} \right), \end{aligned} \quad (2.2)$$

where  $\Phi$  is the blocked field. Consider for simplicity the low-energy limit where the blocked action can be approximated by derivative expansion:

$$\tilde{S}_k[\Phi] = \int_x \left\{ \frac{Z_k(\Phi)}{2} (\partial_\mu \Phi)^2 + U_k(\Phi) + O(\partial^4) \right\}, \quad (2.3)$$

with  $Z_k(\Phi)$  being the wave-function renormalization constant. The leading-order contribution is then the one-loop blocked potential

$$U_k^{(1)}(\Phi) = \frac{1}{2} \int_p' \ln \left( \frac{p^2 + V''(\Phi)}{p^2 + V''(0)} \right). \quad (2.4)$$

Differentiating the above with respect to the arbitrary IR scale  $k$  leads to the flow equation

$$k \frac{\partial U_k(\Phi)}{\partial k} = - \frac{S_d}{2} k^d \ln \left( \frac{k^2 + V''(\Phi)}{k^2 + V''(0)} \right). \quad (2.5)$$

This linear differential equation is obtained based on the so-called “independent-mode approximation” since it incorporates only the contribution from one-loop order and ignores the continuous feedbacks between different modes. A RG-improved equation which takes the interactions between fast and slow modes into consideration is given by the modified expression [15]

$$k \frac{\partial U_k(\Phi)}{\partial k} = -\frac{S_d}{2} k^2 \ln \left( \frac{k^2 + U_k''(\Phi)}{k^2 + U_k''(0)} \right). \quad (2.6)$$

In [16] where field theory at finite temperature was considered, it was found that the independent-mode approximation breaks down in the high-temperature limit and one must resort to the finite-temperature RG improved equation similar to (2.6) in order to account for the important daisy and superdaisy graphs.

However, when gauge theories are considered, the momentum cutoff regulator is not directly applicable for generating a flow equation such as (2.6) since it does not respect gauge symmetry. The key issues which we

wish to explore are the following: Are cutoff scales *truly* in conflict with gauge symmetry? Can we formulate a scheme which contains cutoffs yet allows for the investigation of RG flow for gauge theories in the spirit of Wilson-Kadanoff blocking transformation? We now turn to the operator cutoff regularization which offers the hope of introducing the cutoff scales in a symmetry-preserving manner.

The basis of the operator cutoff formalism is provided by Schwinger’s proper-time regularization [3] in which one employs the following identity for computing the one-loop contribution:

$$\text{Tr}(\ln \mathcal{H} - \ln \mathcal{H}_0) = - \int_0^\infty \frac{ds}{s} \text{Tr}(e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s}). \quad (2.7)$$

The idea of operator cutoff regularization is to modify the above expression by introducing into the proper-time integration a regulating smearing function  $\rho_k^{(d)}(s, \Lambda)$  such that

$$\text{Tr}'(\ln \mathcal{H} - \ln \mathcal{H}_0) \rightarrow \text{Tr}(\ln \mathcal{H} - \ln \mathcal{H}_0)|_{\text{OC}} = - \int_0^\infty \frac{ds}{s} \rho_k^{(d)}(s, \Lambda) \text{Tr}(e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s}), \quad (2.8)$$

i.e., a complete trace can now be taken after inserting  $\rho_k^{(d)}(s, \Lambda)$  into the  $s$  integration. The absence of any cutoff in the  $p$  integration is a *sine qua non* for preserving gauge symmetry. As an illustration, we consider the Lagrangian in (2.1). Following the procedures outlined in [12] and using  $\mathcal{H} = p^2 + V''(\Phi)$ , the one-loop blocked potential becomes

$$\begin{aligned} U_k^{(1)}(\Phi) &= \frac{1}{2} \int_p' \ln \left( \frac{p^2 + V''(\Phi)}{p^2 + V''(0)} \right) \\ &\rightarrow -\frac{1}{2} \int_0^\infty \frac{ds}{s} \rho_k^{(d)}(s, \Lambda) \int_p e^{-p^2 s} (e^{-V''(\Phi)s} - e^{-V''(0)s}) \\ &= -\frac{1}{2(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{1+d/2}} \rho_k^{(d)}(s, \Lambda) (e^{-V''(\Phi)s} - e^{-V''(0)s}), \end{aligned} \quad (2.9)$$

where we have used

$$\int_p (p^2)^m e^{-p^2 s} = \frac{\Gamma(m + d/2)}{(4\pi)^{d/2} \Gamma(d/2)} s^{-(m+d/2)}. \quad (2.10)$$

The cutoffs are now taken over by the smearing function. With the  $k$  dependence of  $\tilde{S}_k[\Phi]$  contained entirely in  $\rho_k^{(d)}(s, \Lambda)$ , probing the RG flow of the theory amounts to studying the change of  $\rho_k^{(d)}(s, \Lambda)$  with varying the IR cutoff  $k$ . Thus, the differential flow equation of the theory can be written as

$$k \frac{\partial U_k(\Phi)}{\partial k} = -\frac{1}{2(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{1+d/2}} \left( k \frac{\partial \rho_k^{(d)}(s, \Lambda)}{\partial k} \right) (e^{-V''(\Phi)s} - e^{-V''(0)s}). \quad (2.11)$$

To deduce the form of  $\rho_k^{(d)}(s, \Lambda)$ , we now equate (2.11) with (2.5) which is derived using the cutoff approach:

$$\begin{aligned} k \frac{\partial U_k(\Phi)}{\partial k} &= -\frac{S_d}{2} k^d \ln \left( \frac{k^2 + V''(\Phi)}{k^2 + V''(0)} \right) = \frac{S_d}{2} k^d \int_0^\infty \frac{ds}{s} e^{-k^2 s} (e^{-V''(\Phi)s} - e^{-V''(0)s}) \\ &= -\frac{1}{2(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{1+d/2}} \left( k \frac{\partial \rho_k^{(d)}(s, \Lambda)}{\partial k} \right) (e^{-V''(\Phi)s} - e^{-V''(0)s}), \end{aligned} \quad (2.12)$$

or, equivalently,

$$k \frac{\partial \rho_k^{(d)}(s, \Lambda)}{\partial k} = -\frac{2}{\Gamma(d/2)} (k^2 s)^{d/2} e^{-k^2 s}. \quad (2.13)$$

We shall choose a set of boundary conditions for  $\rho_k^{(d)}(s, \Lambda)$  which renders (2.9) finite through the calculation. Since the proper-time variable  $s$  has dimension (length)<sup>2</sup>, UV divergence corresponding to the short-distance singularity appears at  $s = 0$ . Thus, setting  $\rho_k^{(d)}(s = 0, \Lambda) = 0$  will eliminate the unwanted UV singularity as  $s \rightarrow 0$ . On the other hand, since we wish to modify only the UV behavior of the theory while leaving the IR physics intact, it is appropriate to have  $\rho_{k=0}^{(d)}(s \rightarrow \infty, \Lambda) = 1$ . Finally, we demand  $\rho_{k=\Lambda}^{(d)}(s, \Lambda) = 0$  since the one-loop contribution must vanish at the UV cutoff scale  $\Lambda$  to give back the original bare theory. Solving (2.13) subject to the conditions imposed above leads to

$$\begin{aligned} \rho_k^{(d)}(s, \Lambda) &= \rho^{(d)}(\Lambda^2 s) - \rho^{(d)}(k^2 s) \\ &= \frac{2s^{d/2}}{\Gamma(d/2)} \int_k^\Lambda dz z^{d-1} e^{-z^2 s} \\ &= \frac{2s^{d/2}}{S_d \Gamma(d/2)} \int_z' e^{-z^2 s} \\ &= \frac{1}{\Gamma(d/2)} \Gamma\left(\frac{d}{2}; k^2 s, \Lambda^2 s\right), \end{aligned} \quad (2.14)$$

where

$$U_k^{(1)}(\Phi) = -\frac{1}{S_d \Gamma(d/2)} \int_x \int_0^\infty \frac{ds}{s^{1-d/2}} (e^{-V''(\Phi)s} - e^{-V''(0)s}) \int_z' e^{-z^2 s} \int_p e^{-p^2 s}. \quad (2.18)$$

The equation readily shows how  $p$  is intimately connected to  $z$  through the transfer of cutoff dependence.

It is instructive to examine how the propagators and the one-loop kernel are modified in the presence of  $\rho_k^{(d)}(s, \Lambda)$ . Straightforward calculation leads to

$$\begin{aligned} \frac{1}{\mathcal{H}^n} &= \frac{1}{\Gamma(n)} \int_0^\infty ds s^{n-1} e^{-\mathcal{H}s} \xrightarrow{\text{OC}} \frac{1}{\mathcal{H}^n} \Big|_{\text{OC}} = \frac{1}{\Gamma(n)} \int_0^\infty ds s^{n-1} \rho_k^{(d)}(s, \Lambda) e^{-\mathcal{H}s} \\ &= \frac{1}{\mathcal{H}^n} \frac{2\Gamma(n+d/2)}{d\Gamma(n)\Gamma(d/2)} \left\{ \left(\frac{\Lambda^2}{\mathcal{H}}\right)^{d/2} F\left(\frac{d}{2}, \frac{d}{2} + n, 1 + \frac{d}{2}; -\frac{\Lambda^2}{\mathcal{H}}\right) - \left(\frac{k^2}{\mathcal{H}}\right)^{d/2} F\left(\frac{d}{2}, \frac{d}{2} + n, 1 + \frac{d}{2}; -\frac{k^2}{\mathcal{H}}\right) \right\}, \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} \text{Tr}'(\ln \mathcal{H} - \ln \mathcal{H}_0) &= -\int_0^\infty \frac{ds}{s} \text{Tr}(e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s}) \\ &\rightarrow \text{Tr}(\ln \mathcal{H} - \ln \mathcal{H}_0)|_{\text{OC}} = -\int_0^\infty \frac{ds}{s} \rho_k^{(d)}(s, \Lambda) \text{Tr}(e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s}) \\ &= -\frac{2}{d} \text{Tr} \left\{ \left(\frac{\Lambda^2}{\mathcal{H}}\right)^{d/2} F\left(\frac{d}{2}, \frac{d}{2}, 1 + \frac{d}{2}; -\frac{\Lambda^2}{\mathcal{H}}\right) - \left(\frac{\Lambda^2}{\mathcal{H}_0}\right)^{d/2} F\left(\frac{d}{2}, \frac{d}{2}, 1 + \frac{d}{2}; -\frac{\Lambda^2}{\mathcal{H}_0}\right) \right. \\ &\quad \left. - \left(\frac{k^2}{\mathcal{H}}\right)^{d/2} F\left(\frac{d}{2}, \frac{d}{2}, 1 + \frac{d}{2}; -\frac{k^2}{\mathcal{H}}\right) + \left(\frac{k^2}{\mathcal{H}_0}\right)^{d/2} F\left(\frac{d}{2}, \frac{d}{2}, 1 + \frac{d}{2}; -\frac{k^2}{\mathcal{H}_0}\right) \right\}, \end{aligned} \quad (2.20)$$

where

$$F(a, b, c; \beta) = B^{-1}(b, c-b) \int_0^1 dx x^{b-1} (1-x)^{(c-b-1)} (1-\beta x)^{-a} \quad (2.21)$$

$$\Gamma[\alpha; x_1, x_2] = \int_{x_1}^{x_2} dx x^{\alpha-1} e^{-x} \quad (2.15)$$

is the generalized incomplete  $\Gamma$  function. Notice that in the physical limit  $k \rightarrow 0$  and  $\Lambda \rightarrow \infty$ , we have

$$\rho_{k=0}^{(d)}(s, \Lambda \rightarrow \infty) = 1, \quad (2.16)$$

and the operator cutoff regularization is reduced to that of Schwinger's proper time. In this limit UV and possible IR divergences may appear and additional counterterms must be added to subtract off the infinities [3].

To explicitly demonstrate that our smearing function  $\rho_k^{(d)}(s, \Lambda)$  simulates a sharp momentum cutoff regulator, we substitute (2.14) into the last expression of (2.9),

$$U_k^{(1)}(\Phi) = -\frac{1}{2} \int_0^\infty \frac{ds}{s} (e^{-V''(\Phi)s} - e^{-V''(0)s}) \int_z' e^{-z^2 s}, \quad (2.17)$$

which, upon switching the order of integrations between  $s$  and  $z$  and equating  $z$  with the momentum variable  $p$ , gives back (2.2). Thus, we conclude that the proper-time smearing function derived in (2.14) completely reproduces the usual blocked potential  $U_k(\Phi)$  at the one-loop level. That the dummy integration variable  $z$  hidden in  $\rho_k^{(d)}(s, \Lambda)$  turns out to be the momentum variable  $p$  can be seen from a direct substitution of (2.14) into the second equation on the right-hand side of (2.9) which yields

is the hypergeometric function symmetric under the exchange between  $a$  and  $b$ , and

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dt t^{x-1}(1-t)^{y-1}, \quad (2.22)$$

the Euler  $\beta$  function. For  $n = 1$  and  $2$ , (2.19) yields, respectively,

$$\begin{aligned} \frac{1}{\mathcal{H}} \Big|_{\text{OC}} &= \int_0^\infty ds \rho_k^{(d)}(s, \Lambda) e^{-\mathcal{H}s} \\ &= \frac{1}{\mathcal{H}} \left\{ \left( \frac{\Lambda^2}{\mathcal{H} + \Lambda^2} \right)^{d/2} - \left( \frac{k^2}{\mathcal{H} + k^2} \right)^{d/2} \right\}, \end{aligned} \quad (2.23)$$

and

$$\begin{aligned} \frac{1}{\mathcal{H}^2} \Big|_{\text{OC}} &= \int_0^\infty ds s \rho_k^{(d)}(s, \Lambda) e^{-\mathcal{H}s} \\ &= \frac{1}{\mathcal{H}^2} \left\{ \left( \frac{\Lambda^2}{\mathcal{H} + \Lambda^2} \right)^{d/2} \left[ 1 + \frac{d}{2} \frac{\mathcal{H}}{\mathcal{H} + \Lambda^2} \right] \right. \\ &\quad \left. - \left( \frac{k^2}{\mathcal{H} + k^2} \right)^{d/2} \left[ 1 + \frac{d}{2} \frac{\mathcal{H}}{\mathcal{H} + k^2} \right] \right\}. \end{aligned} \quad (2.24)$$

By further restricting ourselves to  $d = 4$  where

$$\rho_k^{(4)}(s, \Lambda) = (1 + k^2 s) e^{-k^2 s} - (1 + \Lambda^2 s) e^{-\Lambda^2 s}, \quad (2.25)$$

the operator cutoff regularized propagator and the one-loop kernel take on the structures

$$\begin{aligned} \frac{1}{\mathcal{H}} \Big|_{\text{OC}} &= \frac{1}{\mathcal{H} + k^2} - \frac{1}{\mathcal{H} + \Lambda^2} \\ &\quad - \frac{\Lambda^2}{(\mathcal{H} + \Lambda^2)^2} + \frac{k^2}{(\mathcal{H} + k^2)^2} \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} \text{Tr}(\ln \mathcal{H} - \ln \mathcal{H}_0) \Big|_{\text{OC}} &= \text{Tr} \left\{ \ln \left[ \frac{\mathcal{H} + k^2}{\mathcal{H}_0 + k^2} \frac{\mathcal{H}_0 + \Lambda^2}{\mathcal{H} + \Lambda^2} \right] \right. \\ &\quad - \frac{\Lambda^2(\mathcal{H} - \mathcal{H}_0)}{(\mathcal{H} + \Lambda^2)(\mathcal{H}_0 + \Lambda^2)} \\ &\quad \left. + \frac{k^2(\mathcal{H} - \mathcal{H}_0)}{(\mathcal{H} + k^2)(\mathcal{H}_0 + k^2)} \right\}. \end{aligned} \quad (2.27)$$

The above equations imply that one may regard  $\Lambda$  as the mass of some unitarity-violating ghost states, which can be seen from the relative negative sign in the modified propagator. However, in the limit  $\Lambda \rightarrow \infty$ , the ghosts decouple from the theory, as they should. The IR cutoff scale  $k$  which can also be thought of as being the "fictitious" mass ascribed to the fields is a useful regulator particularly when the theory contains massless particles. Physical observables must be computed, however, by taking the limit  $k \rightarrow 0$ . From (2.26), one may also say that the effect of  $\Lambda$  is equivalent to introducing higher derivative terms into the theory. In other words, the Lagrangian density (2.1) may be replaced by the corresponding regularized counterpart:

$$\begin{aligned} \mathcal{L}_{\text{reg}} &= \frac{1}{2} \phi \left[ -\partial^2 + \frac{2}{\Lambda^2} (-\partial^2)^2 \right. \\ &\quad \left. + \frac{1}{\Lambda^4} (-\partial^2)^3 \right] \phi + V(\phi). \end{aligned} \quad (2.28)$$

The interpretations on the role played by the cutoff scales in the operator cutoff approach are reminiscent to that of the Pauli-Villars regulator. In fact, one can show that the conventional Pauli-Villars scheme is a special case of the proper-time regularization having a smearing function of the form [14]

$$\rho_k^{\text{PV}}(s, \Lambda) = \sum_i (a_i e^{-k_i^2 s} - b_i e^{-\Lambda_i^2 s}), \quad (2.29)$$

with  $\Lambda_i$  and  $k_i$  carrying the same meaning as the operator cutoff scales. In order to render the theory finite, the coefficients  $a_i$  and  $b_i$  as well as  $i$ , the number of ghost terms, must be appropriately chosen. Equation (2.29) implies

$$\begin{aligned} \frac{1}{\mathcal{H}^n} \Big|_{\text{PV}} &= \sum_i \frac{1}{\Gamma(n)} \int_0^\infty ds s^{n-1} (a_i e^{-k_i^2 s} - b_i e^{-\Lambda_i^2 s}) e^{-\mathcal{H}s} \\ &= \sum_i \left[ \frac{a_i}{(\mathcal{H} + k_i^2)^n} - \frac{b_i}{(\mathcal{H} + \Lambda_i^2)^n} \right] \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} \text{Tr} \ln(\mathcal{H} - \mathcal{H}_0) \Big|_{\text{PV}} &= - \sum_i \int_0^\infty \frac{ds}{s} (a_i e^{-k_i^2 s} - b_i e^{-\Lambda_i^2 s}) \text{Tr}(e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s}) \\ &= \text{Tr} \sum_i \ln \left[ \left( \frac{\mathcal{H} + k_i^2}{\mathcal{H}_0 + k_i^2} \right)^{a_i} \left( \frac{\mathcal{H}_0 + \Lambda_i^2}{\mathcal{H} + \Lambda_i^2} \right)^{b_i} \right], \end{aligned} \quad (2.31)$$

which resemble (2.26) and (2.27) by simply taking  $a_i = b_i = i = 1$ . The minute difference can be attributed to the extra terms that are linearly dependent in  $s$  in the definition of the smearing function.

For a general fluctuation operator of the form  $\mathcal{H} = \mathcal{H}_0 + \lambda \delta \mathcal{H}$ , one can easily show that  $\rho_k^{(d)}(s, \Lambda)$  reproduces the usual momentum cutoff results order by order in  $\lambda$  by comparing the general expansion

$$\text{Tr}'(\ln \mathcal{H} - \ln \mathcal{H}_0) = \text{Tr}' \ln[1 + \lambda \mathcal{H}_0^{-1} \delta \mathcal{H}] = - \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n} \text{Tr}' \{ (\mathcal{H}_0^{-1} \delta \mathcal{H})^n \} \quad (2.32)$$

with

$$\begin{aligned}
\text{Tr}(\ln \mathcal{H} - \ln \mathcal{H}_0)|_{\text{OC}} &= - \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n} \text{Tr}\{(\mathcal{H}_0^{-1} \delta \mathcal{H})^n |_{\text{OC}}\} \\
&= - \int_0^{\infty} \frac{ds}{s} \rho_k^{(d)}(s, \Lambda) \text{Tr}\{e^{-(\mathcal{H}_0 + \lambda \delta \mathcal{H})s} - e^{-\mathcal{H}_0 s}\} \\
&= - \int_0^{\infty} \frac{ds}{s} \rho_k^{(d)}(s, \Lambda) \text{Tr}\left\{e^{-\mathcal{H}_0 s} \left[ \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} (\delta \mathcal{H})^n s^n \right] - e^{-\mathcal{H}_0 s}\right\} \\
&= - \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_0^{\infty} ds s^{n-1} \rho_k^{(d)}(s, \Lambda) \text{Tr}\{e^{-\mathcal{H}_0 s} (\delta \mathcal{H})^n\} \\
&= - \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_0^{\infty} ds s^{n-1} \text{Tr}'\{e^{-\mathcal{H}_0 s} (\delta \mathcal{H})^n\} \\
&= - \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n} \text{Tr}'\{(\mathcal{H}_0^{-1} \delta \mathcal{H})^n\} \tag{2.33}
\end{aligned}$$

in the operator cutoff approach. As an explicit check, we again consider the  $\lambda \Phi^4$  theory in  $d = 4$  with  $k = 0$  as an example and obtain the following one-loop corrections to the two- and four-point vertex functions:

$$\begin{aligned}
\delta \Gamma_{\text{OC}}^{(2)} &= \frac{\lambda}{2} \int_p \frac{1}{p^2 + \mu^2} \Big|_{\text{OC}} = \frac{\lambda}{2} \int_p \frac{1}{p^2 + \mu^2} \left( \frac{\Lambda^2}{p^2 + \mu^2 + \Lambda^2} \right)^2 \\
&= \frac{\lambda}{32\pi^2} \left[ \Lambda^2 - \mu^2 \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) \right] = \frac{\lambda}{2} \int_p' \frac{1}{p^2 + \mu^2}, \tag{2.34}
\end{aligned}$$

and

$$\begin{aligned}
\delta \Gamma_{\text{OC}}^{(4)} &= - \frac{3\lambda^2}{2} \int_p \frac{1}{(p^2 + \mu^2)^2} \Big|_{\text{OC}} = - \frac{3\lambda^2}{2} \int_p \frac{1}{(p^2 + \mu^2)^2} \left( \frac{\Lambda^2}{p^2 + \mu^2 + \Lambda^2} \right)^2 \left( 1 + \frac{2(p^2 + \mu^2)}{p^2 + \mu^2 + \Lambda^2} \right) \\
&= - \frac{3\lambda^2}{32\pi^2} \left[ \ln \left( \frac{\Lambda^2 + \mu^2}{\mu^2} \right) - \frac{\Lambda^2}{\mu^2 + \Lambda^2} \right] = - \frac{3\lambda^2}{2} \int_p' \frac{1}{(p^2 + \mu^2)^2}, \tag{2.35}
\end{aligned}$$

which are in complete agreement with the momentum cutoff results. Thus, we see that operator cutoff regularization reproduces the usual momentum cutoff results order by order in the coupling constant expansion.

### III. OPERATOR CUTOFF AND DERIVATIVE EXPANSION

It was demonstrated in the last section that the smearing function  $\rho_k^{(d)}(s, \Lambda)$  derived in (2.14) imitates the momentum cutoff at the level of one-loop blocked potential  $U_k(\Phi)$ . In this section we take into account the  $\partial^2$  derivative term and inquire how to further construct the linkage between the two regularization schemes. For the computation of the wave-function renormalization constant  $Z_k(\Phi)$ , a small inhomogeneity is assumed to be present in the background field. Various methods for deriving  $Z_k(\Phi)$  are available [17]. Below we rederive  $Z_k(\Phi)$  using the two prescriptions prescribed above and compare their results. As we shall see, discrepancy between the two schemes appears already at the level of  $Z_k(\Phi)$  if only one

smearing function  $\rho_k^{(d)}(s, \Lambda)$  is used throughout. Equality between the two formalisms up to  $O(\partial^2)$  is restored provided that an additional smearing function be used.

#### A. Momentum cutoff regularization

To compute  $Z_k(\Phi)$  via the momentum cutoff regularization, we adopt the approach originated by Fraser [18]. The manner in which the derivative terms are extracted is based on the notion of treating the momentum and field variables as noncommuting operators  $\hat{p}_\mu$  and  $\hat{\Phi}$  obeying the commutation relations

$$[\hat{\Phi}, \hat{p}_\mu] = -i \partial_\mu \hat{\Phi} \tag{3.1}$$

and

$$[\hat{\Phi}, \hat{p}^2] = -\partial^2 \hat{\Phi} - 2i \hat{p}_\mu \partial_\mu \hat{\Phi}, \tag{3.2}$$

where a caret has been added to the operators to distinguish them from the ordinary  $c$ -number variables. Repeated use of the above relations leads to

$$\begin{aligned}
(\hat{p}^2 + \hat{V}'')^n &= (\hat{p}^2 + \hat{V}'')^n : -\frac{1}{2}n(n-1) : (\hat{p}^2 + \hat{V}'')^{n-2} : \partial^2 \hat{V}'' \\
&\quad -in(n-1)\hat{p}_\mu : (\hat{p}^2 + \hat{V}'')^{n-2} : \partial_\mu \hat{V}'' \\
&\quad -\frac{1}{3}n(n-1)(n-2) : (\hat{p}^2 + \hat{V}'')^{n-3} : (\partial_\mu \hat{V}'')^2 \\
&\quad -\frac{2}{3}n(n-1)(n-2)\hat{p}_\mu \hat{p}_\nu : (\hat{p}^2 + \hat{V}'')^{n-3} : \partial_\mu \partial_\nu \hat{V}'' \\
&\quad -\frac{1}{2}n(n-1)(n-2)(n-3)\hat{p}_\mu \hat{p}_\nu : (\hat{p}^2 + \hat{V}'')^{n-4} : \partial_\mu \hat{V}'' \partial_\nu \hat{V}'' + O(\partial^4), \tag{3.3}
\end{aligned}$$

where  $::$  implies a “normal ordering” procedure such that all  $\hat{p}$  dependences are moved to the left of the  $\hat{\Phi}$ -dependent terms. This is a necessary step for evaluating the functional trace in (2.2) since the momentum integration is to be performed before  $x$ . Once the normal ordering procedure is done, we may simply drop the carets and treat the quantities on the right-hand side of (3.3) as ordinary  $c$  numbers since any further application of (3.1) or (3.2) will only generate higher-order derivative terms which do not affect the computation of  $Z_k$ .

With the help of the identity

$$\ln(\hat{p}^2 + \hat{V}'') = \lim_{n \rightarrow 0} \frac{\partial}{\partial n} (\hat{p}^2 + \hat{V}'')^n, \quad (3.4)$$

the one-loop contribution to the blocked action becomes (dropping the carets)

$$\begin{aligned} \tilde{S}_k^{(1)}[\Phi] &= \frac{1}{2} \int_x \int_p \left\{ \ln(p^2 + V'') + \frac{(9-2d)p^2 - 2dV''}{3d(p^2 + V'')^4} (\partial_\mu V'')^2 + \frac{(3d-8)p^2 + 3dV''}{6d(p^2 + V'')^3} \partial^2 V'' + O(\partial^4) \right\} \\ &= \frac{1}{2} \int_x \int_p \left\{ \ln(p^2 + V'') + \frac{(d-3)p^2 + dV''}{3d(p^2 + V'')^4} (\partial_\mu V'')^2 + O(\partial^4) \right\}. \end{aligned} \quad (3.5)$$

The above expression is obtained by first dropping the surface terms with

$$0 = \int_x \partial^2 \left[ \frac{1}{(p^2 + V'')^n} \right] = \int_x \left\{ \frac{n(n+1)}{(p^2 + V'')^{n+2}} (\partial_\mu V'')^2 - \frac{n}{(p^2 + V'')^{n+1}} \partial^2 V'' \right\}, \quad (3.6)$$

followed by simplifying the momentum integrations using the  $O(d)$  invariant property:

$$\int_p p_{\mu_1} p_{\mu_2} \cdots p_{\mu_m} f(p^2) = \frac{T_{\mu_1 \mu_2 \cdots \mu_m}^m \Gamma(d/2)}{2^m \Gamma(m + d/2)} \int_p (p^2)^m f(p^2), \quad (3.7)$$

where  $f(p^2)$  is an arbitrary scalar function and

$$T_{\mu_1 \mu_2 \cdots \mu_m}^m = \delta_{\mu_1, \mu_2} \cdots \delta_{\mu_{2m-1}, \mu_{2m}} + \text{permutations}. \quad (3.8)$$

While the first term in the last equation of (3.5) matches the usual one-loop logarithmic contribution for the blocked potential  $U_k(\Phi)$ , the second term represents the correction to  $Z_k(\Phi)$ . Taking the familiar  $\lambda\Phi^4$  theory as an example, with  $V''(\Phi) = \mu^2 + \lambda\Phi^2/2$ , we have

$$Z_k^{(1)}(\Phi) = \frac{\lambda^2 \Phi^2}{3d} \int_p \frac{(d-3)p^2 + dV''}{(p^2 + V'')^4}, \quad (3.9)$$

which, upon differentiating with respect to  $k$ , yields the differential flow equation

$$k \frac{\partial Z_k(\Phi)}{\partial k} = -\frac{S_d k^d \lambda^2 \Phi^2 (d-3)k^2 + dV''}{3d (k^2 + V'')^4}. \quad (3.10)$$

For  $d = 4$ , (3.9) becomes

$$Z_k^{(1)}(\Phi) = \frac{\lambda^2 \Phi^2}{192\pi^2} \frac{k^4 + 3k^2 V'' + V''^2}{(k^2 + V'')^3}, \quad (3.11)$$

which agrees with that obtained in [12,17,18] in the limit  $k \rightarrow 0$ . Since the contribution to  $Z_k(\Phi)$  is UV finite, one can safely take the limit  $\Lambda \rightarrow \infty$ .

## B. Operator cutoff regularization

In the alternative operator cutoff approach, derivative terms also arise from commuting the operators  $\hat{p}_\mu$  and  $\hat{\Phi}$ . The one-loop contribution to the blocked action in the proper-time representation is given by

$$\tilde{S}_k^{(1)}[\hat{\Phi}] = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr}'(e^{-[\hat{p}^2 + \hat{V}''(\hat{\Phi})]s} - e^{-[\hat{p}^2 + \hat{V}''(0)]s}), \quad (3.12)$$

where the normal ordering procedure can be carried out as



$$\begin{aligned}
e^{-(\hat{p}^2 + \hat{V}'')_s} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} s^n (\hat{p}^2 + \hat{V}'')^n \\
&=: e^{-(\hat{p}^2 + \hat{V}'')_s} : - : e^{-(\hat{p}^2 + \hat{V}'')_s} : \left\{ i s^2 p_\mu \partial_\mu \hat{V}'' + \frac{1}{2} s^2 \partial^2 \hat{V}'' - \frac{1}{3} s^3 (\partial_\mu \hat{V}'')^2 \right. \\
&\quad \left. - \frac{2}{3} s^3 \hat{p}_\mu \hat{p}_\nu \partial_\mu \partial_\nu \hat{V}'' + \frac{1}{2} s^4 \hat{p}_\mu \hat{p}_\nu \partial_\mu \hat{V}'' \partial_\nu \hat{V}'' + \dots \right\} \\
&\stackrel{O(d)}{\rightarrow} : e^{-(\hat{p}^2 + \hat{V}'')_s} : - : e^{-(\hat{p}^2 + \hat{V}'')_s} : \left\{ \frac{1}{2} s^2 \partial^2 \hat{V}'' - \frac{1}{3} s^3 (\partial_\mu \hat{V}'')^2 - \hat{p}^2 \left[ \frac{2s^3}{3d} \partial^2 \hat{V}'' - \frac{s^4}{2d} (\partial_\mu \hat{V}'')^2 \right] + \dots \right\}.
\end{aligned} \tag{3.13}$$

Dropping the distinction between operators and  $c$  numbers as before, the one-loop correction to the blocked action becomes

$$\begin{aligned}
\tilde{S}_k^{(1)}[\Phi] &= -\frac{1}{2} \int_x \int_0^\infty \frac{ds}{s} \int_p' e^{-p^2 s} (e^{-V''_s} - 1) + \frac{1}{12} \int_x \int_0^\infty ds s \int_p' e^{-p^2 s} e^{-V''_s} [3\partial^2 V'' - 2s(\partial_\mu V'')^2] \\
&\quad - \frac{1}{12d} \int_x \int_0^\infty ds s^2 \int_p' e^{-p^2 s} p^2 e^{-V''_s} [4\partial^2 V'' - 3s(\partial_\mu V'')^2].
\end{aligned} \tag{3.14}$$

In the above, the second and the third integrals together contribute to  $Z_k^{(1)}(\Phi)$ . Equation (3.14) reduces to (3.5) if the  $s$  integration is carried out first and simplified by the help of (3.6). However, our goal here is to find out how the regulating smearing function(s) should be implemented in the  $s$  integrations in order to allow for a complete  $p$  integration without imposing any cutoff scales. The precaution to be taken here is that due to the different powers of  $p$  dependence, there is no reason *a priori* that the same smearing function  $\rho_k^{(d)}(s, \Lambda)$  can yield a wave-function renormalization constant  $Z_k(\Phi)$  identical to (3.10) which was derived from momentum cutoff. That a new smearing function must be called for at each level of derivative expansion is actually hinted from the integral transformation

$$\begin{aligned}
\int_p' \frac{(p^2)^m}{(p^2 + a)^n} &= \frac{1}{\Gamma(n)} \int_0^\infty ds s^{n-1} e^{-as} \int_p' (p^2)^m e^{-p^2 s} \\
&\rightarrow \frac{1}{\Gamma(n)} \int_0^\infty ds s^{n-1} e^{-as} \rho_k^{(d,m)}(s, \Lambda) \int_p' (p^2)^m e^{-p^2 s} \\
&= \frac{\Gamma(m + d/2)}{(4\pi)^{d/2} \Gamma(d/2) \Gamma(n)} \int_0^\infty ds s^{n-1-m-d/2} e^{-as} \rho_k^{(d,m)}(s, \Lambda),
\end{aligned} \tag{3.15}$$

which is satisfied provided that

$$\rho_k^{(d,m)}(s, \Lambda) = \frac{2s^{m+d/2}}{S_d \Gamma(m + d/2)} \int_z' (z^2)^m e^{-z^2 s} = \frac{1}{\Gamma(m + d/2)} \Gamma\left(m + \frac{d}{2}; k^2 s, \Lambda^2 s\right). \tag{3.16}$$

One may explicitly check that by using  $\rho_k^{(d,1)}(s, \Lambda)$  [with  $\rho_k^{(d,0)}(s, \Lambda) = \rho_k^{(d)}(s, \Lambda)$ ] for the third integral in (3.14), the original cutoff expression is recovered. With

$$\begin{aligned}
\tilde{S}_k^{(1)}[\Phi] &= -\frac{1}{2} \int_x \int_0^\infty \frac{ds}{s} \rho_k^{(d,0)}(s, \Lambda) \int_p' e^{-p^2 s} (e^{-V''_s} - 1) \\
&\quad + \frac{1}{12} \int_x \int_0^\infty ds s \rho_k^{(d,0)}(s, \Lambda) \int_p' e^{-p^2 s} e^{-V''_s} [3\partial^2 V'' - 2s(\partial_\mu V'')^2] \\
&\quad - \frac{1}{12d} \int_x \int_0^\infty ds s^2 \rho_k^{(d,1)}(s, \Lambda) \int_p' e^{-p^2 s} p^2 e^{-V''_s} [4\partial^2 V'' - 3s(\partial_\mu V'')^2],
\end{aligned} \tag{3.17}$$

upon performing the  $p$  integration followed by a differentiation with respect to  $k$ , we arrive at

$$\begin{aligned}
k \frac{\partial \tilde{S}_k[\Phi]}{\partial k} &= \int_x \left\{ k \frac{\partial U_k(\Phi)}{\partial k} + \frac{1}{2} k \frac{\partial Z_k(\Phi)}{\partial k} (\partial_\mu \Phi)^2 + \dots \right\} \\
&= -\frac{1}{2(4\pi)^{d/2}} \int_x \int_0^\infty \frac{ds}{s^{1+d/2}} k \frac{\partial \rho_k^{(d,0)}}{\partial k} (e^{-V''_s} - 1) \\
&\quad + \frac{1}{24(4\pi)^{d/2}} \int_x (\partial_\mu V'')^2 \int_0^\infty ds s^{2-d/2} e^{-V''_s} \left[ 2k \frac{\partial \rho_k^{(d,0)}}{\partial k} - k \frac{\partial \rho_k^{(d,1)}}{\partial k} \right],
\end{aligned} \tag{3.18}$$

where the expression is simplified by the help of

$$0 = \int_x \partial_\mu (e^{-V''s} \partial_\mu V'') = \int_x e^{-V''s} [\partial^2 V'' - (\partial_\mu V'')^2 s]. \quad (3.19)$$

From (3.18), it is clear that despite the presence of the derivative terms  $\rho_k^{(d,0)}(s, \Lambda)$  is still given by (2.14) for the equality in  $k(\partial U_k/\partial k)$  to hold. As for  $Z_k(\Phi)$ , comparison between (3.18) and (3.10) gives

$$\begin{aligned} k \frac{\partial Z_k(\Phi)}{\partial k} &= -\frac{S_d k^d \lambda^2 \Phi^2 (d-3)k^2 + dV''}{3d (k^2 + V'')^4} \\ &= \frac{\lambda^2 \Phi^2}{12(4\pi)^{d/2}} \int_0^\infty ds s^{2-d/2} e^{-V''s} \left[ 2k \frac{\partial \rho_k^{(d,0)}}{\partial k} - k \frac{\partial \rho_k^{(d,1)}}{\partial k} \right]. \end{aligned} \quad (3.20)$$

Use of the identity

$$\frac{1}{\mathcal{H}^n} = \frac{1}{\Gamma(n)} \int_0^\infty ds s^{n-1} e^{-\mathcal{H}s} \quad (3.21)$$

then leads to

$$\begin{aligned} k \frac{\partial \rho_k^{(d,1)}}{\partial k} &= 2k \frac{\partial \rho_k^{(d,0)}}{\partial k} + \frac{4(k^2 s)^{d/2}}{d\Gamma(d/2)} (d - k^2 s) e^{-k^2 s} \\ &= -\frac{4(k^2 s)^{1+d/2}}{d\Gamma(d/2)} e^{-k^2 s} \end{aligned} \quad (3.22)$$

or

$$\begin{aligned} \rho_k^{(d,1)}(s, \Lambda) &= \frac{2s^{1+d/2}}{S_d \Gamma(1+d/2)} \int_z^1 z^2 e^{-z^2 s} \\ &= \frac{1}{\Gamma(1+d/2)} \Gamma\left(1 + \frac{d}{2}; k^2 s, \Lambda^2 s\right), \end{aligned} \quad (3.23)$$

which agrees with (3.15) for  $m = 1$  and confirms that more than one smearing function must be used to yield the same  $Z_k(\Phi)$  in both formalisms. Had  $\rho_k^{(4)}(s, \Lambda)$  been used alone, the resulting one-loop correction which we denote with a bar symbol would have been

$$\begin{aligned} \bar{Z}_k^{(1)}(\Phi) &= \frac{\lambda \Phi^2}{192\pi^2} \frac{2k^2 + V''}{(k^2 + V'')^2} \\ &= \frac{\lambda \Phi^2}{192\pi^2} \frac{2k^4 + 3k^2 V'' + V''^2}{(k^2 + V'')^3}, \end{aligned} \quad (3.24)$$

instead of (3.11) even though the limit  $\bar{Z}_{k=0}^{(1)}(\Phi) = \lambda^2 \Phi^2 / 192\pi^2 V''$  is insensitive to whether  $\rho_k^{(4,1)}(s, \Lambda)$  is actually employed or not. A comparison between (3.24) and (3.11) reveals that the discrepancy between  $Z_k(\Phi)$  and  $\bar{Z}_k(\Phi)$  comes from an  $O(k^4)$  mismatch in the numerator. The discrepancy can be traced to be originated from terms that are multiplied by  $p^2$  in the derivative expansion, i.e., the last two terms inside the curly brackets in (3.13). These are the quantities that vanish most rapidly in the IR limit  $k \rightarrow 0$  in the differential flow equation. The insufficiency of using just  $\rho_k^{(d)}(s, \Lambda)$  can be understood as follows: From (2.10), one readily sees that after the  $p$  integration, all  $p^2$ -dependent terms generated from derivative expansion will acquire an extra factor of  $s^{-1}$  compared with the ones without the  $p^2$  dependence. Subsequent  $s$  integration using (2.14) then yields one power

$(z^2 + V'')$  for each  $s^{-1}$ . However, when  $z$  is equated with  $p$ , the cutoff result can no longer be recovered without the unjustified substitution of  $p^2$  by  $p^2 + V''$ . Therefore, to account for those higher-order contributions that vanish more rapidly as  $k \rightarrow 0$ , it is necessary to introduce  $\rho_k^{(d,1)}(s, \Lambda)$ .

In the case where  $d = 4$ , we have

$$\begin{aligned} \rho_k^{(4,1)}(s, \Lambda) &= (1 + k^2 s + \frac{1}{2} k^4 s^2) e^{-k^2 s} \\ &\quad - (1 + \Lambda^2 s + \frac{1}{2} \Lambda^4 s^2) e^{-\Lambda^2 s}, \end{aligned} \quad (3.25)$$

which differs from  $\rho_k^{(4)}(s, \Lambda)$  by the higher-order  $s^2$ -dependent terms. These terms, as already demonstrated, are essential for regaining the expected cutoff dependence. That  $\rho_k^{(d,1)}(s, \Lambda)$  provides a faster UV and IR convergence can be seen from the fact that the smearing functions are of the form of the generalized incomplete  $\Gamma$  function which can be expanded as [19]

$$\begin{aligned} \rho_k^{(4,m)}(s, \Lambda) &= e^{-k^2 s} \sum_{l=0}^{m+1} \frac{1}{l!} (k^2 s)^l \\ &\quad - e^{-\Lambda^2 s} \sum_{l=0}^{m+1} \frac{1}{l!} (\Lambda^2 s)^l. \end{aligned} \quad (3.26)$$

This yields

$$\begin{aligned} \int_0^\infty ds \rho_k^{(4,m)}(s, \Lambda) e^{-\mathcal{H}s} \\ = \frac{1}{\mathcal{H}} \left[ \left( \frac{\Lambda^2}{\mathcal{H} + \Lambda^2} \right)^{m+2} - \left( \frac{k^2}{\mathcal{H} + k^2} \right)^{m+2} \right], \end{aligned} \quad (3.27)$$

which indicates that the larger the  $m$ , the more rapid the convergence [12]. We also comment that choosing  $m$  in  $\rho_k^{(4,m)}(s, \Lambda)$  is analogous to choosing the number of ghost terms in the Pauli-Villars regularization since in both approaches the divergence is eradicated by increasing the power of  $p$  dependence in the denominator. For example, while only one ghost term is sufficient to regularize the logarithmic divergence found in the four-point vertex function for the  $\lambda\Phi^4$  theory in  $d = 4$ , the quadratically divergent integral characteristic of the two-point function calls for at least two ghost terms to ensure the proper convergence [20].

The requirement of using more than one smearing function to attain equality between the momentum cutoff and operator cutoff regularizations at the level of  $Z_k(\Phi)$  may seem disturbing at first glance since its generalization to higher orders of derivative expansion will become more complicated. However, we remark that the computation of  $Z_k(\Phi)$  is generally dependent on how the derivative terms are isolated; disagreements exist even within momentum cutoff regularization itself. For example, imposing momentum cutoff on the approach used by Fraser in [18] would have led to

$$\tilde{Z}_k^{(1)}(\Phi) = \frac{\lambda\Phi^2}{192\pi^2} \frac{3k^2V'' + V'''^2}{(k^2 + V'')^3}, \quad (3.28)$$

which also differs from (3.11) by a higher order  $O(k^4)$  term in the numerator. To reconcile the difference, we first observe that since the flow pattern of  $Z_k(\Phi)$  allows for the determination of the anomalous dimension  $\gamma_k$  via

$$\gamma_k = k \frac{\partial \ln Z_k(\Phi)}{\partial k}, \quad (3.29)$$

the ambiguity in the  $k$  dependence of  $Z_k(\Phi)$  must be intimately connected with the scheme dependence in the computation of  $\gamma_k$ . This is precisely what was concluded in [21] where the explicit cutoff dependence in  $\gamma_k$  beyond the lowest order was demonstrated. Such a dependence should come as no surprise since the RG coefficient functions such as the anomalous dimension and  $\beta$  functions are generally regularization dependent beyond the leading order. Nonetheless, taking the physical limit  $k \rightarrow 0$  for all three cases, we have

$$Z_{k=0}^{(1)}(\Phi) = \tilde{Z}_{k=0}^{(1)}(\Phi) = \tilde{\tilde{Z}}_{k=0}^{(1)}(\Phi) = \frac{1}{192\pi^2} \frac{\lambda^2\Phi^2}{V''}. \quad (3.30)$$

Even though the one-loop correction to the wavefunction renormalization exhibits different  $k$  dependence for different methods, we argue that when dealing with real physical situations the precise form of the differential flow equations for the higher-order derivative terms should not be taken too seriously. The concept of derivative expansion carried out in (2.3) has practical use only when one is interested in exploring the large-distance effects in the IR regime. The expansion of  $\tilde{S}_k[\Phi]$  in powers of  $\partial_\mu\Phi$  can only generate higher-order corrections to the dominant blocked potential  $U_k(\Phi)$ . In addition, we have seen that after taking the physical limit  $\Lambda \rightarrow \infty$  and  $k \rightarrow 0$ ,  $U_{k=0}(\Phi)$  and  $Z_{k=0}(\Phi)$  are all the same irrespective of how they are computed. Hence, for all practical purposes one may safely ignore the small higher-order mismatches and employ  $\rho_k^{(d)}(s, \Lambda)$  alone to compute  $U_k(\Phi)$ ,  $Z_k(\Phi)$  as well as other higher-order coefficients. While  $\rho_k^{(d)}(s, \Lambda)$  can be regarded as a sharp momentum cutoff for the leading order  $U_k(\Phi)$ , it corresponds to a *smooth* momentum regulator for  $Z_k(\Phi)$  and beyond. In the next section where gauge theories are explored, we shall see how these small mismatches precisely correspond to the gauge-noninvariant contributions that must be purged in order to preserve gauge symmetry.

Another aspect concerning the use of a sharp cutoff

in the derivative expansion is the emergence of nonlocal interactions as we lower the scale  $k$  which defines the sharp boundary between the high and the low modes. The presence of nonlocality in  $\tilde{S}_k[\Phi]$  is reflected by the necessity of incorporating interactions to all ranges, and hence, the simplifying picture of utilizing a reduced set of degrees of freedom to characterize  $\tilde{S}_k[\Phi]$  may be lost. However, the objection against the use of a sharp cutoff in conjunction with derivative expansion can be overcome by the following argument: In the perturbative approach the loop integrations are performed between the IR and UV cutoffs for summing up an infinite number of Feynman graphs for the partition function. While the UV cutoff is eliminated by renormalization, we would like to remove the IR cutoff as well in order to study the theory in the thermodynamical limit  $k \rightarrow 0$  where all physical observables take on their limiting values. The goal of using a sharp IR cutoff precisely allows us to explore these physical observables in the vicinity  $k \sim 0$ . Whatever nonlocality may arise from this sharp cutoff regularization will also be present in the thermodynamical limit with any other regularization schemes since the physics in this regime is independent of how one achieves the elimination of all the high modes [22]. Similar viewpoints have also been presented by Morris in [23], where the difficulties and inadequacy of choosing a smooth regulator were addressed. In fact the ‘‘most promising’’ method proposed there coincides with the formalism we have developed earlier [15] and presented here, namely, a derivative expansion around  $k = 0$  with a sharp cutoff as the candidate for the low-energy effective theory.

#### IV. OPERATOR CUTOFF AND GAUGE SYMMETRY

In the previous sections we have seen that by employing a set of smearing functions  $\rho_k^{(d,m)}(s, \Lambda)$  in the operator cutoff regularization, the one-loop cutoff structure can be recovered to arbitrary order in the derivative expansion. However, the major distinction between operator cutoff and momentum cutoff regularizations is that while the former is a special case of the proper-time regulator and thus symmetry preserving, the latter is not. We now turn to gauge theories and explore the role dictated by symmetry.

In the course of evaluating the effective action for gauge theories, one frequently encounters the fluctuation operator

$$\mathcal{H} = -D^2 + \mu^2 + Y(x), \quad (4.1)$$

where  $D_\mu$  is the covariant derivative for the gauge group,  $\mu^2$  may be the mass for the scalar field coupled to the gauge field  $A_\mu^a(x)$ , and  $Y(x)$  a matrix-valued function of  $x$  describing, say, the interaction between the scalar particles. The index  $a$  runs over the dimension of the gauge group. One may also write  $Y = Y^a T^a$  where  $T^a$  are the generators of the gauge group satisfying  $\text{tr}(T^a T^b) = -\delta^{ab}/2$  with  $\text{tr}$  denoting the summation over internal indices only. When operating on  $Y$ , the covariant derivative gives  $D_\mu Y = \partial_\mu Y + [A_\mu, Y]$ , where  $A_\mu = g A_\mu^a T^a$  with  $g$  being a coupling constant. The unregularized one-loop contribution to the effective action can be written as

$$\tilde{S}^{(1)} = \frac{1}{2} \text{Tr}(\ln \mathcal{H} - \ln \mathcal{H}_0) = -\frac{1}{2} \int_x \int_0^\infty \frac{ds}{s} \text{tr} \langle x | (e^{-\mathcal{H}s} - e^{-\mathcal{H}_0 s}) | x \rangle. \quad (4.2)$$

The diagonal part of the “heat kernel” in the above can be written as

$$\begin{aligned} h(s; x, x) &= \langle x | e^{-\mathcal{H}s} | x \rangle = \int_p \langle x | p \rangle e^{-\mathcal{H}_x s} \langle p | x \rangle = \int_p e^{-ipx} e^{-\mathcal{H}_x s} e^{ipx} \\ &= \int_p e^{-(p^2 - 2ip \cdot D + \mathcal{H}_x)s} \mathbb{1} = \int_p e^{-(p^2 + \mu^2)s} e^{(2ip \cdot D + D^2 - Y)s} \mathbb{1} \\ &= \sum_{n=0}^{\infty} \frac{s^n}{n!} \int_p e^{-(p^2 + \mu^2)s} (2ip \cdot D + D^2 - Y)^n \mathbb{1}, \end{aligned} \quad (4.3)$$

where we have employed the plane wave basis  $|p\rangle$  with  $\langle x|p\rangle = e^{-ipx}$  and the commutation relations [13,24]

$$[D_\mu, e^{ipx}] = ip_\mu, \quad [\mathcal{H}_x, e^{ipx}] = p^2 - 2ip \cdot D. \quad (4.4)$$

The factor  $\mathbb{1}$  indicates that the operator  $D_\mu$  acts on the identity. Inserting (4.3) into (4.2), we are led to

$$\begin{aligned} \tilde{S}^{(1)} &= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \int_x \int_0^\infty ds s^{n-1} \int_p \text{tr} \{ e^{-(p^2 + \mu^2)s} (2ip \cdot D + D^2 - Y)^n \} \mathbb{1} \\ &= \frac{1}{2} \int_x \int_p \text{tr} \left\{ \ln(p^2 + \mu^2) - \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2ip \cdot D + D^2 - Y}{p^2 + \mu^2} \right)^n \right\} \mathbb{1}, \end{aligned} \quad (4.5)$$

where the first term in the integrand represents an overall constant and can be subsequently dropped. The above expression suggests that it is possible to expand the effective action in terms of inverse mass provided that the background fields  $A_\mu(x)$  and  $Y(x)$  vary slowly on the scale  $\mu^{-1}$ . Thus, we may write the effective action as [25,26]

$$\tilde{S}^{(1)} = \int_x \sum_{n=1}^{\infty} \frac{a_n}{(\mu^2)^{n-2}} \mathcal{O}_n, \quad (4.6)$$

where  $\mathcal{O}_n$  are the traces of dimension  $2n$  gauge-invariant operators and can in general be written as

$$\mathcal{O}_n = \sum_i \gamma_i^{(n)} \tilde{\mathcal{O}}_n^{(i)}, \quad (4.7)$$

with  $\gamma_i^{(n)}$  being the numerical coefficients associated with  $\tilde{\mathcal{O}}_n^{(i)}$ , the set of linearly independent traces. As an example, using the field strength  $F_{\mu\nu}$ ,  $Y$ , and  $D_\mu$  one can construct the following:

$$\begin{aligned} \tilde{\mathcal{O}}_1 &= \{\text{tr}(Y)\}, \\ \tilde{\mathcal{O}}_2 &= \{\text{tr}(Y^2), \text{tr}(F_{\mu\nu} F_{\mu\nu})\}, \\ \tilde{\mathcal{O}}_3 &= \{\text{tr}(Y^3), \text{tr}(D_\mu Y)^2, \text{tr}(F_{\mu\nu} Y F_{\mu\nu}), \text{tr}(D_\mu F_{\mu\nu} D_\sigma F_{\sigma\nu}), \text{tr}(F_{\mu\nu} F_{\nu\sigma} F_{\sigma\mu})\}. \end{aligned} \quad (4.8)$$

With the integrand written explicitly in terms of symmetry-preserving quantities, gauge invariance of the effective action is automatically satisfied by choosing an invariant regularization scheme. One possible candidate for regulating the second expression on the right-hand side of (4.5) is by dimensional regularization since the noninvariant momentum cutoff is undesirable here. However, if we consider the first equation in (4.5) instead, the most natural way to do away with the divergence is to introduce a set of smearing functions  $\rho_k^{(d,m)}(s, \Lambda)$  for the proper-time integration, as suggested before. The advantage of going to the proper-time formalism is that in addition to allowing the theory to be regulated in a completely invariant manner by preserving the full symmetries, even gauge symmetry, of the original Lagrangian, it also admits cutoff scales. The symmetry-preserving nature of the regulator can easily be seen from the ab-

sence of cutoff scales in the  $p$  integration and the transfer of spacetime singularity into a singularity in the proper-time variable which is independent of symmetry transformation on the background fields. The insertion of the regulating function may be thought of as solely for the purpose of “technical” convenience to cope with the divergence manifested in the  $s$  integration. Any possible violation of symmetry within the operator cutoff formalism can take place only if the smearing function depends on certain parameters such as the background fields or  $p$ . In such case, the regularized action will vary under symmetry transformation.

Below we regularize the theory with both the gauge-noninvariant momentum cutoff and the invariant operator cutoff regulators to establish a connection between them.

### A. Momentum cutoff regularization

The one-loop contribution to the effective blocked action regularized by momentum cutoff can be written as

$$\begin{aligned}\tilde{S}_k^{(1)} &= \frac{1}{2} \int_x \int_p' \text{tr} \{ \ln(p^2 + \mu^2 - 2ip \cdot D - D^2 + Y) \} \mathbb{1} \\ &= -\frac{1}{2} \int_x \int_0^\infty \frac{ds}{s} \int_p' \text{tr} \{ e^{-(p^2 + \mu^2)s} e^{(2ip \cdot D + D^2 - Y)s} \} \mathbb{1}.\end{aligned}\quad (4.9)$$

Following the details presented by Nepomechie [24] and Mukku [27], we first employ the Baker-Campbell-Hausdorff formulas

$$e^{A+B} = e^A \left\{ 1 + B + \frac{[B, A]}{2} + \frac{B^2}{2} + \frac{[[B, A] + B^2, A]}{3!} + \frac{[B, A]B}{3!} + \frac{B^3}{3!} + \dots \right\}, \quad (4.10)$$

$$\begin{aligned}e^{A+B+C} &= e^{A+B} \left\{ 1 + C + \frac{[C, A]}{2} + \frac{[C, B]}{2} + \frac{C^2}{2} + \frac{[[C, A], A]}{3!} + \frac{[[C, A], B]}{3!} + \frac{[[C, B], A]}{3!} \right. \\ &\quad \left. + \frac{[[C, B], B]}{3!} + \frac{[C, [C, A]]}{3!} + \frac{[C, [C, B]]}{3!} + \frac{[C, A]C}{2} + \frac{[C, B]C}{2} + \frac{C^3}{3!} + \dots \right\},\end{aligned}\quad (4.11)$$

and expand the heat kernel as

$$\begin{aligned}h(s; x, x) &= \int_p' e^{-(p^2 + \mu^2)s} e^{(2ip \cdot D + D^2 - Y)s} \mathbb{1} \\ &= e^{-(\mu^2 + Y)s} \int_p' e^{-p^2 s} \left\{ 1 + D^2 s - 2p_\mu p_\nu D_\mu D_\nu s^2 + \frac{D^4 s^2}{2} - \frac{[D^2, Y] s^2}{2} \right. \\ &\quad \left. - \frac{2p_\mu p_\nu}{3} \{ [[D^2, D_\mu], D_\nu] + 3D_\nu [D^2, D_\mu] + 3D_\mu D_\nu D^2 - [D_\mu D_\nu, Y] \right. \\ &\quad \left. - [D_\mu, Y] D_\nu - p_\alpha p_\beta D_\mu D_\nu D_\alpha D_\beta \} s^3 + \dots \right\} \mathbb{1},\end{aligned}\quad (4.12)$$

where the contributions with odd powers of  $p$  are neglected since they give vanishing contribution after momentum integrations. Substituting the above into (4.9) and carrying out the  $s$  integration, we obtain

$$\begin{aligned}\tilde{S}_k^{(1)} &= \frac{1}{2} \int_x \int_p' \text{tr} \left\{ \ln(p^2 + \mu^2 + Y) - \frac{1}{p^2 + \mu^2 + Y} D^2 + \frac{2p_\mu p_\nu}{(p^2 + \mu^2 + Y)^2} D_\mu D_\nu \right. \\ &\quad \left. - \frac{1}{2(p^2 + \mu^2 + Y)^2} (D^4 - [D^2, Y]) + \frac{4p_\mu p_\nu}{3(p^2 + \mu^2 + Y)^3} ([[D^2, D_\mu], D_\nu] + 3D_\nu [D^2, D_\mu] \right. \\ &\quad \left. + 3D_\mu D_\nu D^2 - [D_\mu D_\nu, Y] - [D_\mu, Y] D_\nu) - \frac{4p_\mu p_\nu p_\alpha p_\beta}{(p^2 + \mu^2 + Y)^4} D_\mu D_\nu D_\alpha D_\beta + \dots \right\} \mathbb{1},\end{aligned}\quad (4.13)$$

which by rotational  $O(d)$  symmetry can be further simplified to

$$\begin{aligned}\tilde{S}_k^{(1)} &= \frac{1}{2} \int_x \int_p' \text{tr} \left\{ \ln(p^2 + \mu^2 + Y) + \frac{1}{(p^2 + \mu^2 + Y)^2} \left[ \frac{(2-d)p^2 - d(\mu^2 + Y)}{d} D^2 - \frac{1}{2} (D^4 - [D^2, Y]) \right] \right. \\ &\quad \left. + \frac{4p^2}{3d(p^2 + \mu^2 + Y)^3} ([[D^2, D_\mu], D_\mu] + 3D_\mu [D^2, D_\mu] + 3D^4 - [D^2, Y] - [D_\mu, Y] D_\mu) \right. \\ &\quad \left. - \frac{4(p^2)^2}{d(d+2)(p^2 + \mu^2 + Y)^4} [D^4 + (D_\mu D_\nu)^2 + D_\mu D^2 D_\mu] \right\} \mathbb{1},\end{aligned}\quad (4.14)$$

and evaluated with (3.7). Finally, using

$$D_\mu Y = (D_\mu Y) + Y D_\mu, \quad D^2 Y = (D^2 Y) + 2D_\mu Y D_\mu - Y D^2, \quad F_{\mu\nu} = [D_\mu, D_\nu], \quad (4.15)$$

Eq. (4.14) becomes

$$\begin{aligned} \tilde{S}_k^{(1)} = & \frac{1}{2} \int_x \int_p \text{tr} \{ \ln(p^2 + \mu^2 + Y) + a_1 D^2 + a_2 (D^2 Y) + a_3 D_\mu Y D_\mu + a_4 Y D^2 \\ & + a_5 D^4 + a_6 D_\mu D^2 D_\mu + a_7 F_{\mu\nu} F_{\mu\nu} \} \mathbb{1}, \end{aligned} \quad (4.16)$$

where

$$a_1 = \frac{(2-d)p^2 - d(\mu^2 + Y)}{d(p^2 + \mu^2 + Y)^2}, \quad (4.17)$$

$$a_2 = \frac{(3d-8)p^2 + 3d(\mu^2 + Y)}{6d(p^2 + \mu^2 + Y)^3}, \quad (4.18)$$

$$a_3 = -a_4 = \frac{(d-4)p^2 + d(\mu^2 + Y)}{d(p^2 + \mu^2 + Y)^3}, \quad (4.19)$$

$$a_5 = -\frac{(3d-2)(d-4)p^4 + 2p^2(d+2)(3d-8)(\mu^2 + Y) + 3d(d+2)(\mu^2 + Y)^2}{6d(d+2)(p^2 + \mu^2 + Y)^4}, \quad (4.20)$$

$$a_6 = \frac{4p^2[(d-4)p^2 + (d+2)(\mu^2 + Y)]}{3d(d+2)(p^2 + \mu^2 + Y)^4}, \quad (4.21)$$

and

$$a_7 = -\frac{2(p^2)^2}{d(d+2)(p^2 + \mu^2 + Y)^4}. \quad (4.22)$$

The additional noninvariant operators  $D^2$ ,  $D_\mu Y D_\mu$ ,  $Y D^2$ ,  $D^4$ , and  $D_\mu D^2 D_\mu$  generated by the momentum cutoff regulator in (4.16) can be readily seen from a simple comparison with (4.6) which consists of gauge-invariant quantities only. Nevertheless, taking the limit  $\Lambda \rightarrow \infty$  and  $k \rightarrow 0$ , the coefficients associated with these gauge-noninvariant contributions are identically zero, i.e.,

$$\int_p a_i = 0, \quad (4.23)$$

for  $i = 1, 3, 4, 5$ , and  $6$ . The presence of noninvariant operators for theories regularized by momentum cutoff makes it difficult to extend the regularization to gauge theories. One must therefore resort to other methods which contain the cutoff scales and yet preserve the symmetries of the original Lagrangian. A promising candidate which encompasses both features is the operator cutoff regularization which we next turn to.

### B. Operator cutoff regularization

We now apply the alternative operator cutoff formalism to regularize the divergence found in (4.2). Following the methodology outlined in the previous section, the heat kernel (4.12) may be modified as

$$\begin{aligned} h(s; x, x) \rightarrow h(s; x, x)|_{\text{OC}} = & e^{-(\mu^2 + Y)s} \int_p e^{-p^2 s} \left\{ \rho_k^{(d,0)}(s, \Lambda) \left( 1 + D^2 s + \frac{D^4}{2} s^2 - \frac{[D^2, Y]}{2} s^2 \right) \right. \\ & \left. - \frac{2p^2}{d} \rho_k^{(d,1)}(s, \Lambda) \left[ D^2 s^2 + \frac{1}{3} ([D^2, D_\mu], D_\mu) + 3D_\mu [D^2, D_\mu] + 3D^4 - [D^2, Y] - [D_\mu, Y] D_\mu \right] s^3 \right\} \\ & + \frac{2(p^2)^2}{3d(d+2)} \rho_k^{(d,2)}(s, \Lambda) [D^4 + (D_\mu D_\nu)^2 + D_\mu D^2 D_\mu] s^4 \left. \right\} \mathbb{1} \\ = & \frac{e^{-(\mu^2 + Y)s}}{(4\pi s)^{d/2}} \left\{ \rho_k^{(d,0)}(s, \Lambda) + b_1 D^2 + b_2 (D^2 Y) + b_3 D_\mu Y D_\mu + b_4 Y D^2 + b_5 D^4 \right. \\ & \left. + b_6 D_\mu D^2 D_\mu + b_7 F_{\mu\nu} F_{\mu\nu} + O(s^3) \right\} \mathbb{1}, \end{aligned} \quad (4.24)$$

where, to have agreement with the cutoff results, one requires

$$b_1 = [\rho_k^{(d,0)}(s, \Lambda) - \rho_k^{(d,1)}(s, \Lambda)]s = \frac{2s^{1+d/2}}{S_d \Gamma(d/2)} \int_z' \left(1 - \frac{2}{d} z^2 s\right) e^{-z^2 s}, \quad (4.25)$$

$$b_2 = -\frac{1}{8} [3\rho_k^{(d,0)}(s, \Lambda) - 2\rho_k^{(d,1)}(s, \Lambda)]s^2 = -\frac{s^{2+d/2}}{S_d \Gamma(d/2)} \int_z' \left(1 - \frac{4}{3d} z^2 s\right) e^{-z^2 s}, \quad (4.26)$$

$$b_3 = -b_4 = -[\rho_k^{(d,0)}(s, \Lambda) - \rho_k^{(d,1)}(s, \Lambda)]s^2 = -\frac{2s^{2+d/2}}{S_d \Gamma(d/2)} \int_z' \left(1 - \frac{2}{d} z^2 s\right) e^{-z^2 s}, \quad (4.27)$$

$$\begin{aligned} b_5 &= \frac{1}{8} [3\rho_k^{(d,0)}(s, \Lambda) - 4\rho_k^{(d,1)}(s, \Lambda) + \rho_k^{(d,2)}(s, \Lambda)]s^2 \\ &= \frac{S^{2+d/2}}{3S_d \Gamma(d/2)} \int_z' \left(3 - \frac{8}{d} z^2 s + \frac{4}{d(d+2)} (z^2 s)^2\right) e^{-z^2 s}, \end{aligned} \quad (4.28)$$

$$b_6 = -\frac{1}{3} [\rho_k^{(d,1)}(s, \Lambda) - \rho_k^{(d,2)}(s, \Lambda)]s^2 = -\frac{4s^{2+d/2}}{3dS_d \Gamma(d/2)} \int_z' \left(1 - \frac{2}{d+2} z^2 s\right) z^2 s e^{-z^2 s}, \quad (4.29)$$

and

$$b_7 = \frac{1}{12} \rho_k^{(d,2)}(s, \Lambda) s^2 = \frac{s^{2+d/2}}{6S_d \Gamma(d/2)} \int_z' (z^2 s)^2 e^{-z^2 s}. \quad (4.30)$$

Notice that there exists a one-to-one correspondence between the coefficients  $b_i$ 's in the proper-time formulation and the  $a_i$ 's in the momentum cutoff approach. In arriving at (4.24), we again have inserted the regulating smearing functions  $\rho_k^{(d,m)}(s, \Lambda)$  whose general form has been derived in (3.16), and performed the momentum integrations using

$$\int_p p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{2m}} e^{-p^2 s} = \frac{T_{\mu_1 \mu_2 \cdots \mu_{2m}}^m \Gamma(d/2)}{2^m \Gamma(m + d/2)} \int_p (p^2)^m e^{-p^2 s} = \frac{T_{\mu_1 \mu_2 \cdots \mu_{2m}}^m}{(4\pi s)^{d/2} (2s)^m}. \quad (4.31)$$

However, our aim here is not to reproduce the sharp cutoff results, but to formulate a regularization scheme which leads to a gauge-invariant blocked action  $\tilde{S}_k$ . Gauge symmetry remains unbroken only if the contributions from  $b_i$  for  $i = 1, 3, 4, 5,$  and  $6$  vanish. This requirement is easily attained if instead of using  $\rho_k^{(d,m)}(s, \Lambda)$ , only  $\rho_k^{(d,0)}(s, \Lambda) = \rho_k^{(d)}(s, \Lambda)$  is utilized in the expansion of the heat kernel in (4.24). The resulting gauge-invariant "blocked" heat kernel then takes the form

$$h_k(s; x, x) = \frac{e^{-(\mu^2 + Y)s}}{(4\pi)^{d/2}} \rho_k^{(d)}(s, \Lambda) \left\{ 1 + \frac{1}{12} [F_{\mu\nu} F_{\mu\nu} - 2(D^2 Y)] s^2 + O(s^3) \right\}, \quad (4.32)$$

which, for  $\rho_k^{(d)}(s, \Lambda) \rightarrow 1$ , agrees with that obtained in [24,27]. The one-loop blocked action becomes

$$\begin{aligned} \tilde{S}_k^{(1)} &= -\frac{1}{2(4\pi)^{d/2}} \int_x \int_0^\infty \frac{ds}{s^{1+d/2}} \rho_k^{(d)}(s, \Lambda) e^{-(\mu^2 + Y)s} \left\{ 1 + \frac{1}{12} [F_{\mu\nu} F_{\mu\nu} - 2(D^2 Y)] s^2 \right\} \\ &= -\frac{1}{2} \int_x \int_0^\infty \frac{ds}{s} e^{-(\mu^2 + Y)s} \left\{ 1 + \frac{1}{12} [F_{\mu\nu} F_{\mu\nu} - 2(D^2 Y)] s^2 \right\} \int_z' e^{-z^2 s}. \end{aligned} \quad (4.33)$$

The RG flow equation can subsequently be obtained by varying (4.33) with respect to the IR scale  $k$ .

What we have seen here is that by using just one smearing function  $\rho_k^{(d)}(s, \Lambda)$ , an effective blocked action having a momentum cutoff regularized scalar sector as well as the symmetry-preserving contributions from the gauge fields is obtained. The invariant prescription adopted here departs from the usual momentum cutoff regularization in the sense that the gauge-noninvariant contri-

butions are effectively subtracted off. Any dependence on the UV cutoff  $\Lambda$  present in (4.33) can subsequently be absorbed with the usual procedure of renormalization, i.e., redefinition of parameters. In particular, the familiar  $\ln \Lambda^2$  divergence coming from the  $F_{\mu\nu} F_{\mu\nu}$  term for  $d = 4$  can be dialed away via the coupling constant renormalization. From the modification induced by  $\rho_k^{(d)}(s, \Lambda)$  on the fluctuation operator  $\mathcal{H}$ :

$$\mathcal{H} \rightarrow \mathcal{H}|_{\text{OC}} = \mathcal{H} \left( 1 + \frac{\mathcal{H}}{\Lambda^2} \right)^2 = \mathcal{H} + \frac{2\mathcal{H}^2}{\Lambda^2} + \dots, \quad (4.34)$$

one readily notices the similarity between the effect brought about by the operator cutoff regularization and the gauge-invariant method of higher covariant derivatives [28].

We conclude this section with the remark that the task of preserving gauge symmetry for any given regularization often amounts to finding a proper way of transferring the singularity accompanied in the trace operation to some parameters which are independent of gauge transformation. For example, the spirit of dimensional regularization is to displace  $d$ , the space dimensionality in which the system is defined, by a small positive quantity  $\epsilon$ . Since gauge symmetry is not influenced by the value of  $d$ , gauge invariance is readily fulfilled by transforming the divergent structures of the theory into a pole term  $\sim \epsilon^{-1}$ . In a similar fashion, the operator cutoff approach transfers the divergences to the proper-time parameter  $s$ . Embedding the cutoff scales tactically in the regulating smearing function  $\rho_k^{(d)}(s, \Lambda)$  leads to an effective blocked action which is manifestly gauge invariant. That the momentum cutoff regularization fails to be an invariant prescription is seen here from its requirement of having to employ more than one smearing function when expressed in the proper-time representation, and hence it does not fall into the generalized class of invariant proper-time regularization.

## V. SUMMARY AND DISCUSSIONS

In this paper we have followed the formalism developed in [12] and illustrated how a theory regularized with momentum cutoff can be represented by the proper-time parametrization using a set of regulating smearing functions  $\rho_k^{(d,m)}(s, \Lambda)$ . These smearing functions incorporate cutoff scales and reproduce the essential features of blocking transformation, albeit in the less obvious proper-time coordinate. The modifications induced by  $\rho_k^{(d)}(s, \Lambda)$  in our operator cutoff regularization are seen to be reminiscent to that of the Pauli-Villars, or the generalized method of higher (covariant) derivatives.

Equivalence between the two regulators was demonstrated for the one-loop effective blocked action expanded

in a number of (covariant) derivatives. We also computed the one-loop corrections to the two- and four-point functions for the  $\lambda\Phi^4$  theory and showed that the cutoff expressions can indeed be reproduced order by order in terms of coupling constant  $\lambda$  using the operator cutoff formalism.

The most important feature of the operator cutoff regularization is that it is a gauge-invariant prescription even when momentum cutoff scales are present. The symmetry-preserving nature of the formalism is attributed to its capacity of transferring the singularity that arises from taking the spacetime trace to the proper-time variable  $s$  which is independent of gauge transformation. Gauge invariance is ensured by retaining the  $s$  integration to be performed last.

Instead of employing the entire set of smearing functions  $\rho_k^{(d,m)}(s, \Lambda)$ , the invariant prescription we proposed here is to utilize only  $\rho_k^{(d)}(s, \Lambda) = \rho_k^{(d,0)}(s, \Lambda)$ . While the momentum cutoff structure for the leading order blocked potential is automatically reproduced with  $\rho_k^{(d)}(s, \Lambda)$  alone, discrepancies between the two formalisms inevitably arise in the high-order (covariant) derivative terms. The differences, as noted in Sec. IV, are precisely the gauge-noninvariant contributions that are generated in the momentum cutoff prescription. The virtue of  $\rho_k^{(d)}(s, \Lambda)$  is that it is chosen in a such a way that the gauge-noninvariant sector in the effective theory is completely relegated. Therefore, our invariant regularization resembles a sharp cutoff for the blocked potential and a smooth regulator for the derivative terms. The requirement of using the complete set of  $\rho_k^{(d,m)}(s, \Lambda)$  in order to reproduce the cutoff results term by term in the derivative expansion provides another indication that momentum cutoff does not belong to the generalized class of proper-time regularization and hence cannot be a gauge-invariant prescription.

With momentum scales implemented in a symmetry-preserving manner, operator cutoff regularization offers a promising method for exploring the RG flow of gauge theories in the spirit of Wilson-Kadanoff blocking transformation. The evolution of the theory will now be characterized by the variation of the blocked action in response to the change in the proper-time smearing function  $\rho_k^{(d)}(s, \Lambda)$ . For example, the full nonlinear RG flow equation for the scalar theory explored in Sec. II can be written as

$$k \frac{\partial U_k(\Phi)}{\partial k} = -\frac{1}{2(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{1+d/2}} \left( k \frac{\partial \rho_k^{(d)}(s, \Lambda)}{\partial k} \right) (e^{-U_k'(\Phi)s} - e^{-U_k''(0)s}). \quad (5.1)$$

The precaution to be taken when employing operator cutoff regularization for Yang-Mills theory is that Becchi-Rouet-Stora (BRS) symmetry is generally violated unless a covariant background field gauge is chosen [29]. The general framework of the covariant technique for computing the one-loop effective action can be found in [30].

In a preliminary work [31], we adopted the spirit of operator cutoff outlined above and demonstrated how the expected  $\beta$  function and the corresponding RG flow of the Yang-Mills theories can be obtained with the Wilson-Kadanoff blocking approach. A more thorough study for the non-Abelian theories is under way.



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