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Black-hole collisions from Brill-Lindquist initial data: Predictions of perturbation theory

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The Misner initial value solution for two momentarily stationary black holes has been the focus of much numerical study. We report here analytic results for an astrophysically similar initial solution, that of Brill and Lindquist (BL). Results are given from perturbation theory for initially close holes and are compared with available numerical results. A comparison is made of the radiation generated from the BL and the Misner initial values, and the physical meaning is discussed.

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I. INTRODUCTION

Black-hole collisions are presently of great interest as one of the "grand challenges" in high performance computing [1]. The results of those studies, in turn, can be important to the understanding of detectable sources of gravitational waves [2].

To the present date, the only case that has been extensively studied is the head-on collision, from rest, of two holes starting with the initial value solution given by Misner [3]. The spacetime growing out of those initial data has been computed by the techniques of numerical relativity [4], and has been studied by analytic means [5–7].

The initial value solution of Brill and Lindquist [8] (BL), like the Misner solution, represents two initially stationary nonspinning holes. Neither solution contains any initial radiation of short wavelength compared to the characteristic size of the throats. Outside the horizon, the two initial value solutions can be thought of as differing in the initial distortion of each throat caused by the presence of the other throat. There is, in fact, no solution of the initial value equations of general relativity that is uniquely singled out as representing two initially stationary holes. The Misner solution and BL solution are special only in their mathematical convenience, and in the topological properties of the geometry of the initial surface extended inside the throats. Specifically, the Misner solution may be thought of as having a two-sheeted topology. The two throats, representing the two black holes, connect an upper "physical" sheet to a single lower sheet, isometric to the upper one. In contrast, in the three-sheeted BL solution, each of the throats connects from the upper sheet to a separate lower sheet. The isometry between the two sheets in the Misner solution results from an infinite series of image terms in the solution to the Hamiltonian constraint. It is reasonable to expect that these terms might lead to additional gravitational radiation, not present in the BL solution. Other physical consequences of the image terms have been studied in Ref. [9].

Here, we extend the analytic study of collisions of holes to the case of BL initial data. There are two main justifications for doing this. The first is that analytic answers are a useful aid to development of the codes used in numerical relativity. The values reported here for radiated energy can be tested against numerical codes for evolution of axisymmetric initial data. For initially close black holes, it will be interesting to see whether those codes agree with the analytic answers as well as they do in the case of Misner initial data.

The second reason for some interest in the evolution of BL data is the general question of the relationship of initial data to the generation of gravitational radiation. In astrophysically realistic problems, the initial data will necessarily come from some approximation scheme, such as post-Newtonian solutions. Such an approach is justified if the gravitational wave signal generated depends only on certain general features of the initial data and is insensitive to many details (e.g., topology). The comparison of the evolution of the BL and the Misner data gives us a simple model for studying this question, and an interestingly simple (though limited) answer.

In the next section, we describe the application of close-limit perturbation theory to the evolution of the BL initial data. In Sec. III results are given for the radiation predicted by perturbation theory. These results are compared with available fully numerical results for the BL case, and are compared with analogous results previously reported for collisions from the Misner initial data. We summarize our results in Sec. IV.

II. CLOSE-LIMIT PERTURBATION THEORY FOR THE BL INITIAL DATA

Like the Misner solution, the BL geometry is conformally flat and takes the form $ds^2 = \Phi^4\, ds^2_{\rm flt}$, where $ds^2_{\rm flt}$ is the line element for flat three-dimensional space, and where Φ satisfies the Laplace equation in the flat space.

In terms of spherical coordinates R,θ,ϕ, for $ds^2_{\rm flt}$, the Misner or BL metrics can be written

$$ds^2 = \Phi^4(R,\theta;\mu_0)\left(dR^2 + R^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) \ . \quad (1)$$

For the BL geometry, the form of Φ, aside from a factor of 2, corresponds to the potential of Newtonian theory, with points of mass m at positions $z = \pm z_0$ on the z axis:

$$\Phi_{\rm BL} = 1 + \frac{1}{2}\left(\frac{m}{\sqrt{R^2\sin^2\theta + (R\cos\theta - z_0)^2}} + \frac{m}{\sqrt{R^2\sin^2\theta + (R\cos\theta + z_0)^2}}\right) . \quad (2)$$

For $R > z_0$, the square roots can be expanded in a power series in z_0/R and the BL three-geometry written as

$$ds^2_{\rm BL} = \left[1 + \frac{M}{2R}\sum_{\ell=0,2,\ldots}\left(\frac{z_0}{R}\right)^\ell P_\ell(\cos\theta)\right]^4 \times\left(dR^2 + R^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) , \quad (3)$$

where the P_ℓ are the Legendre polynomials, and where $M \equiv 2m$.

We next make a transformation of the radial coordinate R to a new coordinate r, as if we were transforming, in the Schwarzschild spacetime, from isotropic coordinates to Schwarzschild coordinates:

$$R = \left(\sqrt{r} + \sqrt{r - 2M}\right)^2/4 \ . \quad (4)$$

It is convenient now to rewrite the line element for the three-geometry as

$$ds^2_{\rm BL} = \left[1 + \frac{M/(2R)}{1+M/(2R)}\sum_{\ell=2,4,\ldots}\left(\frac{z_0}{M}\right)^\ell\left(\frac{M}{R}\right)^\ell P_\ell(\cos\theta)\right]^4\left(\frac{dr^2}{1-2M/r} + r^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) , \quad (5)$$

where the meaning of R is given by (4).

The geometry in (5) reduces to the Schwarzschild geometry if the summation in the leading factor on the right-hand side is ignored. That summation, then, contains the information about the deviations from sphericity and is the starting point for close-limit nonspherical perturbation calculations [10]. In particular, the parameter $\epsilon \equiv z_0/M$ can be considered an expansion parameter for perturbation theory. If, for each multipole index ℓ, we keep only the leading order in ϵ, the approximation to the BL initial geometry takes the form

$$ds^2_{\rm BL} \approx \left[1 + \frac{2M/R}{1+M/(2R)}\sum_{\ell=2,4,\ldots}\left(\frac{z_0}{M}\right)^\ell\left(\frac{M}{R}\right)^\ell P_\ell(\cos\theta)\right]\left(\frac{dr^2}{1-2M/r} + r^2\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]\right) . \quad (6)$$

In principle, for each multipole index ℓ, one can read off the metric perturbations (which are purely even parity) from (6), can construct Moncrief's [11] gauge-invariant perturbation wave function $\psi_{\rm pert}$, and can evolve that wave function with the Zerilli equation [12]. In practice, this need not be explicitly carried out. There is a striking similarity between the expressions in (3)–(6) and the equivalent expressions for the Misner geometry [5,6]. The single difference is the coefficients in the series appearing in (3)–(6). For the Misner initial geometry, the coefficients are $\kappa_\ell(\mu_0)$. The dimensionless quantity μ_0 parametrizes the initial separation of the throats, and the κ's are functions given in Ref. [6]. The single change

$$(z_0/M)^\ell \to 4\kappa_\ell(\mu_0) \quad (7)$$

converts (3)–(6) to their equivalent form for the Misner case. This means, for a given ℓ, that $\psi_{\rm pert}$ for the BL case has precisely the same form as for the Misner case; the outgoing gravitational waves, according to perturbation theory, are identical in shape. They differ only in a multiplicative factor. Since power carried by outgoing waves is proportional to the square of $\psi_{\rm pert}$, the results for BL infall, for each ℓ, can be found by multiplying the Misner results by $[(z_0/M)^\ell/4\kappa_\ell(\mu_0)]^2$. We note in passing that the "forced linearization" procedure discussed in Ref. [10] is, of course, also applicable to the BL data.

This Misner-BL equivalence applies for any separation of the holes. For large separations of the throats, it is not surprising that the gravitational waves generated by BL and by Misner initial data should be similar. For small initial separations, however, there is a significant difference between the three-geometries of Misner and BL, and it does seem strange that the gravitational wave forms should be identical. Furthermore, it is for close initial separation that perturbation theory is most applicable, so the prediction of identical linearized wave forms is also a prediction about the actual wave forms. How can such different initial conditions give rise to identical outgoing wave forms?

It is important to realize that the linearized outgoing waves are identical in form for each ℓ, but the ratio of multipole contributions differs for BL and Misner. In Fig. 1, this difference in multipoles is shown quantitatively. For a given value of μ_0 in the Misner geometry, an equivalent configuration for the BL geometry is defined by setting the quadrupole amplitudes of $\psi_{\rm pert}$ equal, i.e., by setting $(z_0/M) = 2\sqrt{\kappa_2(\mu_0)}$. The ratios of the BL amplitude to the Misner amplitude are then computed for $\ell = 4$ and $\ell = 6$. [These amplitude ratios are in fact simply $4\kappa_2(\mu_0)^2/\kappa_4(\mu_0)$ and $16\kappa_2(\mu_0)^3/\kappa_4(\mu_0)$.] At large separation, the amplitude ratios approach unity; this shows that in the limit of large separation, the external fields become identical in the two initial geometries. For small separations, however, the BL solution has a relatively smaller contribution because of higher multipole moments; its geometry is more quadrupole dominated. Though this is a relatively important difference in the initial geometry near the throats, it is of little importance for the gravitational radiation. Even for the Misner initial conditions, the radiation is heavily quadrupole dominated. It is possible that the lesson of this example has a broader generality: the outgoing radiation can be insensitive to many details of the initial data and even for strong field sources, a knowledge of the quadrupole moment may be all that is needed.

It is worth asking whether there is any deep physical meaning in the fact that the only difference between the BL and Misner linear perturbations is the ratio of the multipole amplitudes. This follows from the fact that for a conformally flat three-metric, with the form (1), the factor Φ satisfies the flat space Laplacian. If the solution is axisymmetric and asymptotically flat, it must be of the form $\sum(\alpha_\ell/R^{\ell+1})P_\ell(\cos\theta)$; solutions can differ only in the values of the constants α_ℓ. So the striking similarity of the BL and the Misner perturbations is a direct result of the choice of the conformally flat form (1). This choice is dictated by convenience, and need not be made in principle. For more general momentarily stationary initial geometries, the linearized wave forms for each multipole will have different appearance. For example, one could generate valid initial data representing a Schwarzschild spacetime with a nonconformally flat perturbation by choosing an arbitrary (small) metric perturbation and solving the linearized Hamiltonian constraint for the conformally flat part of the perturbation. The gauge-invariant function would then be computed from the full perturbation.

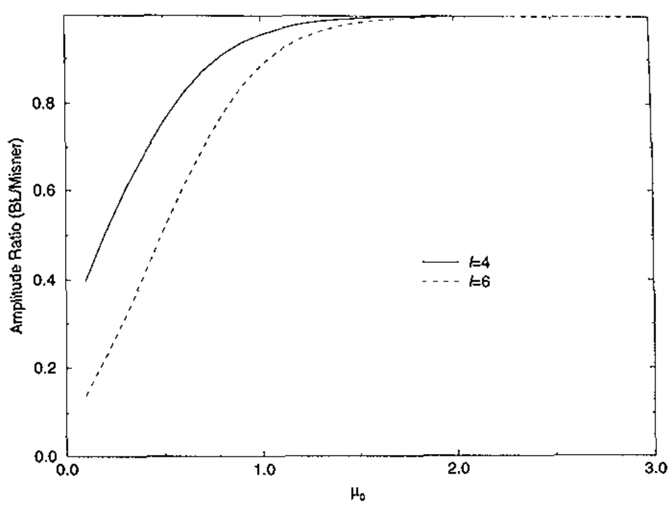

FIG. 1. Ratio of amplitudes of $\psi_{\rm pert}$ for BL and Misner geometries. For equal amplitudes of $\ell = 2$, amplitude ratios are shown for $\ell = 4$ and $\ell = 6$.

III. RADIATION ENERGY: BL VS MISNER

The first, and most difficult, step in comparing radiation from the two initial value sets is to decide on the basis for comparison: How does one compare a BL problem with a particular value of z_0/M with a Misner problem of a particular μ_0? At large separations, it is not difficult; one can compare BL and Misner configurations in which the masses and separation of the holes are identical. For small separations, however, the separation of the holes is somewhat ambiguous. To deal with small, as well as large, separations we choose a reasonably natural and convenient specific measure of the separation L: the proper distance along the symmetry axis, between the outermost disjoint marginally outer-trapped surfaces around each throat. (For z_0/M less than about 0.4, a single apparent horizon encompasses both holes.) The locations of the marginally outer-trapped surfaces was found, using a standard shooting technique applicable to axisymmetric spatial slices [13]. We characterize both BL and Misner configurations with L/M, where M is the mass of the spacetime. It is, of course, interesting not only to compare the linearized predictions for BL against those for Misner, but also to compare both against the results of numerical solutions of the fully nonlinear field equations. For the Misner initial geometry, the numerical results are known from the work reported in Ref. [4]. For BL initial conditions, two data points are available: cases c2 and c4 from Ref. [14]. These numerically generated spacetimes have Euclidean spatial topology, with initial data consisting of spherical (in the conformal space) collisionless matter configurations. When the initial configurations are sufficiently compact, the matter is all inside disjoint apparent horizons and the external three-geometry is identical to the BL data.

For clarity, the results are presented in three separate figures. Figure 2 shows the comparison of perturbation results and numerical results for the Misner case. The

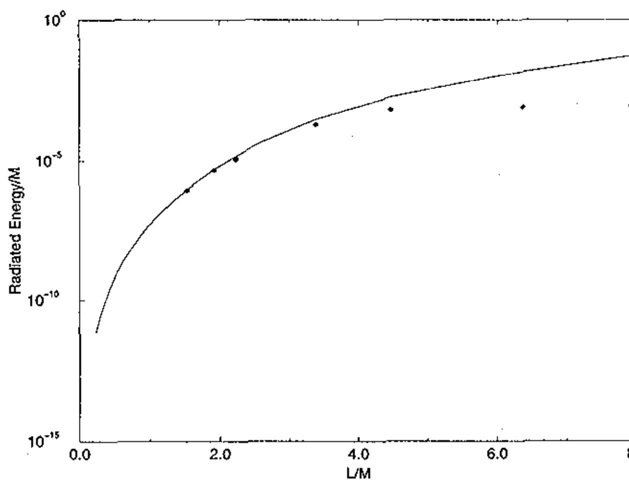

FIG. 2. Gravitational radiation energy emitted during the head-on collision of two black holes starting from the Misner initial conditions. Results are shown for close-limit perturbation theory (continuous curve) and for numerical relativity (isolated points).

perturbation energies [$E/M \approx 0.0251\kappa_2^2(\mu_0)$] are those of Ref. [5], except that energy has been plotted as a function of L/M, rather than of μ_0. The numerical data are those of Refs. [4,6]. Figure 3 shows the analogous results for the BL case, for which the energy is $E/M \approx 0.0251[(z_0/M)^2/4]^2$. The two "numerical" data points here are those of Ref. [14].

Figure 2 shows that for the Misner case, linearized predictions begin to diverge from the fully numerical results at around $L/M = 4$. It is fortunate that the numerical results available for the BL case are for L/M in the range 3–4. From Fig. 3 we can infer that for L/M less than around 3, the agreement between linearized and numerical results is very good for BL collisions, and for L/M above 4 there is significant disagreement. In this sense there is little difference between Misner and BL cases. Figure 4 shows the perturbation theory comparison of Misner and BL cases. This figure shows that there is little difference between the predicted radiation when L/M is greater than around 2. It is, therefore, not surprising that the agreement between numerical and perturbation results, which breaks down well above $L/M = 2$, does not distinguish between BL and Misner collisions. It is also not surprising that in BL collisions, as in Misner collisions [6], the radiation is always quadrupole dominated. (The large values of hexdecapole energy in Fig. 3 occur only at separations large enough that linearized theory wildly overestimates radiation.)

The results in Fig. 4 would seem to suggest that, for black holes initially close, BL initial conditions lead to less radiation than that with Misner black holes as expected by the presence of image terms in the Misner solution. An alternative interpretation is that for equal radiation, the initial separation of the apparent horizons is greater in the BL case than that in the Misner case.

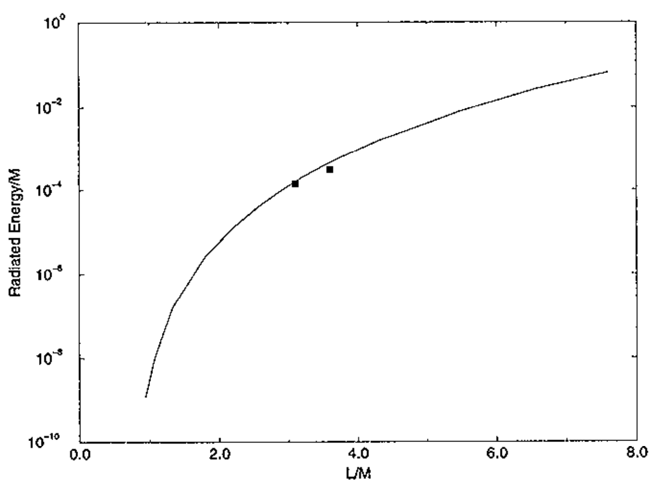

FIG. 3. Gravitational radiation emitted during the head-on collision of two black holes starting from BL initial conditions. Results are shown for close-limit perturbation theory (continuous curve), and two values are shown from numerical relativity.

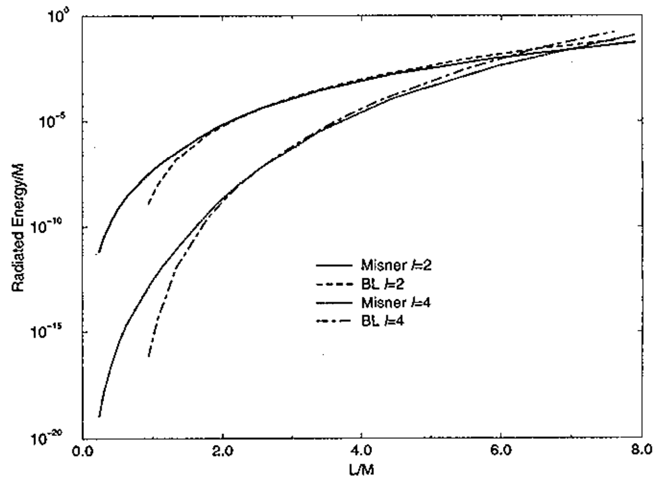

FIG. 4. Comparison of perturbation theory predictions for radiated energy from Misner and from BL initial conditions. Results are given for both $\ell = 2$ and 4 multipoles.

Since equal radiation implies equal quadrupole moments, this means that the different multipole structure of the BL and Misner geometries makes the proper distance between apparent horizons larger in the BL case when quadrupole moments are equal. In this sense then, Fig. 4 is more of a depiction of proper distances than of radiation.

This motivates asking whether there is a way of comparing BL and Misner scenarios that is better, or at least different, from that using L/M. Another physically meaningful measure of how close the initial throats are is the gravitational binding energy. The gravitational binding energy is the difference between the Arnowitt-Deser-Misner (ADM) energy of an initial data set representing two black holes at finite separation and the energy of an initial data set with the holes infinitely separated (the sum of the bare masses of the holes). For BL data this is given by [8]

$$\frac{E_B}{M} = -\frac{M}{8z_0}. \tag{8}$$

For Misner data one has [15]

$$\frac{E_B}{M} = -\frac{\sum_{n=1}^{\infty}(n-1)\mathrm{csch}n\mu_0}{\sum_{n=1}^{\infty}\mathrm{csch}n\mu_0}. \tag{9}$$

Radiated energy is plotted against binding energy in Fig. 5, but the results give a picture very much like that of Fig. 4. In particular, for small initial separations (tightly bound initial configurations), there is less energy radiated from a BL collision than that from a Misner collision. For large initial separations (small binding energies), the difference in radiated energy is small for configurations with the same binding energy. The BL and Misner cases become significantly different (say by a factor of 2) for binding energy (binding energy/$M \approx -1.5$) that corresponds roughly to the point ($L/M \approx 1.3$) at which the BL and Misner energies separate in Fig. 4.

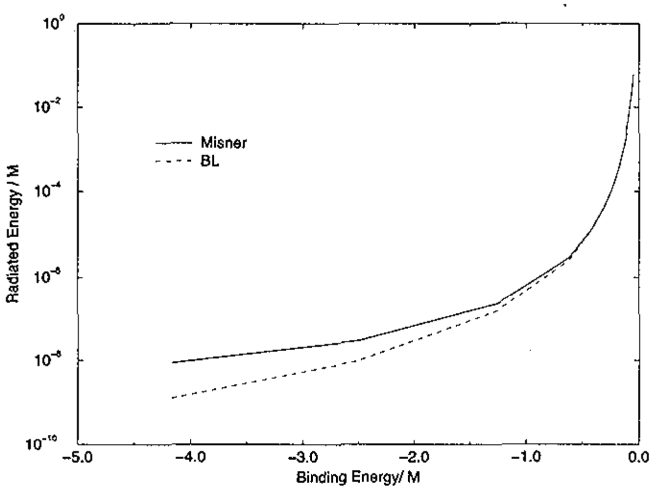

FIG. 5. Gravitational radiations emitted from Misner and BL initial conditions are plotted as a function of the binding energy of the initial configuration divided by the mass of the spacetime.

IV. SUMMARY

The result of applying close-limit methods to BL initial data shows that the single difference between radiation from Misner and BL data lies in the relative strengths of multipoles. In view of the very different nature of the BL and Misner spacetimes close to the holes, this discovery is at first surprising. It turns out, however, not to be of any deep physical significance, but rather an artifact of the use, in both cases, of conformally flat initial data. There may be some interest, therefore, in investigating initial black hole solutions which are not conformally flat, to see whether there is any significant new feature of the emerging radiation.

There are two aspects of the present results for BL which are potentially interesting. First, these results afford an opportunity to make relatively simple checks of numerical relativity codes. It should be possible to evolve BL data with the codes that have been used to evolve Misner data, particularly if causal ("apparent horizon") boundary conditions are used. In addition to tests for the wave form shape, radiation intensity, etc., a test of particular interest would be the ratio of $\ell = 4$, to $\ell = 2$ radiation. The results of numerical relativity for $\ell = 4$ radiation from Misner data have large uncertainty when the holes are initially close [6]. It would, therefore, be comforting to see that the codes can find that the ratio of $\ell = 4$ to $\ell = 2$ radiation, is smaller for BL data than that for Misner data.

Aside from their value in connection with code checks, the results are primarily of interest for the way in which they show the care that must be used in making a physical comparison of two different sets of initial data representing holes which are initially close. When we try to use both BL and Misner data to describe two black holes at a certain separation, how do we give specific meaning to the idea of "two black holes at a certain separation"? The result of the comparison may depend more on our choice of meaning for "separation" than on something more physically meaningful. We have used two measures of separation: (i) the distance between apparent horizons, and (ii) the binding energy of the system. For either criteria, we find that at very small initial separation, BL initial data generates less radiation than that by Misner initial data for the "same" configuration. This would seem to suggest that the details of the BL data make it, in some sense, really less efficient in generating radiation. An alternate viewpoint is that there is less spurious radiation present in the initial data.

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