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Alternative derivation of the Hu-Paz-Zhang master equation of quantum Brownian motion

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Hu, Paz, and Zhang [B.L. Hu, J.P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992)] have derived an exact master equation for quantum Brownian motion in a general environment via path integral techniques. Their master equation provides a very useful tool to study the decoherence of a quantum system due to the interaction with its environment. In this paper we give an alternative and elementary derivation of the Hu-Paz-Zhang master equation, which involves tracing the evolution equation for the Wigner function. We also discuss the master equation in some special cases.

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I. INTRODUCTION

Quantum Brownian motion (QBM) models provide a paradigm of open quantum systems that has been very useful in quantum measurement theory [1], quantum optics [2], and decoherence [3–5]. One of the advantages of the QBM models is that they are reasonably simple, yet sufficiently complex to manifest many important features of realistic physical processes.

Central to the study of QBM is the master equation for the reduced density operator of the Brownian particle, derived by tracing out the environment in the evolution equation for the combined system plus environment. A variety of such derivation have been given [6–9]. The most general is that of Hu, Paz, and Zhang [10,11], who used path integral techniques and in particular, the Feynman-Vernon influence functional.

The purpose of this paper is to provide an alternative and elementary derivation of the Hu-Paz-Zhang master equation for QBM, by tracing the evolution equation for the Wigner function of the whole system.

II. MASTER EQUATION FOR QUANTUM BROWNIAN MOTION

The system we considered is a harmonic oscillator with mass M and bare frequency Ω, in interaction with a thermal bath consisting of a set of harmonic oscillators with mass m_n and natural frequency ω_n. The Hamiltonian of the system plus environment is given by

$$H = \frac{p^2}{2M} + \frac{1}{2}M\Omega^2 q^2 + \sum_n \left(\frac{p_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2 q_n^2\right) + q\sum_n C_n q_n, \tag{1}$$

where q, p and q_n, p_n are the coordinates and momenta of the Brownian particle and oscillators, respectively, and C_n are coupling constants.

The state of the combined system (1) is most completely described by a density matrix $\rho(q, q_i; q', q'_i, t)$ where q_i denotes $(q_1, \ldots, q_N)$, and ρ evolves according to

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho]. \tag{2}$$

The state of the Brownian particle is described by the reduced density matrix, defined by tracing over the environment:

$$\rho_r(q, q', t) = \int \prod_n [dq_n dq'_n \delta(q_n - q'_n)]\rho(q, q_i; q', q'_i, t). \tag{3}$$

The equation of time evolution for the reduced density matrix is called the master equation. For a general environment, Hu, Paz, and Zhang [10] derived the following master equation by using path integral techniques:

$$i\hbar\frac{\partial\rho_r}{\partial t} = -\frac{\hbar^2}{2M}\left(\frac{\partial^2\rho_r}{\partial q^2} - \frac{\partial^2\rho_r}{\partial q'^2}\right) + \tfrac{1}{2}M\Omega^2(q^2 - q'^2)\rho_r + \tfrac{1}{2}M\delta\Omega^2(t)(q^2 - q'^2)\rho_r$$
$$-i\hbar\Gamma(t)(q - q')\left(\frac{\partial\rho_r}{\partial q} - \frac{\partial\rho_r}{\partial q'}\right) - iM\Gamma(t)h(t)(q - q')^2\rho_r + \hbar\Gamma(t)f(t)(q - q')\left(\frac{\partial\rho_r}{\partial q} + \frac{\partial\rho_r}{\partial q'}\right). \tag{4}$$

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The explicit form of the coefficients of the above equation will be given later on. The coefficient $\delta\Omega^2(t)$ is the frequency shift term, the coefficient $\Gamma(t)$ is the "quantum dissipative" term, and the coefficients $\Gamma(t)h(t), \Gamma(t)f(t)$ are "quantum diffusion" terms. Generally, these coefficients are time dependent and of quite complicated behavior.

We find it convenient to use the Wigner function of the reduced density matrix:

$$\tilde{W}(q,p,t) = \frac{1}{2\pi}\int du\, e^{iup/\hbar}\rho_r\left(q-\frac{u}{2}, q+\frac{u}{2}, t\right). \quad (5)$$

Taking the Wigner transform of (4) we obtain[1]

$$\frac{\partial\tilde{W}}{\partial t} = -\frac{1}{M}p\frac{\partial\tilde{W}}{\partial q} + M[\Omega^2+\delta\Omega^2(t)]q\frac{\partial\tilde{W}}{\partial p} + 2\Gamma(t)\frac{\partial(p\tilde{W})}{\partial p} + \hbar M\Gamma(t)h(t)\frac{\partial^2\tilde{W}}{\partial p^2} + \hbar\Gamma(t)f(t)\frac{\partial^2\tilde{W}}{\partial q\partial p}. \quad (6)$$

The inverse transformation of (5) is given by

$$\rho_r(q,q',t) = \int dp\, e^{-ip(q-q')/\hbar}\tilde{W}\left(\frac{q+q'}{2}, p, t\right). \quad (7)$$

Our strategy for deriving the master equation (4) is to derive the Fokker-Planck type equation (6) from the Wigner equation for the total system. The master equation can be obtained from the Wigner equation for the system by using the transformation (7).

We shall make the following two assumptions.

(1) The system and the environment are initially uncorrelated; i.e., the initial Wigner function factors

$$W_0(q,p;q_i,p_i) = W_0^s(q,p)W_0^b(q_i,p_i), \quad (8)$$

where W_0^s and W_0^b are the Wigner functions of the system and the bath, respectively, at $t=0$.

(2) The heat bath is initially in a thermal equilibrium state at temperature $T=(k_B\beta)^{-1}$. This means that the initial Wigner function of bath is of Gaussian form

$$W_0^b = \prod_n W_{n0}^b = \prod_n N_n \exp\left(-\frac{2}{\omega_n\hbar}\tanh(\tfrac{1}{2}\hbar\omega_n\beta)H_n\right), \quad (9)$$

where H_n is the Hamiltonian of the nth oscillator in the bath:

$$H_n = \frac{p_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2 q_n^2. \quad (10)$$

In addition, one can easily see that the initial moments of the bath are

$$\langle q_n(0)\rangle = \langle p_n(0)\rangle = 0, \quad (11)$$

$$\langle q_n(0)q_m(0)\rangle = 0 \quad (\text{if } m\neq n), \quad (12)$$

$$\langle p_n(0)p_m(0)\rangle = 0 \quad (\text{if } m\neq n), \quad (13)$$

$$\langle q_n(0)p_m(0) + p_m(0)q_n(0)\rangle = 0, \quad (14)$$

and

$$\langle q_n^2(0)\rangle = \frac{\hbar}{2m_n\omega_n}\coth(\tfrac{1}{2}\hbar\omega_n\beta), \qquad \langle p_n^2(0)\rangle = \tfrac{1}{2}\hbar m_n\omega_n\coth(\tfrac{1}{2}\hbar\omega_n\beta). \quad (15)$$

For the QBM problem derived by (1) and (2), the Wigner function of the combined system plus environment satisfies

$$\frac{\partial W}{\partial t} = -\frac{p}{M}\frac{\partial W}{\partial q} + M\Omega^2 q\frac{\partial W}{\partial p} + \sum_n\left(-\frac{p_n}{m_n}\frac{\partial W}{\partial q_n} + m_n\omega_n^2 q_n\frac{\partial W}{\partial p_n}\right) + \sum_n C_n\left(q_n\frac{\partial W}{\partial p} + q\frac{\partial W}{\partial p_n}\right). \quad (16)$$

By integrating over the bath variables on the both sides of the above equation one obtains

$$\frac{\partial\tilde{W}}{\partial t} = -\frac{p}{M}\frac{\partial\tilde{W}}{\partial q} + M\Omega^2 q\frac{\partial\tilde{W}}{\partial p} + \sum_n C_n\int\prod_i dq_i dp_i q_n\frac{\partial W}{\partial p}, \quad (17)$$

where $\tilde{W}(q,p)$ is the reduced Wigner function:

$$\tilde{W}(q,p) = \int_{-\infty}^{+\infty}\prod_i dq_i dp_i W(q,p;q_i,p_i). \quad (18)$$

[This definition is equivalent to Eqs. (3) and (5).] The first two terms on the right-hand side of Eq. (17) give rise to the standard evolution equation of the system. The last term contains all the information about the behavior of the system in the presence of interaction with environment.

In what follows we shall demonstrate that the quantity

$$G(q,p) = \sum_n C_n\int\prod_i dq_i dp_i q_n W \quad (19)$$

appearing (differentiated with respect to p) in (17) can be expressed in terms of $\tilde{W}$ and its derivatives. To this end we first perform Fourier transform of $G(q,p)$:

[1] We believe that Eq. (2.48) in Ref. [10] contains some incorrect numerical factors.

$$G(k,k') = \int dq\, dp\, e^{ikq+ik'p} G(q,p) = \sum_n C_n \int dq\, dp \prod_i dq_i dp_i q_n e^{ikq+ik'p} W(q,p;q_i,p_i). \tag{20}$$

It is well known that $q(t), p(t)$ and $q_n(t), p_n(t)$ are related to the classical evolution of their initial values $q(0), p(0)$ and $q_n(0), p_n(0)$ through a canonical transformation:

$$\mathbf{z}(t) = U(t)\mathbf{z}(0), \tag{21}$$

where

$$\mathbf{z}(t) = (q(t), q_1(t), \ldots, q_N(t); p(t), p_1(t), \ldots, p_N(t)).$$

Since the Hamiltonian (1) is quadratic, Eq. (16) has the same form as the classical Liouville equation, so the solution of Eq. (16) is of the form

$$W_t(\mathbf{z}) = W_0(U^{-1}(t)\mathbf{z}). \tag{22}$$

Changing the integration variables into their initial values by this canonical transformation, we obtain

$$\begin{aligned} G(k,k') = &\int dq(0)dp(0) \prod_i dq_i(0)dp_i(0) \\ &\times \left[fq(0) + gp(0) + \sum_n (f_n q_n(0) + g_n p_n(0)) \right] \exp\left[ik \left((\alpha q(0) + \beta p(0)) + \sum_n (a_n q_n(0) + b_n p_n(0)) \right) \right] \\ &\times \exp\left[ik'M \left(\dot\alpha q(0) + \dot\beta p(0) + \sum_n (\dot a_n q_n(0) + \dot b_n p_n(0)) \right) \right] W_0^s(q(0),p(0)) W_0^b(q_i(0),p_i(0)). \end{aligned} \tag{23}$$

Here the coefficients $f, g, f_n, g_n, \alpha, \beta, a_n, b_n$ are time dependent. Their explicit values are not required. Similarly, the Fourier transform of the reduced Wigner function is

$$\begin{aligned} \tilde W(k,k') &= \int dq\, dp\, e^{ikq+ik'p} \tilde W(q,p) \\ &= \int dq(0)dp(0) \prod_i dq_i(0)dp_i(0) \exp\left[ik \left((\alpha q(0) + \beta p(0)) + \sum_n (a_n q_n(0) + b_n p_n(0)) \right) \right] \\ &\quad \times \exp\left[ik'M \left(\dot\alpha q(0) + \dot\beta p(0) + \sum_n (\dot a_n q_n(0) + \dot b_n p_n(0)) \right) \right] W_0^s(q(0),p(0)) W_0^b(q_i(0),p_i(0)). \end{aligned} \tag{24}$$

Now compare $G(k,k')$ and $\tilde W(k,k')$. They differ by the terms linear in $q(0), p(0), q_n(0), p_n(0)$ in the preexponential factor in $G(k,k')$. Consider the factors $f_n q_n(0)$ and $g_n p_n(0)$ in $G(k,k')$. Since they multiply $W_0^b(q_i(0), p_i(0))$, and since $W_0^b(q_i(0), p_i(0))$ is Gaussian in $q_n(0), p_n(0)$, the terms $f_n q_n(0) W_0^b$ and $g_n p_n(0) W_0^b$ may be replaced by terms of the form $\partial W_0^b/\partial q_n(0), \partial W_0^b/\partial p_n(0)$ up to time-dependent factors. An integration by parts then may be performed, and these factors are then effectively replaced by multiplicative factors of k, k'.

Similarly, the factors $fq(0), gp(0)$ in the prefactor in $G(k,k')$ may be replaced by $\partial/\partial k, \partial/\partial k'$ (plus some more factors of k and k'). Hence, it is readily seen that $G(k,k')$ is a linear combination of terms of the form $k, k', \partial/\partial k, \partial/\partial k'$ operating on $\tilde W(k,k')$, with time-dependent coefficients.

Inverting the Fourier transform, it follows that

$$G = A(t) q \tilde W + B(t) p \tilde W + C(t) \frac{\partial \tilde W}{\partial q} + D(t) \frac{\partial \tilde W}{\partial p}, \tag{25}$$

for some coefficients $A(t), B(t), C(t), D(t)$ to be determined. This result immediately leads to the general form Wigner equation

$$\begin{aligned} \frac{\partial \tilde W}{\partial t} = &-\frac{p}{M}\frac{\partial \tilde W}{\partial q} + M\Omega^2 q \frac{\partial \tilde W}{\partial p} + A(t) q \frac{\partial \tilde W}{\partial p} \\ &+ B(t) \frac{\partial (p\tilde W)}{\partial p} + C(t) \frac{\partial^2 \tilde W}{\partial p \partial q} + D(t) \frac{\partial^2 \tilde W}{\partial p^2}. \end{aligned} \tag{26}$$

III. DETERMINATION OF THE COEFFICIENTS (GENERAL CASE)

Having found the functional form of the Wigner equation (26) of the Brownian particle, the next step is to determine the coefficients in the equation. Undoubtedly, there is more than one way to do this. Here we shall choose a way which is both mathematically simple and physically heuristic. Toward this direction let us consider the time evolution of the expectation values of the system variables: q, p, q^2, p^2, and $\frac{1}{2}(pq+qp)$.

By using Eq. (16) we have

$$\frac{d}{dt}\langle q\rangle = \frac{1}{M}\langle p\rangle, \tag{27}$$

$$\frac{d}{dt}\langle p\rangle = -M\Omega^2\langle q\rangle - \sum_n C_n\langle q_n\rangle, \tag{28}$$

$$\frac{d}{dt}\langle q^2\rangle = \frac{1}{M}\langle pq+qp\rangle, \tag{29}$$

$$\frac{d}{dt}\langle p^2\rangle = -M\Omega^2\langle pq+qp\rangle - 2\sum_n C_n\langle pq_n\rangle, \tag{30}$$

$$\frac{d}{dt}\langle pq+qp\rangle = \frac{2}{M}\langle p^2\rangle - 2M\Omega^2\langle q^2\rangle - 2\sum_n C_n\langle qq_n\rangle. \tag{31}$$

Similarly, using Eq. (26) yields

$$\frac{d}{dt}\langle q\rangle = \frac{1}{M}\langle p\rangle, \tag{32}$$

$$\frac{d}{dt}\langle p\rangle = -(M\Omega^2+A)\langle q\rangle - B\langle p\rangle, \tag{33}$$

$$\frac{d}{dt}\langle q^2\rangle = \frac{1}{M}\langle pq+qp\rangle, \tag{34}$$

$$\frac{d}{dt}\langle p^2\rangle = -(M\Omega^2+A)\langle pq+qp\rangle - 2B\langle p^2\rangle + 2D, \tag{35}$$

$$\frac{d}{dt}\langle pq+qp\rangle = \frac{2}{M}\langle p^2\rangle - 2(M\Omega^2+A)\langle q^2\rangle - B\langle pq+qp\rangle + 2C. \tag{36}$$

Since the evolution equations of the expectation values are confined to the system variables, the above two sets of equations must be identical.

Now by comparing (28) with (33) we see that

$$\sum_n C_n\langle q_n\rangle = A\langle q\rangle + B\langle p\rangle. \tag{37}$$

Similarly, by comparing (31) with (36), (30) with (35), respectively, we get

$$\sum_n C_n\langle qq_n\rangle = A\langle q^2\rangle + \frac{B}{2}\langle qp+pq\rangle - C, \tag{38}$$

$$\sum_n C_n\langle pq_n\rangle = \frac{A}{2}\langle pq+qp\rangle + B\langle p^2\rangle - D. \tag{39}$$

The coefficients A, B, C, D may now be determined from (37)–(39) by regarding the expectation values $\langle q\rangle$, $\langle q_n q\rangle$, etc., as expectation values of Heisenberg picture operators, and by solving the operator equation of motion. For simplicity we still use ordinary notation to represent an operator without adding a hat on it.

The solution to the equation of motion may be written

$$q_n(t) = \alpha_n q(t) + \beta_n p(t) + \sum_m (a_{nm}q_m(0) + b_{nm}p_m(0)), \tag{40}$$

$$q(t) = \alpha q(0) + \beta p(0) + \sum_n (a_n q_n(0) + b_n p_n(0)), \tag{41}$$

for some time-dependent coefficients $\alpha_n, \beta_n, a_{nm}, b_{nm}, \alpha, \beta, a_n, b_n$. Note that $q_n(t)$ has been expressed in terms of the final, not initial values of q, p. By substituting Eq. (40) into (37), keeping (11) in mind, and comparing the two sides of the resulting equation, we have

$$A = \sum_n C_n\alpha_n, \quad B = \sum_n C_n\beta_n. \tag{42}$$

Similarly, substituting (40) and (41) into (38) and (39), respectively, we get

$$C = -\sum_{mn} C_n(a_{nm}a_m\langle q_m^2(0)\rangle + b_{nm}b_m\langle p_m^2(0)\rangle), \tag{43}$$

$$D = -M\sum_{mn} C_n(a_{nm}\dot{a}_m\langle q_m^2(0)\rangle + b_{nm}\dot{b}_m\langle p_m^2(0)\rangle). \tag{44}$$

Here we have made use of $p = M\dot{q}$. The coefficients A, B, C, D are therefore completely determined by solving the equation of motion. We now do this explicitly.

We have

$$\ddot{q}(t) + \Omega^2 q(t) = -\frac{1}{M}\sum_n C_n q_n(t), \tag{45}$$

$$\ddot{q}_n(t) + \omega_n^2 q_n(t) = -\frac{C_n}{m_n}q(t). \tag{46}$$

The solution to Eq. (46) is

$$q_n(t) = q_n(0)\cos(\omega_n t) + \frac{p_n(0)}{m_n}\frac{\sin(\omega_n t)}{\omega_n} - C_n\int_0^t ds\frac{\sin[\omega_n(t-s)]}{\omega_n}\frac{q(s)}{m_n}. \tag{47}$$

Combining (45) and (47) gives

$$\ddot{q}(t) + \Omega^2 q(t) + \frac{2}{M}\int_0^t d\tau\,\eta(t-s)q(s) = \frac{f(t)}{M}, \tag{48}$$

where

$$f(t) = -\sum_n C_n\left(q_n(0)\cos(\omega_n t) + \frac{p_n(0)}{m_n}\frac{\sin(\omega_n t)}{\omega_n}\right). \tag{49}$$

The kernel $\eta(s)$ is defined as

$$\eta(s) = \frac{d}{ds}\gamma(s), \tag{50}$$

where

$$\gamma(s) = \int_0^{+\infty} d\omega \frac{I(\omega)}{\omega}\cos(\omega s). \tag{51}$$

Here $I(\omega)$ is any spectral density of the environment:

$$I(\omega) = \sum_n \delta(\omega - \omega_n)\frac{C_n^2}{2m_n\omega_n}. \tag{52}$$

In order to get expressions (40) and (41) we solve Eq. (48) with the following two different initial conditions:

$$q(s=0) = q(0), \quad \dot{q}(s=0) = \frac{p(0)}{M}, \tag{53}$$

and

$$q(s=t) = q(t), \quad \dot{q}(s=t) = \frac{p(t)}{M}, \tag{54}$$

where t is any given time point. In doing so we consider the elementary functions $u_i(s)$ $(i = 1, 2)$ introduced by Hu, Paz, and Zhang [10] which satisfy the homogeneous integro-differential equation

$$\ddot{\Sigma}(s) + \Omega^2\Sigma(s) + \frac{2}{M}\int_0^s d\lambda\, \eta(s-\lambda)\Sigma(\lambda) = 0 \tag{55}$$

with the boundary conditions

$$u_1(s=0) = 1, \quad u_1(s=t) = 0, \tag{56}$$

and

$$u_2(s=0) = 0, \quad u_2(s=t) = 1. \tag{57}$$

The solution to Eq. (55) with the initial condition (53) is obtained as the linear combination of u_1, u_2:

$$w(s) = \left(u_1(s) - \frac{\dot{u}_1(0)}{\dot{u}_2(0)}u_2(s)\right) q(0) + \frac{u_2(s)}{\dot{u}_2(0)}\frac{p(0)}{M}. \tag{58}$$

The solution to Eq. (48) with the homogeneous initial conditions can be formally written as

$$\tilde{w}(s) = \frac{1}{M}\int_0^{+\infty} d\tau\, G_1(s,\tau)f(\tau), \tag{59}$$

where Green function $G_1(s,\tau)$ satisfies

$$\frac{d^2}{ds^2}G_1(s,\tau) + \Omega^2 G_1(s,\tau) + \frac{2}{M}\int_0^s d\lambda\, \eta(s-\lambda)G_1(\lambda,\tau) = \delta(s-\tau) \tag{60}$$

with

$$G_1(s=0,\tau) = 0, \quad \frac{d}{ds}G_1(s=0,\tau) = 0. \tag{61}$$

Then the solution to Eq. (48) with initial conditions (53) reads

$$q(s) = w(s) + \tilde{w}(s), \tag{62}$$

explicitly,

$$q(s) = \left(u_1(s) - \frac{\dot{u}_1(0)}{\dot{u}_2(0)}u_2(s)\right) q(0) + \frac{u_2(s)}{\dot{u}_2(0)}\frac{p(0)}{M} - \sum_n \frac{C_n}{M}\int_0^{+\infty} d\tau\, G_1(s,\tau)\cos(\omega_n\tau)q_n(0) - \sum_n \frac{C_n}{M}\int_0^{+\infty} d\tau\, G_1(s,\tau)\frac{\sin(\omega_n\tau)}{\omega_n}\frac{p_n(0)}{m_n}. \tag{63}$$

It can be shown that the solution to the homogeneous equation (55) with the initial conditions (54) is

$$u(s) = \left(u_s(s) - \frac{\dot{u}_2(t)}{\dot{u}_1(t)}u_1(s)\right) q(t) + \frac{u_1(s)}{\dot{u}_1(t)}\frac{p(t)}{M} \tag{64}$$

and

$$\tilde{u}(s) = \frac{1}{M}\int_0^t d\tau\, G_2(s,\tau)f(\tau) \tag{65}$$

is the solution to the inhomogeneous equation (48) with the homogeneous initial conditions

$$\tilde{u}(t) = 0, \quad \dot{\tilde{u}}(t) = 0. \tag{66}$$

The equation for Green function $G_2(s,\tau)$ is analogous. Hence, we get the solution to Eq. (48) with the initial conditions (54):

$$
\begin{aligned}
q(s) &= u(s) + \tilde{u}(s) \\
&= \left(u_2(s) - \frac{\dot{u}_2(t)}{\dot{u}_1(t)} u_1(s) \right) q(t) + \frac{u_1(s)}{\dot{u}_1(t)} \frac{p(t)}{M} - \sum_n \frac{C_n}{M} \int_0^t d\tau\, G_2(s,\tau) \cos(\omega_n \tau) q_n(0) \\
&\quad - \sum_n \frac{C_n}{M} \int_0^t d\tau\, G_2(s,\tau) \frac{\sin(\omega_n \tau)}{\omega_n} \frac{p_n(0)}{m_n}.
\end{aligned} \tag{67}
$$

Substituting (67) into (47) one obtains

$$
\begin{aligned}
q_n(t) &= -\frac{C_n}{m_n \omega_n} \int_0^t ds \sin[\omega_n(t-s)] \left(u_2(s) - \frac{\dot{u}_2(t)}{\dot{u}_1(t)} u_1(s) \right) q(t) \\
&\quad - \frac{C_n}{m_n \omega_n} \int_0^t ds \sin[\omega_n(t-s)] \frac{u_1(s)}{\dot{u}_1(t)} \frac{p(t)}{M} + q_n(0) \cos(\omega_n t) + \frac{p_n(0)}{m_n} \frac{\sin(\omega_n t)}{\omega_n} \\
&\quad + \frac{1}{M} \sum_m \frac{C_n C_m}{m_n \omega_n} \int_0^t ds \int_0^t d\tau \sin[\omega_n(t-s)] G_2(s,\tau) \cos(\omega_m \tau) q_m(0) \\
&\quad + \frac{1}{M} \sum_m \frac{C_n C_m}{m_n \omega_n} \int_0^t ds \int_0^t d\tau \sin[\omega_n(t-s)] G_2(s,\tau) \frac{\sin(\omega_m \tau)}{m_m \omega_m} p_m(0).
\end{aligned} \tag{68}
$$

By using (42) we immediately arrive at

$$
A(t) = -\sum_n \frac{C_n^2}{m_n \omega_n} \int_0^t ds \sin[\omega_n(t-s)] \left(u_2(s) - \frac{\dot{u}_2(t)}{\dot{u}_1(t)} u_1(s) \right), \tag{69}
$$

$$
B(t) = -\frac{1}{M} \sum_n \frac{C_n^2}{m_n \omega_n} \int_0^t ds \sin[\omega_n(t-s)] \frac{u_1(s)}{\dot{u}_1(t)}. \tag{70}
$$

Furthermore, A, B can be written as

$$
A(t) = 2 \int_0^t ds\, \eta(t-s) u_2(s) - 2 \frac{\dot{u}_2(t)}{\dot{u}_1(t)} \int_0^t ds\, \eta(t-s) u_1(s), \tag{71}
$$

$$
B(t) = \frac{2}{M \dot{u}_1(t)} \int_0^t ds\, \eta(t-s) u_1(s). \tag{72}
$$

From (63), the momentum of the Brownian particle is then

$$
\begin{aligned}
p(t) &= M \dot{q}(t) \\
&= \left(\dot{u}_1(t) - \frac{\dot{u}_1(0)}{\dot{u}_2(0)} \dot{u}_2(t) \right) M q(0) + \frac{\dot{u}_2(t)}{\dot{u}_2(0)} p(0) - \sum_n \int_0^{+\infty} d\tau\, G_1'(t,\tau) \cos(\omega_n \tau) q_n(0) \\
&\quad - \sum_n \int_0^{+\infty} d\tau\, G_1'(t,\tau) \frac{\sin(\omega_n \tau)}{\omega_n} \frac{p_n(0)}{m_n}.
\end{aligned} \tag{73}
$$

Here "prime" stands for derivative with respect to the first variable of $G_1(s,\tau)$. With these results [see Eqs. (43) and (44)], it can be easily shown that

$$
C(t) = \frac{\hbar}{M} \int_0^{+\infty} d\lambda\, G_1(t,\lambda) \nu(t-\lambda) + \frac{2\hbar}{M^2} \int_0^t ds \int_0^t d\tau \int_0^{+\infty} d\lambda\, \eta(t-s) G_1(t,\lambda) G_2(s,\tau) \nu(\tau-\lambda) \tag{74}
$$

and

$$
D(t) = \hbar \int_0^{+\infty} d\lambda\, G_1'(t,\lambda) \nu(t-\lambda) + \frac{2\hbar}{M} \int_0^t ds \int_0^t d\tau \int_0^{+\infty} d\lambda\, \eta(t-s) G_1'(t,\lambda) G_2(s,\tau) \nu(\tau-\lambda), \tag{75}
$$

where $\nu(s)$ is defined as

$$
\nu(s) = \int_0^{+\infty} d\omega\, I(\omega) \coth(\tfrac{1}{2}\hbar\omega\beta) \cos(\omega s). \tag{76}
$$

It is seen that the coefficients $A(t), B(t), C(t), D(t)$ are dependent only on the kernels $\eta(s)$ and $\nu(s)$ and the initial state of the bath, not dependent on the initial state of the system. Once the spectral density of the environment is given, in principle, the elementary functions u_i ($i = 1,2$) and Green functions G_i ($i = 1,2$) can be solved from Eqs. (55), (60), etc. Then the coefficients of master equation can be determined.

IV. PARTICULAR CASES

In this section we will consider some special cases. Let us at first treat a special case in which we assume that the interaction between the system and environment is weak, so the C_n are small. In this case, the coefficients are of simple forms, and the determination of these coefficients is very simple and straightforward. We shall work out these coefficients directly using the method in the last section, rather than the general formulas.

The solution to Eq. (46) may be written as

$$q_n(t) = q_n(0)\cos(\omega_n t) + \frac{p_n(0)}{m_n}\frac{\sin(\omega_n t)}{\omega_n} - \frac{C_n}{m_n}\int_0^t dt'\frac{\sin[\omega_n(t-t')]}{\omega_n}\cos[\Omega(t'-t)]q(t) - \frac{C_n}{m_n}\int_0^t dt'\frac{\sin[\omega_n(t-t')]}{\omega_n}\frac{\sin[\Omega(t'-t)]}{\Omega}\frac{p(t)}{M} + O(C_n^2). \tag{77}$$

Using Eq. (37) and ignoring terms with higher than the second order of C_n we get

$$\sum_n C_n\langle q_n(t)\rangle = \left(\sum_n -\frac{C_n^2}{m_n}\int_0^t dt'\frac{\sin[\omega_n(t-t')]}{\omega_n}\cos[\Omega(t'-t)]\right)\langle q(t)\rangle + \left(\frac{1}{M}\sum_n -\frac{C_n^2}{m_n}\int_0^t dt'\frac{\sin[\omega(t-t')]}{\omega_n}\frac{\sin[\Omega(t'-t)]}{\Omega}\right)\langle p(t)\rangle. \tag{78}$$

Then we immediately get

$$A(t) = 2\int_0^t ds\,\eta(s)\cos(\Omega s), \tag{79}$$

$$B(t) = -\frac{2}{M\Omega}\int_0^t ds\,\eta(s)\sin(\Omega s). \tag{80}$$

We next evaluate $\Sigma_n C_n\langle q(t)q_n(t)\rangle$ and $\Sigma_n C_n\langle p(t)q_n(t)\rangle$. After a few manipulations we arrive at the expressions

$$C(t) = -\sum_n C_n\left(\langle q(t)q_n(0)\rangle\cos(\omega_n t) + \langle q(t)p_n(0)\rangle\frac{\sin(\omega_n t)}{m_n\omega_n}\right) \tag{81}$$

and

$$D(t) = -\sum_n C_n\left(\langle p(t)q_n(0)\rangle\cos(\omega_n t) + \langle p(t)p_n(0)\rangle\frac{\sin(\omega_n t)}{m_n\omega_n}\right). \tag{82}$$

To calculate $C(t)$ and $D(t)$ we need to expand $q(t)$ up to the second order of C_n:

$$q(t) = q(0)\cos(\Omega t) + \frac{p(0)}{M}\frac{\sin(\Omega t)}{\Omega} - \sum_n\frac{C_n}{M}\int_0^t ds\frac{\sin[\Omega(t-s)]}{\Omega}\cos(\omega_n s)q_n(0) - \sum_n\frac{C_n}{M}\int_0^t ds\frac{\sin[\Omega(t-s)]}{\Omega}\frac{\sin(\omega_n s)}{\omega_n}\frac{p_n(0)}{m_n} + O(C_n^2). \tag{83}$$

The expansion of $p(t)$ is easily obtained from that of $q(t)$:

$$p(t) = M\dot{q}(t). \tag{84}$$

With these results it is easy to compute $C(t)$ and $D(t)$:

$$C(t) = \frac{\hbar}{M\Omega}\int_0^t ds\,\nu(s)\sin(\Omega s), \tag{85}$$

$$D(t) = \hbar\int_0^t ds\,\nu(s)\cos(\Omega s). \tag{86}$$

This simple example exhibits the time dependency of the coefficients of the master equation in a general environment. Equations (79), (80), (85), and (86) are in agreement with Hu, Paz, and Zhang [10].

As another example, we briefly discuss the purely

Ohmic case in the Fokker-Planck limit (a particular form of high temperature limit), which has been extensively discussed in the literature [6,10]. In this case one has

$$\eta(s-s') = M\gamma\delta'(s-s'), \quad (87)$$

$$\nu(s-s') = \frac{2M\gamma k_B T}{\hbar}\delta(s-s'). \quad (88)$$

Then Eq. (55) reduces to

$$\ddot{u}(s) + \Omega_{\rm ren}^2 u(s) + \gamma\dot{u}(s) = -2\gamma\delta(s)u(0), \quad (89)$$

where $\Omega_{\rm ren}^2 = \Omega^2 - 2\gamma\delta(0)$. After solving this equation a few calculations give

$$A(t) = -2M\gamma\delta(0), \quad (90)$$

$$B(t) = 2\gamma, \quad (91)$$

$$C(t) = 0, \quad (92)$$

$$D(t) = 2M\gamma k_B T. \quad (93)$$

Then the Wigner equation reads

$$\frac{\partial\tilde{W}}{\partial t} = -\frac{p}{M}\frac{\partial\tilde{W}}{\partial q} + M\Omega_{\rm ren}^2 q\frac{\partial\tilde{W}}{\partial p} + 2\gamma\frac{\partial\tilde{W}}{\partial p} + 2M\gamma k_B T\frac{\partial^2\tilde{W}}{\partial p^2}. \quad (94)$$

In this regime the coefficients of this Wigner equation are constants.

V. DISCUSSION

We have shown how to derive the Hu-Paz-Zhang master equation by tracing the evolution equation for the Wigner function of the whole system. Although actually quite lengthy we referred to it earlier as "elementary" because it is conceptually so. The length comes largely from the simple but tedious job of solving the classical equation of motion for prescribed boundary conditions, Eqs. (45)–(76).

Our evolution equation, Eq. (26), is in general non-Markovian, because the time-dependent coefficients depend on a fiducial moment of time, namely, the initial time at which the Wigner function is assumed to factor. Generally, one would expect a non-Markovian evolution equation at a particular moment of time t to involve the integral of the Wigner function over times to the past of t. The possibility that this non-Markovian equation can be written in the simpler, superficially "memoryless" form (26) was first emphasized by Shibata *et al.* [12–14].

After completion of this work we become aware of a paper by Anglin and Habib [15], who also consider the derivation of the Wigner equation Eq. (6) by tracing the Wigner equation for the whole system. Their approach is very similar to our derivation of Eq. (26) in Sec. II. They also emphasized that, at least as far as solving equations goes, the derivation is an essentially classical calculation. They do not, however, give explicitly the detailed derivation of the coefficients, as we do in Sec. III.

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