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Back reaction on the topological degrees of freedom in (2+1)-dimensional spacetime

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We investigate the back-reaction effect of the quantum field on the topological degrees of freedom in a (2+1)-dimensional toroidal universe, $\mathcal{M} \simeq T^2 \times \mathbf{R}$. Constructing a homogeneous model of the toroidal universe, we examine explicitly the back-reaction effect of the Casimir energy of a massless, conformally coupled scalar field, with a conformal vacuum. The back reaction causes an instability of the universe: The torus becomes thinner and thinner as it evolves, while its total two-volume (area) becomes smaller and smaller. The back reaction caused by the Casimir energy can be compared with the influence of the negative cosmological constant: Both of them make the system unstable and the torus becomes thinner and thinner in shape. On the other hand, the Casimir energy is a complicated function of the Teichmüller parameters (τ^1, τ^2) causing highly nontrivial dynamical evolutions, while the cosmological constant is simply a constant. Since the spatial section is a two-torus, we shall write down the partition function of this system, fixing the path-integral measure for gravity modes, with the help of the techniques developed in string theories. We show explicitly that the partition function expressed in terms of the canonical variables corresponding to the (redundantly large) original phase space is reduced to the partition function defined in terms of the physical-phase-space variables with a standard Liouville measure. This result is compatible with the general theory of the path integral for the first-class constrained systems.

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I. INTRODUCTION

Topological considerations are necessary in many situations. Since physical laws are usually expressed in terms of local, differential equations, their importance is not prominent at first sight. However, once one proceeds to solve the equations, one has to take boundary conditions into account, which allow the topological information to enter in the theory. In general relativity, which handles the dynamics of spacetime, the topological properties acquire dynamical meaning and their consideration becomes more significant. The aim of this paper is to present an explicit, detailed investigation of the dynamics of topological degrees of freedom in spacetime, in the context of the back-reaction problem in semiclassical gravity. We concentrate on the case of (2+1)-dimensional toroidal spacetime $\mathcal{M} \simeq T^2 \times \mathbf{R}$ and make use of various techniques developed for two- and three-dimensional gravity. Here we do not discuss the topology change [1,14]. The term "topological degrees of freedom" indicates those global parameters describing the global deformations of the spatial hypersurface which are of topological origin (the moduli deformations) (Sec. III A and Appendix A).

As a first preliminary study for the full quantum gravity, it is reasonable to consider the effect of the curvature of a fixed background spacetime on the behavior of quantum matter field, which is the subject of quantum field theory on a curved spacetime [2,3]. Then the next natural step is the investigation of the influence of such a quantum field on classical spacetime geometry, which is called the back-reaction problem in semiclassical gravity. Usually, one tries to describe this effect by the semiclassical Einstein equation

$$G_{\alpha\beta} = \alpha\langle T_{\alpha\beta}\rangle \ , \tag{1}$$

where $\langle T_{\alpha\beta}\rangle$ is some c number, obtained from the energy-momentum tensor operator and the inner product of some quantum states, and α is an appropriate gravitational constant with physical dimension $[\alpha] = [(\text{length})^{n-2}]$. (Here, n is the spacetime dimension. We treat $\hbar$ as $[\hbar] = [1]$ and set $c = 1$ in this paper.) There are several uncertain issues and technically complicated points about this treatment. First, it is not clear what kind of quantity should be chosen for $\langle T_{\alpha\beta}\rangle$ [4]. Here we regard that $\langle T_{\alpha\beta}\rangle$ should be some expectation value, rather than the quantity $\langle \text{out}|T_{\alpha\beta}|\text{in}\rangle$, since the latter harms the reality and causal nature of Eq. (1) [5–7]. Then, if one regards the path-integral formalism as fundamental for quantum gravity, the so-called in-in formalism [8,5,6] should be of more importance than the standard in-out formalism [7]. Second, the regularization of $\langle T_{\alpha\beta}\rangle$ requires complicated, though well-established, techniques, which itself is one main topic of the quantum field theory on a curved spacetime [2,3]. Third, Eq. (1) in general becomes complicated, even though $\langle T_{\alpha\beta}\rangle$ has been successfully computed, so that it is difficult to solve it and study the effect of the back reaction in detail. Fourth, one can show that Eq. (1) can be obtained from

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the first variation of the phase part in the in-in path-integral expression [5,7], in which the matter part has been integrated out formally, while the gravity part is left unintegrated without explicit fixation of the measure. If one wants to go one step further, however, one should also take care of the effect coming from the path-integral measure for the gravity part. It is usually difficult since a reasonable, general measure has not been fixed yet. Fifth, to speak rigorously, Eq. (1) itself contains an inconsistency from the very beginning. Since gravity and matter couple, quantum fluctuations of matter cause corresponding quantum fluctuations of gravity. Thus there is a limitation in principle to the semiclassical treatment [Eq. (1)], because we try to treat gravity classically, while matter is treated by quantum theory [7]. Specifying the exact validity conditions for Eq. (1) is one of the main topics of semiclassical gravity [7,9,10].

In this paper, we consider a (2+1)-dimensional spacetime $\mathcal{M} \simeq \Sigma \times \mathbf{R}$, with $\Sigma \simeq T^2$ a torus. We choose, as a matter field, a massless conformally coupled scalar field with a conformal vacuum and investigate explicitly the back-reaction effect resulting from the Casimir energy of matter on the topological degrees of freedom, i.e., the modular deformations of the torus. As stated above, the topological degrees of freedom are one of the essential ingredients of spacetime dynamics. However, the back reaction on topological modes has seldom been discussed so far, partially because such a finite number of degrees of freedom is hidden in an infinite number of gravity modes in four-dimensional spacetime. One advantage of the reduction of the number of dimension from 4 to 3 is that only finite topological modes plus a spatial volume remain dynamical for the case of pure gravity, due to the dimensionality [11–13]. One can understand this point as follows. When $n = 3$, the spatial metric h_{ab} has three independent components at each spatial point, while there are three constraints at each point. Thus a redundant infinite number of modes is gauged away and only a finite number of modes remains. Here we want to investigate the back-reaction effect from matter onto the topological degrees of freedom of spacetime, which would force us to take the matter field into account. To preserve the above-mentioned nice property of the finiteness of the number of degrees of freedom, we choose a model in which the matter field is in a vacuum state on a spatially homogeneous (2+1)-dimensional spacetime. Another advantage of the reduction of dimension in the discussion of topological aspects comes from the fact that two-dimensional topology is completely classified in a simple manner so that it is easy to construct various topologies [14].

Another good point of this model is that some difficulties and complications stated above of the semiclassical Einstein equation, Eq. (1), become simplified and tractable to a great extent in this case.

First, we choose a conformal vacuum $|0\rangle$ as a natural candidate for a vacuum state of matter in our case and use $\langle 0|T_{\alpha\beta}|0\rangle$ on the right-hand side of Eq. (1).

Second, since the background spacetime shall be chosen as (conformally) flat and the matter field is conformally invariant, $\langle T_{\alpha\beta}(g)\rangle$ can be calculated from $\langle T_{\alpha\beta}(\eta)\rangle$ (η a flat metric) along with the trace anomaly [2], which simplifies the manipulation. Furthermore, in our case, the spacetime dimension is odd, $n = 3$, so that there is no trace anomaly [2]. Thus $\langle T_{\alpha\beta}(g)\rangle$ is related to $\langle T_{\alpha\beta}(\eta)\rangle$ in a simple manner.

Third, because of the dimensionality, Eq. (1) is reduced to a set of six first-order ordinary differential equations and we can investigate the effect of the back-reaction explicitly.

Fourth, we restrict the metrics to a special class, with the spatial part being the one for the locally flat metrics on a torus. Thus we can fix the path-integral measure by the use of the techniques developed in string theories [15,16]. Within this model, we can discuss explicitly the influence of matter on the semiclassical dynamics of gravity. Our treatment corresponds to the minisuperspace approach in quantum cosmology: Putting restrictions on the variables to be quantized (e.g., spatial homogeneity), which is compatible with the classical dynamics, quantum theory is to be constructed within this restricted subclass of variables. Though this treatment is self-consistent as a quantum system, one significant question naturally arises: To what extent does such a treatment reflect faithfully the original full quantum theory? From the viewpoint of the original full system, the restrictions are regarded as constraints on the phase space, which can modify the path integral measure for the reduced variables (minisuperspace variables). Our model may be a good test candidate to investigate this point in detail.

Fifth, the (in-in) effective action for gravity, $W[g_+ : g_-]$, becomes relatively simple in our case, and this reduces to $W[\tau^1_+, \tau^2_+, V_+ : \tau^1_-, \tau^2_-, V_-]$, a functional of six functions of t ($\tau^1_\pm, \tau^2_\pm, V_\pm$), where $V_\pm$ indicate the spatial two-volume (area) and $(\tau^1_\pm, \tau^2_\pm)$ are the Teichmüller parameters describing the topological degrees of freedom of a torus. Although the exact calculation W has already become difficult, we can still estimate its functional form to leading order in $\hbar$. In computing W, our model reveals explicitly the peculiarity of the semiclassical gravity, compared with the standard treatment of the quantum dissipative system, e.g., Brownian motion [17]: There is no linear coupling between the subsystem (gravity) and the environment (matter field). Their coupling is put in the kinetic term of the matter field. This model might provide the simplest nontrivial example for the investigation of the quantum dissipative system including gravity.

In Sec. II, we recapitulate how to handle quantum fields on topologically nontrivial spaces: Construct the quantum field theory on $\mathcal{M} \simeq T^2 \times \mathbf{R}$ and calculate the Casimir energy of a massless, conformally coupled scalar field with a conformal vacuum [2,3,19].

In Sec. III, we extract explicitly the topological degrees of freedom of a torus and reduce Eq. (1) to a canonical system with a finite number of degrees of freedom [13]. Then we investigate explicitly the effect of the back reaction of matter on the dynamics of the topological degrees of freedom. We shall see that the back reaction makes the system unstable and the torus becomes thinner and thinner as it evolves, while its two-volume becomes smaller and smaller. These behaviors are universal, that is, independent of the initial conditions. The asymptotic analysis of the set of dynamical equations justifies this point. We shall also compare our case of the Casimir energy with the case of the negative cosmological constant, since both of them can be regarded as negative energies. Most significant difference is that the Casimir energy is a complicated function of the Teichmüller parameters (τ^1, τ^2), while the negative cosmological constant is just a constant.

In Sec. IV, we investigate the partition function of this system, fixing the measure with the help of the techniques in string theories. We show explicitly that gauge fixing reduces the partition function formally expressed in terms of the canonical variables for the (redundantly large) original phase space, to the partition function defined in terms of the physical-phase-space variables with a standard Liouville measure. This result is compatible with the general theory of the path integral for first-class constrained systems. We also estimate the functional form of W to leading order in $\hbar$. Section V is reserved for a discussion.

II. QUANTUM FIELD THEORY ON A (2+1)-DIMENSIONAL TOROIDAL SPACETIME

This section is for defining the model to be considered and calculating the energy-momentum tensor in our model as a preliminary for the next section, where the back-reaction effect is analyzed in detail. Calculating $\langle T_{\alpha\beta}\rangle$ is now a well-established topic, and we just sketch the essence in the context of our model for later uses.

A. Scalar field on a torus

We consider a (2+1)-dimensional spacetime with topology $T^2 \times \mathbf{R}$. We concentrate on the case when the geometry of the space $\Sigma \simeq T^2$ is locally flat. A flat two-geometry is endowed on Σ by giving a metric[1]

$$dl^2 = \hat{h}_{ab} d\xi^a d\xi^b \ , \tag{2}$$

where

$$\hat{h}_{ab} = \frac{1}{\tau^2} \begin{pmatrix} 1 & \tau^1 \\ \tau^1 & |\tau|^2 \end{pmatrix} \ , \tag{3}$$

and the periodicities in the coordinates ξ^1 and ξ^2 with period 1 are understood. Here[2] (τ^1, τ^2) are the Teichmüller parameters [15,16] independent of spatial coordinates (ξ^1, ξ^2) and $\tau := \tau^1 + i\tau^2$, $\tau^2 > 0$. Note that $\sqrt{\hat{h}} := (\det \hat{h}_{ab})^{1/2} = 1$.

The Laplacian operator $\Delta := -1/\sqrt{h}\,\partial_a(h^{ab}\sqrt{h}\,\partial_b)$ on Σ with the line element dl^2 [Eqs. (2),(3)] gives the normalized eigenfunctions

$$f_{n_1 n_2}(\xi) = \exp(i2\pi n_1 \xi^1) \exp(i2\pi n_2 \xi^2) \quad (n_1, n_2 \in \mathbf{Z}) \tag{4}$$

and the eigenvalues

$$\lambda_{n_1 n_2} = \frac{4\pi^2}{\tau^2}(|\tau|^2 n_1^2 - 2\tau^1 n_1 n_2 + n_2^2) \ . \tag{5}$$

Now let us consider a spacetime $\mathcal{M} \simeq \Sigma \times \mathbf{R}$, with a line element $ds^2 = -dt^2 + \hat{h}_{ab} d\xi^a d\xi^b$. The fundamental positive frequency solutions for $\Box u(t, \xi^1, \xi^2) = 0$ are[3]

$$\bar{u}_A(t, \xi) = \frac{1}{\sqrt{2\omega_A}} e^{-i\omega_A t} f_A(\xi) \ , \tag{6}$$

where A stands for $n_1 n_2$ and $\omega_A := \sqrt{\lambda_A} = \sqrt{\lambda_{n_1 n_2}}$. Afterwards, we follow the standard procedure for the field quantization [2,3].

B. Model

We shall investigate the back reaction of the matter field on the topological degrees of freedom (τ^1, τ^2). The most ideal treatment of the back reaction described by Eq. (1) may be the self-consistent determination of the geometry $g_{\alpha\beta}$ through Eq. (1): $\langle T_{\alpha\beta}\rangle$ depends on $g_{\alpha\beta}$, and this $g_{\alpha\beta}$ is self-consistently determined by Eq. (1). However, it turns out that such a treatment becomes highly complicated even in our simple model. To make our analysis tractable, then, we treat the back reaction in the following sense, which is usually adopted in the back-reaction problems [2,3]: We prepare a background spacetime and calculate $\langle T_{\alpha\beta}\rangle$ on it. Then we discuss the modification of the background geometry due to the $\langle T_{\alpha\beta}\rangle$, using Eq. (1).

Now, as a background spacetime, we choose a solution of the vacuum Einstein equation $G_{\alpha\beta} = 0$. More specifically, we prepare a locally flat spacetime $ds^2 = -dt'^2 + V\,dl^2 = V(-dt^2 + dl^2)$, where dl^2 is given by Eqs. (2),(3) and V, τ^1, and τ^2 are chosen to be constant for the background spacetime. (Below, we occasionally treat this flat spacetime as conformally flat, just for mathematical convenience.) We choose as a matter field a massless conformally coupled scalar field ψ:

$$S_m = -\tfrac{1}{2} \int (g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi + \tfrac{1}{8}R\psi^2)\sqrt{-g}\, d^3x \ . \tag{7}$$

The (improved) energy-momentum tensor operator [3] becomes

[1]For definiteness, we shall use the symbol $\hat{h}_{ab}$ to represent the particular matrix given by (3), while the symbol h_{ab} shall be reserved for more general context, representing a general spatial metric induced on a spatial surface Σ.

[2]Throughout this paper, τ^2 always indicates the second component of (τ^1, τ^2) and not the square of τ. The latter never appears in the formulas.

[3]In connection with later applications, it is worthwhile to note that, even though τ would depend on t, the form of the equation $\Box\psi = 0$ would not change, because of the form of the metric, $g_{\alpha\beta} = (-1, \hat{h}_{ab})$ with $\det g_{\alpha\beta} = -1$.

$$T_{\alpha\beta}(g) = \tfrac{3}{4}\partial_\alpha\psi\partial_\beta\psi - \tfrac{1}{4}\partial_\gamma\psi\partial^\gamma\psi g_{\alpha\beta} - \tfrac{1}{4}\psi\partial_\alpha\partial_\beta\psi + \tfrac{1}{12}\psi\,\Box\,\psi g_{\alpha\beta} + \tfrac{1}{8}\psi^2(R_{\alpha\beta} - \tfrac{1}{3}g_{\alpha\beta}R) \,. \quad (8)$$

We choose the conformal vacuum as a vacuum state for the matter field. Then $\langle T_{\alpha\beta}(g)\rangle$ is simply related to $\langle T_{\alpha\beta}(\eta)\rangle$ as

$$\langle T_{\alpha\beta}(g)\rangle = V^{-1/2}\langle T_{\alpha\beta}(\eta)\rangle \,, \quad (9)$$

when the metric $g_{\alpha\beta}$ and the flat metric $\eta_{\alpha\beta}$ are related[4] as $g_{\alpha\beta} = V\eta_{\alpha\beta}$. On flat spacetime, the field equation for ψ becomes $\Box\psi = 0$ and Eq. (6) can be used as fundamental solutions. In this manner, the time evolution of V causes no direct complication in the analysis.

However, the time dependence of (τ^1, τ^2) caused by the back reaction harms the self-consistency of the analysis, which is inevitable if the tractability of the back-reaction problem, described by Eq. (1), is to be maintained. When (τ^1, τ^2) evolve in time, the functions in Eq. (6) are no longer exact solutions for $\Box\psi = 0$, because $\omega_A := \sqrt{\lambda_A}$ becomes t dependent, through the t dependence of (τ^1, τ^2) [Eq. (7)]. Furthermore, the spacetime described by $ds^2 = -dt^2 + V(t)\hat{h}_{ab}d\xi^a d\xi^b$ is no longer conformally flat when (τ^1, τ^2) evolves, because of the t dependence of $\hat{h}_{ab}$. Thus we should look at the results of the analysis in an adiabatic sense, i.e., valid when terms including $\dot{\tau}^1$ and $\dot{\tau}^2$ are not dominant in the formulas prominently. Such a conflict between self-consistency and the tractability of the analysis always occurs in the back-reaction problem. In our present model, this adiabatic treatment provides a good approximation because $\dot{\tau}^1$ and $\dot{\tau}^2$, caused by the back reaction, turn out to be sufficiently small (see Sec. III C).

We next need Hadamard's elementary function [2,3] $G^{(1)}(x)$ for $ds^2 = -dt^2 + dl^2$ to calculate $\langle T_{\alpha\beta}(\eta)\rangle$. This function and the related energy-momentum tensor have already been extensively investigated [19]. We first compute $G^{(1)}(x)$ for $\mathcal{M} \simeq \mathbf{R}^3$ and afterwards take care of the periodicity in $\mathcal{M} \simeq T^2 \times \mathcal{R}$, adding all contributions from points which should be identified [18,19]. For the three-dimensional Minkowski space, $G^{(1)}(x)$ is

$$G^{(1)}(x) := \langle 0|\{\psi(x), \psi(y)\}|0\rangle = \frac{\hbar}{2\pi}(2\sigma)^{-1/2} \quad (\sigma > 0) \,, \quad (10)$$

where $\sigma := \frac{1}{2}x^2 = \frac{1}{2}\eta_{\alpha\beta}x^\alpha x^\beta$, $\frac{1}{2}$ times a square of a world distance. Thus we get[5]

$$G^{(1)}(x) = \frac{\hbar}{2\pi}\sum_{n_1,n_2=-\infty}^{\infty}{}' \, [2\sigma_{n_1 n_2}(x)]^{-1/2} \,, \quad (11)$$

where

$$2\sigma_{n_1 n_2}(x) := -t^2 + \frac{1}{\tau^2}|(\xi^1 + n_1) + \tau(\xi^2 + n_2)|^2 \,.$$

Now it is straightforward to compute $\langle T_{\alpha\beta}(\eta)\rangle$ explicitly[6] [$\eta_{\alpha\beta} = (-1, \hat{h}_{ab})$ with (3)]. The result is

$$\langle T_{00}\rangle = -\frac{\hbar(\tau^2)^{3/2}}{4\pi}\sum_{n_1 n_2}{}' \frac{1}{|n_1 + \tau n_2|^3} \,, \quad (12a)$$

$$\langle T_{11}\rangle = \frac{\hbar(\tau^2)^{1/2}}{4\pi}\sum_{n_1 n_2}{}' \frac{1}{|n_1 + \tau n_2|^3} - \frac{3\hbar(\tau^2)^{1/2}}{4\pi}\sum_{n_1 n_2}{}' \frac{(n_1 + \tau^1 n_2)^2}{|n_1 + \tau n_2|^5} \,, \quad (12b)$$

$$\langle T_{22}\rangle = \frac{\hbar(\tau^2)^{1/2}|\tau|^2}{4\pi}\sum_{n_1 n_2}{}' \frac{1}{|n_1 + \tau n_2|^3} - \frac{3\hbar(\tau^2)^{1/2}}{4\pi}\sum_{n_1 n_2}{}' \frac{(\tau^1 n_1 + |\tau|^2 n_2)^2}{|n_1 + \tau n_2|^5} \,, \quad (12c)$$

$$\langle T_{12}\rangle = \langle T_{21}\rangle = \frac{\hbar\tau^1(\tau^2)^{1/2}}{4\pi}\sum_{n_1 n_2}{}' \frac{1}{|n_1 + \tau n_2|^3} - \frac{3\hbar(\tau^2)^{1/2}}{4\pi}\sum_{n_1 n_2}{}' \frac{(n_1 + \tau^1 n_2)(\tau^1 n_1 + |\tau|^2 n_2)}{|n_1 + \tau n_2|^5} \,, \quad (12d)$$

$$\langle T_{0a}\rangle = \langle T_{a0}\rangle = 0 \quad (a = 1, 2) \,. \quad (12e)$$

[4] This simplification occurs because $\langle T_{\alpha\beta}(g)\rangle$ for a conformally invariant field, with the conformal vacuum, on a conformally flat spacetime is completely determined by $\langle T_{\alpha\beta}(\eta)\rangle$ and the trace anomaly $\langle T^\alpha_\alpha(g)\rangle$, while the latter vanishes when the spacetime dimension is odd [2].

[5] The prime attached to the Σ symbol, as in Eq. (11), indicates that the zero mode ($n_1 = n_2 = 0$) should be excluded from the summation whenever it causes a divergence.

[6] For computation it is helpful to note that $\langle T_{\alpha\beta}(\eta)\rangle = \frac{1}{2}\partial_\alpha\partial_{\beta'}G^{(1)}_{\sim}$, where $\partial_\alpha\partial_{\beta'}G^{(1)}_{\sim} := \partial_{x^\alpha}\partial_{x'^\beta}G^{(1)}(x-x')_{|x'\to x}$ and $x^\alpha := (t, \xi^1, \xi^2)$.

For a metric $g_{\alpha\beta} = V(-1, \hat{h}_{ab})$, $\langle T_{\alpha\beta}(g)\rangle$ can be obtained by Eq. (9). Since the Planck scale is the only scale which comes into our model, we understand that a suitable power of $\alpha := l_{\rm Planck}$ is multiplied to quantities like those in Eqs. (12a)–(12e), if necessary, in order to adjust their physical dimensions. These contributions of order $\hbar$ to $\langle T_{\alpha\beta}\rangle$ in Eqs. (12a)–(12e) originate from a nontrivial spatial topology $\Sigma \simeq T^2$ and are well known as the Casimir effect [2,3].

III. BACK REACTION OF THE CASIMIR EFFECT ON THE TOPOLOGICAL DEGREES OF FREEDOM

A. Extraction of dynamics of the modular deformations

Having computed $\langle T_{\alpha\beta}(g)\rangle$ in the previous section, we shall now investigate the back reaction of $\langle T_{\alpha\beta}(g)\rangle$ on the evolution of the spacetime. We consider the Einstein gravity on $\mathcal{M} \simeq T^2 \times \mathbf{R}$ and a massless conformally coupled scalar field on it, $S = \frac{1}{\alpha}\int R\sqrt{-g} + S_m$, where $\alpha := l_{\rm Planck}$ and S_m is given by Eq. (7). The canonical formulation is suitable to investigate the temporal evolution of the spacetime. We thus perform a (2+1) decomposition, but care should be taken because of the presence of the conformally coupled field. In the back-reaction problem, we regard that $\psi^2(x)$ is replaced by a vacuum expectation value $\langle\psi^2(x)\rangle$, which is independent of spatial coordinates. Furthermore, we shall finally choose the spatial coordinates such that $N^a = 0$ so that $n^a = (-1/N, \vec{0})$. These facts simplify the procedure of (2+1) decomposition.

Following the standard manipulation [20], we finally get the total action in canonical form

$$S = \int (\pi^{ab}\dot{h}_{ab} - N\mathcal{H} - N^a\mathcal{H}_a) , \qquad (13a)$$

where the Hamiltonian constraint and the momentum constraint become, respectively,

$$\mathcal{H} = \{(K_{ab}K^{ab} - K^2 - {}^{(2)}R)/\alpha + \langle T_{\alpha\beta}\rangle n^\alpha n^\beta\}\sqrt{h} , \qquad (13b)$$

$$\mathcal{H}_a/\sqrt{h} = -2D_b(K_a{}^b - \delta_a{}^b K)/\alpha - \langle T_{a\beta}\rangle n^\beta . \qquad (13c)$$

Here N and N_a are the lapse and the shift functions, $n^\alpha = (-1/N, N^a/N)$ is the normal unit vector of the spatial surface, and ${}^{(2)}R$ stands for the scalar curvature for the spatial surface Σ. The operator D_a is the covariant derivative with respect to h_{ab} and $\pi^{ab} := (K^{ab} - Kh^{ab})\sqrt{h}/\alpha$; K_{ab} is the extrinsic curvature of a spatial surface.[7]

[7]Throughout this paper, we use a spatial metric h_{ab}, an induced metric on a spatial surface Σ, to raise and lower the spatial indices $a, b, c, \ldots$ and to define the spatial covariant derivative D_a. In particular, the geometry of our concern is given by the line element $ds^2 = -dt^2 + V\,dl^2$, with (2). Thus the spatial metric in our model is $h_{ab} = V\hat{h}_{ab}$, with (3).

We choose a coordinate system such that $N^a = 0$ so that $n^\alpha = (-1/N, 0)$. Thus $\langle T_{a\beta}\rangle n^\beta = -1/N$. $\langle T_{a0}\rangle = 0$ $(a = 1, 2)$ from Eq. (12e). In our case, thus, the momentum constraint becomes

$$\mathcal{H}_a/\sqrt{h} = -2D_b(K_a{}^b - \delta_a{}^b K)/\alpha = 0 . \qquad (13c')$$

Then we can extract the moduli degrees of freedom (corresponding to the global deformations of a torus) by solving Eq. (13c′) explicitly [13].

The system of coordinates in our model [$ds^2 = -dt^2 + V\,dl^2$ with (2), (3)] corresponds to York's time slicing [21], i.e., the time slicing by the spatial surfaces on which

$$\sigma := -K/\alpha = \text{const} . \qquad (14)$$

Thus Eq. (13c′) is equivalent to

$$\alpha\mathcal{H}_a/\sqrt{h} = -2D_b\tilde{K}_a^b = 0 , \qquad (13c'')$$

where $\tilde{K}_a^b := K_a{}^b - \frac{1}{2}\delta_a{}^b K$, the traceless part of $K_a{}^b$. It means that[8] $\tilde{K}^{ab} \in \text{Ker}P_1^\dagger$, $\{\Psi^{Aab}\}_{A=1,2}$:

$$\tilde{K}^{ab} = \frac{1}{\alpha}\sum_{A=1}^{2} p_A \Psi^{Aab} . \qquad (15)$$

In our case, $\Sigma \simeq T^2$, we can choose the lapse function N as $N = N(t)$ without any contradiction with the York's slices. This is shown almost in the same manner as for the case of pure (2+1) gravity [12,13]. Now, using some basic facts on the moduli space $\mathcal{M}_{g=1}$ (see Appendix A), it is straightforward [13] to show that our system is reduced to

$$S = \int dt\, \sigma\frac{dV}{dt} + \sum_{A=1}^{2} p_A \frac{d\tau^A}{dt}$$

$$-\frac{N(t)}{\alpha}\left(\sum_{A,B=1}^{2} G^{AB}p_A p_B - \tfrac{1}{2}\alpha^2\sigma^2 V + \alpha\langle T_{\alpha\beta}\rangle n^\alpha n^\beta V\right) . \qquad (16)$$

Note that the contribution from the spatial diffeomorphism has been eliminated from dynamics by solving the momentum constraint (13c′) explicitly. Only the Weyl deformations and the modular deformations have remained.

B. Evolution of the Teichmüller parameters caused by the back reaction

In our model, $ds^2 = V(-dt'^2 + \hat{h}_{ab}d\xi^a d\xi^b)$. Thus, from Eq. (9) and $n^\alpha = (-1/\sqrt{V}, 0)$, we get $\langle T_{\alpha\beta}\rangle n^\alpha n^\beta =$

[8]See Appendix A for the terminology and notations related to the moduli space.

$V^{-3/2}\langle T_{00}\rangle$, where $\langle T_{00}\rangle$ is given by Eq. (12a). (Note that this combination is coordinate independent.)

By setting $N(t)=1$, we get the canonical equations of motion described by the constraint function

$$\alpha H = \sum_{A,B=1}^{2} \mathcal{G}^{AB} p_A p_B - \tfrac{1}{2}\alpha^2\sigma^2 V - \hbar\alpha(\tau^2)^{3/2} f(\tau) V^{-1/2} = 0\,, \tag{17}$$

where

$$f(\tau^1,\tau^2) := \frac{1}{4\pi}\sum_{n_1,n_2=-\infty}^{\infty}{}' \frac{1}{|n_1+\tau n_2|^3} = \frac{1}{4\pi}\sum_{n_1,n_2=-\infty}^{\infty}{}' \frac{1}{(n_1^2+2\tau^1 n_1 n_2+|\tau|^2 n_2^2)^{3/2}}\,. \tag{18}$$

Clearly, $f(-\tau^1,\tau^2)=f(\tau^1,\tau^2)$, $f(\tau^1+n,\tau^2)=f(\tau^1,\tau^2)$, $f(n+a,\tau^2)=f(n-a,\tau^2)$ (n an integer, a real) and $f(\tau^1,\tau^2)$ is singular at $(\tau^1,\tau^2)=(n,0)$. Furthermore, the combination $2\pi(\tau^2)^{3/2}f(\tau^1,\tau^2)$ appearing in (17) is equivalent to the nonholomorphic Eisenstein series $G(\tau,3/2)$, whose modular invariance as well as other properties are well known [22]. The first term in Eq. (17) is also modular invariant, since it behaves as a scalar field on the moduli space.[9] Thus the Hamiltonian constraint, Eq. (17), is modular invariant as it should be. Figures 1(a), 1(b) show the behavior of the function $f(\tau^1,\tau^2)$.

For the explicit investigation of the dynamics, let us first calculate $\mathcal{G}^{AB}$ according to Eqs. (A5) and (A2c) with

$$h_{ab} = \frac{V}{\alpha^2\tau^2}\begin{pmatrix} 1 & \tau^1 \\ \tau^1 & |\tau|^2 \end{pmatrix}\,.$$

(Note that $ds^2=-dt^2+V\,dl^2$.) Then we get

$$\mathcal{T}_{1ab} = \frac{V}{\alpha^2\tau^2}\begin{pmatrix} 0 & 1 \\ 1 & 2\tau^1 \end{pmatrix}\,,$$

$$\mathcal{T}_{2ab} = \frac{V}{\alpha^2(\tau^2)^2}\begin{pmatrix} -1 & -\tau^1 \\ -\tau^1 & (\tau^2)^2-(\tau^1)^2 \end{pmatrix}\,. \tag{19a}$$

Note that $\{\mathcal{T}_{Aab}\}_{A=1,2}$ are symmetric, traceless two-tensors satisfying $-2D_b\mathcal{T}_{Aa}{}^b = -2\partial_b\mathcal{T}_{Aa}{}^b = 0$. Thus $\{\mathcal{T}_{Aab}\}_{A=1,2}$ can also be utilized to form a basis for $\mathrm{Ker}P_1^\dagger$, $\{\Psi^{Aab}\}_{A=1,2}$. By normalizing them to satisfy $(\Psi^A,\mathcal{T}_B)=\delta^A{}_B$, we obtain

$$\Psi^1_{ab} = \frac{1}{2}\begin{pmatrix} 0 & \tau^2 \\ \tau^2 & 2\tau^1\tau^2 \end{pmatrix}\,,$$

$$\Psi^2_{ab} = \frac{1}{2}\begin{pmatrix} -1 & -\tau^1 \\ -\tau^1 & (\tau^2)^2-(\tau^1)^2 \end{pmatrix}\,. \tag{19b}$$

Thus the Weil-Peterson metric reduces to the one which is conformally equivalent to the Poincaré metric:

$$\mathcal{G}_{AB} = (\mathcal{T}_A,\mathcal{T}_B) = \frac{2V}{\alpha^2(\tau^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

$$\mathcal{G}^{AB} = (\Psi^A,\Psi^B) = \frac{\alpha^2(\tau^2)^2}{2V}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{20}$$

Hence the geometry conformal to the Poincaré geometry

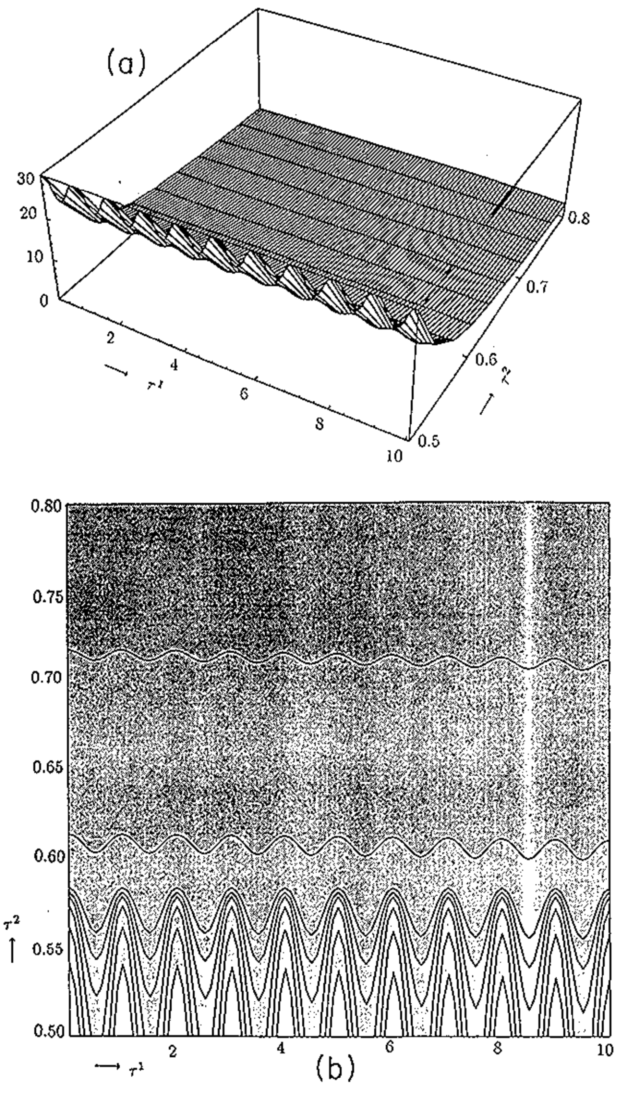

FIG. 1. Plot of the function $f(\tau^1,\tau^2)$ for the range τ^1:0–1 and τ^2:0.5–0.8. The infinite summation has been truncated at −200 and 200. (b) Contour plot of $f(\tau^1,\tau^2)$, with the same range and the truncation points as in (a). The lines indicate the values (from bottom to top) 30, 28, 26, 24, 22, 20, 15, and 5.

[9] Another convenient way for discussing the invariance is to perform the Legendre transformation of the action in concern and to look at the action in terms of the configuration variables. In this case, the kinetic term for (τ^1,τ^2) becomes proportional to $\sum \mathcal{G}_{AB}\dot\tau^A\dot\tau^B$, which is clearly modular invariant. For the discussion in the context of the path integral, including the discussion on the path-integral measure, see Sec. IV A.

[23] (negative constant curvature geometry) is endowed on the Teichmüller space, which is equivalent to the upper half-plane H_+ ($(\tau^1, \tau^2) \in \mathbf{R} \times \mathbf{R}_+$). Then the system has been finally reduced to the constrained system $((V,\sigma),(\tau^1,p_1),(\tau^2,p_2);H=0)$ with

$$\alpha H = \frac{\alpha^2(\tau^2)^2}{2V}(p_1^2+p_2^2) - \tfrac{1}{2}\alpha^2\sigma^2 V - \hbar\alpha(\tau^2)^{3/2}f(\tau)V^{-1/2} = 0\ . \qquad (21)$$

The equation of motion for (V,σ) are

$$\dot{V} = -\alpha\sigma V\ , \qquad (22a)$$

$$\dot{\sigma} = \frac{\alpha}{2}\sigma^2 + \frac{\alpha(\tau^2)^2}{2V^2}(p_1^2+p_2^2) - \frac{\hbar}{2}(\tau^2)^{3/2}f(\tau)V^{-3/2}\ . \qquad (22b)$$

The equations of motion for (τ^1,p_1) and (τ^2,p_2) are

$$\dot{\tau}^1 = \frac{\alpha}{V}(\tau^2)^2 p_1\ , \qquad (23a)$$

$$\dot{p}_1 = \hbar(\tau^2)^{3/2}\frac{\partial f(\tau)}{\partial\tau^1}V^{-1/2}\ , \qquad (23b)$$

$$\dot{\tau}^2 = \frac{\alpha}{V}(\tau^2)^2 p_2\ , \qquad (24a)$$

$$\dot{p}_2 = -\frac{\alpha}{V}\tau^2(p_1^2+p_2^2) + \frac{3\hbar}{2}(\tau^2)^{1/2}f(\tau)V^{-1/2} + \hbar(\tau^2)^{3/2}\frac{\partial f(\tau)}{\partial\tau^2}V^{-1/2}\ . \qquad (24b)$$

First, we should note that the time evolution becomes trivial when there is no matter field, $f(\tau)\equiv 0$, in the following sense: In this case, Eqs. (22a),(22b) allow a solution, $\sigma\equiv 0$, $V=\text{const}$, $p_1=p_2\equiv 0$. It is clear that, from Eqs. (21), (23a), (23b), (24a), and (24b), equations of motion do not allow any solution, compatible with $\sigma\equiv 0$, $V=\text{const}$, other than $\tau^1=\text{const}$, $\tau^2=\text{const}$. This corresponds to the three-dimensional Minkowski space in the standard coordinates (T,X^1,X^2) with suitable identifications in spatial section (X^1,X^2) described by (τ^1,τ^2). The unique solution above shows that there is no time evolution with respect to the standard time slice, $T=\text{const}$ ($\sigma=0$). This configuration is what we have chosen as a background spacetime. [However, there are different solutions characterized by the initial condition $\sigma\neq 0$. In these cases, (τ^1,τ^2) evolve in time.]

The back reaction of the quantum field causes a nontrivial evolution of (τ^1,τ^2), i.e., global deformations of a torus. It is clear from Eq. (21) that even when $\sigma\approx 0$ so that the term $-\frac{1}{2}\alpha^2\sigma^2V$ in Eq. (21) can be neglected, a nontrivial evolution of (τ^1,τ^2) occurs because of the negativity of the term $-\hbar\alpha(\tau^2)^{3/2}f(\tau)V^{-1/2}$ in Eq. (21). The choice of the solution $\sigma\equiv 0$, $V\equiv\text{const}$ is not allowed any more, as is seen from Eqs. (22a), (22b).

Figures 2(a), 2(b), and 2(c) show a typical example of the evolution of (τ^1,τ^2), (p_1,p_2), and (V,σ), respectively. Units such that $\hbar=1$ and $\alpha=1$ have been chosen. We have set the initial conditions for (τ^1,τ^2), p_1, σ, and V. The initial condition for p_2 has been decided using the constraint equation (21). We can observe the universal asymptotic behavior of the system, which arises irrespective of the initial conditions, due to the back reaction: The back reaction drives the system into the direction corresponding to a thinner torus, i.e., $\tau^2\to 0$, while $\tau^1\to$ finite. At the same time, the two-volume V asymptotically approaches zero. We find out that this behavior is universal by setting various generic initial conditions. This universal behavior can also be understood by investigating the qualitative characteristics of Eqs. (21)–(24), which shall be done in the next subsection.

We should also note a special class of trajectories characterized by the initial condition[10] $\tau^1=n/2$ (n an integer), $p_1=0$. The (τ^1,τ^2) trajectory becomes parallel to the τ^2 axis and (p_1,p_2) trajectory is on the p_2 axis. Depending on whether $p_2>0$ or $p_2>0$, τ^2 tends to ∞ or 0, respectively. In any case, the shape of the torus becomes thinner and thinner as it evolves. (Note the modular invariance of the system.)

C. Asymptotic behavior of the system

We can understand the universal behavior of the system by looking at Eqs. (21)–(24) and investigating the asymptotic behavior of the system as $t\to\infty$. Some key types of behavior are (i) $V\to 0$, $\sigma\to\infty$, $\sigma V^n\to\infty$ ($n=1,2,3,\ldots$), (ii) $(\tau^2)^2(p_1^2+p_2^2)$ increases, at least as fast as[11] σ^2V^2, (iii) $\tau^2\downarrow 0$ or $\tau^2\to\infty$, and (iv) $p_1\dot{\tau}^1+p_2\dot{\tau}^2$ increases at least as σ^2V, and $\frac{1}{(\tau^2)^2}[(\dot{\tau}^1)^2+(\dot{\tau}^2)^2]$ increases at least as σ^2.

Now let us derive the above results. First of all, Eq. (22b) can be written with the help of Eq. (21) as

$$\dot{\sigma} = \alpha\sigma^2 + \frac{\hbar}{2}(\tau^2)^{3/2}f(\tau)V^{-3/2}\ . \qquad (22b')$$

Thus $\dot{\sigma}>0$, so that σ always increases and becomes positive at some stage. Then V decreases because of Eq. (22a). Furthermore, it is easily shown that

[10] Because of the modular invariance of the system, the cases of $\tau^1=$ integer are equivalent to the case of $\tau^1=0$ and those of $\tau^1=$ half integer are equivalent to the case of $\tau^1=1/2$. The trajectories of the former cases are stable against perturbations, while the trajectories of the latter cases are unstable. This can be seen from Figs. 1(a), 1(b) along with Eq. (17).

[11] Here "$y(t)$ increases at least as fast as $x(t)$" or "$y(t)$ increases at least as $x(t)$" means that, $|x(t)/y(t)|\to c$, $0\le c<\infty$ when $t\to\infty$. In other words, $1/y(t)=O(1/x(t))$ when $t\to\infty$.

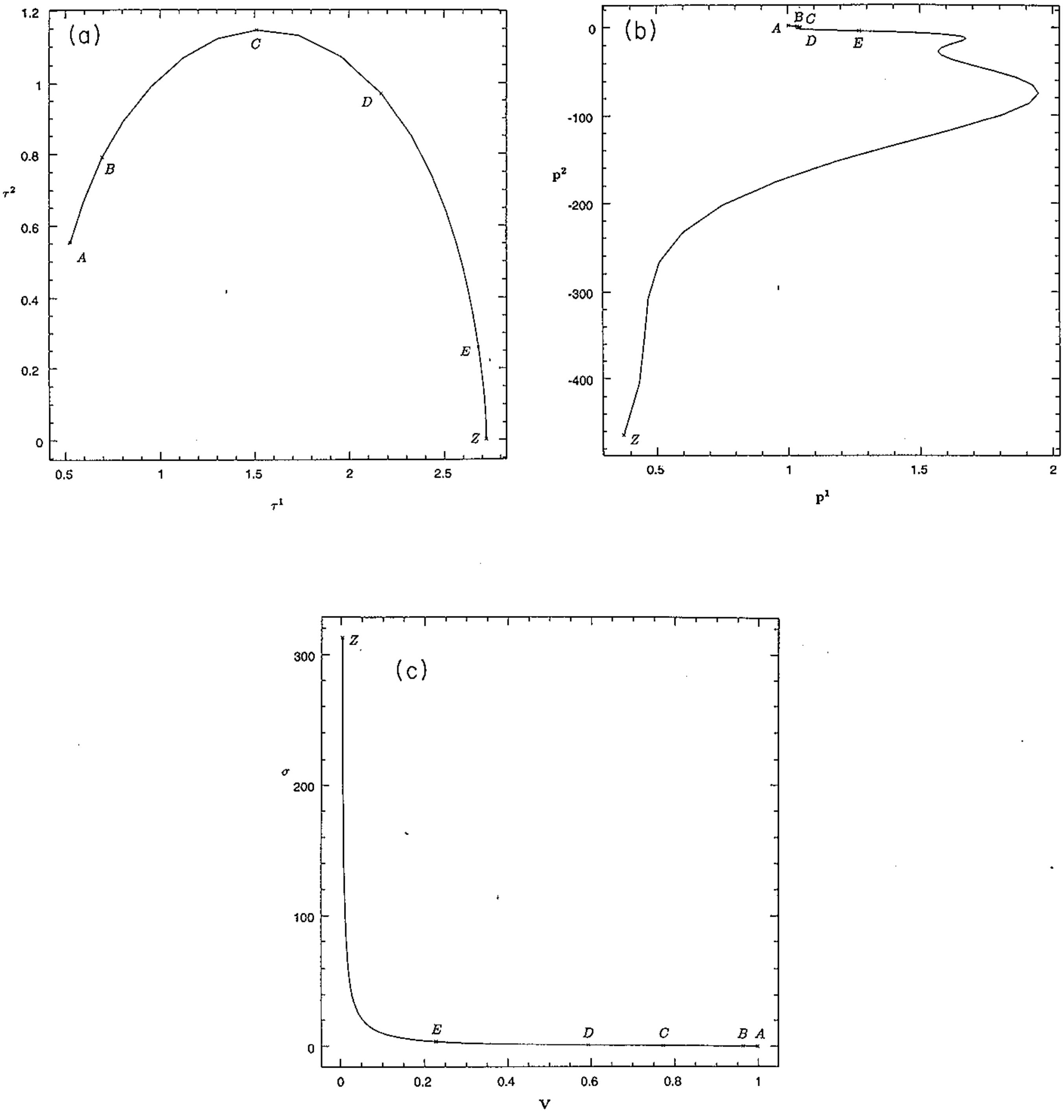

FIG. 2. (a) Trajectory of (τ^1, τ^2) determined by Eqs. (21)–(24). The infinite summation in the definition of $f(\tau)$ has been truncated at -200 and 200. $\hbar$ and α have been set to unity. The initial conditions are $\tau^1 = 0.500$, $\tau^2 = 0.500$, $p_1 = 1.000$, $p_2 = 2.175$, $V = 1.000$, and $\sigma = 0.000$. Points A–E and Z indicate typical points on the trajectory. A, the initial point; B–E, the points for which t is near half integer; Z, the end point of the calculation. C is also a point near the turning point ($p_2 = 0$); A, (0.500,0.500) at $t = 0.000$; B, (0.695,0.791) at $t = 0.454$; C, (1.510,1.145) at $t = 1.116$; D, (2.170,0.970) at $t = 1.480$; E, (2.681,0.260) at $t = 1.995$; and Z, $(2.723, 2.574 \times 10^{-3})$ at $t = 2.223$. (b) Trajectory of (p_1, p_2) determined by Eqs. (21)–(24). The initial conditions are the same as in (a). A, (1.000,2.175) at $t = 0.000$; B, (1.037,1.101) at $t = 0.454$; C, $(1.047, 2.873) \times 10^{-2}$ at $t = 1.116$; D, (1.047,-0.648) at $t = 1.480$; E, (1.271,-4.527) at $t = 1.995$; and Z, $(0.374, -4.641 \times 10^2)$ at $t = 2.223$. (c) Trajectory of (V, σ) determined by Eqs. (21)–(24). The initial conditions are the same as in (a). A, (1.000,0.000) at $t = 0.000$; B, (0.963,0.170) at $t = 0.454$; C, (0.773,0.539) at $t = 1.116$; D, (0.593,0.969) at $t = 1.480$; E, (0.228,3.817) at $t = 1.995$; and Z, $(3.711 \times 10^{-3}, 3.129 \times 10^2)$ at $t = 2.223$.

$(\sigma^2V^2)^{\cdot} = \hbar(\tau^2)^{3/2}f(\tau)\sigma V^{1/2} > 0$, so that the combination σ^2V^2 always increases in time. (Therefore, σ^2V increases more strongly.) Afterwards, it is easy to get (i) by induction. Next, from Eq. (21), $(\tau^2)^2(p_1^2+p_2^2)$ increases, at least in the same manner as σ^2V^2. Thus we get (ii). Now, from Eqs. (24b) and (21),

$$\dot{p}_2 = -\frac{\alpha}{4V}\tau^2(p_1^2+p_2^2) - \frac{3\alpha}{4\tau^2}\sigma^2V + \hbar(\tau^2)^{3/2}\frac{\partial f(\tau)}{\partial\tau^2}V^{-1/2} , \qquad (24b')$$

so that $\dot{p}_2 < 0$ [note that $\frac{\partial f(\tau)}{\partial\tau^2} < 0$]. Furthermore, $|\dot{p}_2| > \frac{3\alpha}{4\tau^2}\sigma^2V$, so that $\dot{p}_2$ decreases faster than $-\frac{\sigma^2V}{\tau^2}$. (Note that, if $\tau^2 \to$ finite, this implies a strong deceleration of p_2.) This fact excludes the behavior $\tau^2 \to$ finite, for, if so, $p_2 \propto V\frac{(\dot{\tau}^2)^2}{(\tau^2)^2}$ [Eq. (24a)] should tend to zero, which contradicts with the deceleration of p_2. Therefore τ^2 always behaves as $\tau^2 \downarrow 0$ or $\to \infty$. Thus we get (iii). Next, we see that $p_1\dot{\tau}^1 + p_2\dot{\tau}^2$ $\left[= \frac{\alpha(\tau^2)^2}{V}(p_1^2+p_2^2)\right]$ [Eqs. (23a),(24a)] increases at least as σ^2V, with the help of Eq. (21). Finally, $\frac{V}{(\tau^2)^2}[(\dot{\tau}^1)^2+(\dot{\tau}^2)^2](\propto p_1\dot{\tau}^1+p_2\dot{\tau}^2)$ increases at least as σ^2V, so that $\frac{1}{(\tau^2)^2}[(\dot{\tau}^1)^2+(\dot{\tau}^2)^2]$ increases at least as σ^2. Thus we get (iv).

The generic trajectories are the ones for which $\tau^1 \to$ finite and $\tau^2 \to 0$, like Fig. 2. We can understand this behavior as follows: Suppose that $|\dot{\tau}^1|$ is at most comparable with $|\dot{\tau}^2|$. Then, from (iv), we can make an estimation as $\frac{\dot{\tau}^2}{\tau^2} \sim -\sigma$, so that τ^2 rapidly approaches 0 [faster than $\exp(-\sigma t)$ since σ is increasing]. Noting that V^{-1} increases much slower than σ [(i)], the combinations of the form $(\tau^2)^nV^{-m}$ in (23a), (23b) become strong suppression factors. This is compatible with the assumption that $|\dot{\tau}^1|$ is not so large compared with $|\dot{\tau}^2|$. Therefore, in (24b), only the term proportional to p_2^2 on the right-hand side dominates and determines the gross properties of the equation, which gives rise to the universal behavior.

There is a special class of trajectories determined by the initial condition $\tau^1 = 0$ [or, in general, $\tau^1 = n/2$ (integer)] and[12] $p_1 = 0$. Because of the property of $f(\tau^1,\tau^2)$ [Eq. (18)] with Eqs. (23a), (23b), this implies that $\tau^1 \equiv 0$ (or $\equiv n/2$), $p_1 \equiv 0$; i.e., the trajectories of (τ^1,τ^2) and (p_1,p_2) form a line segment on (or parallel to) the τ^2 axis and p_2 axis, respectively. Combining (iv) with $p_1 \equiv 0$, we see that $p_2\dot{\tau}^2$ always increases. It means that any (τ^1,τ^2) trajectory which is parallel to the τ^2 axis has no turning point and that τ^2 tends to 0 or ∞, depending on the initial condition. Furthermore, combining again (iv) with $p_1 \equiv 0$ and $\tau^1 =$ const, we see that $\frac{\dot{\tau}^2}{\tau^2} \sim \pm\sigma$, so that τ^2 approaches rapidly ∞ or 0 [faster than $\exp(\pm\sigma t)$ since σ is increasing].

As is noted previously, our treatment is based on the adiabatic approximation. Thus the results should always be taken with a caveat. In general, when instability is observed in the adiabatic treatment, it implies the unstable tendency of the system and it suggests the necessity of a further investigation beyond the adiabatic approximation, rather than just neglecting the resultant instability. Furthermore, in the present case, there are good reasons to regard the unstable behavior as a real one. First, as investigated above, the universal asymptotic behavior of the generic trajectories implies that $\dot{\tau}^1 \to 0$, $\dot{\tau}^2 \to 0$ [and $\dot{V} \sim o(\sigma)$], although $p_2 \to -\infty$. This is because $\dot{\tau}^1 \propto \frac{(\tau^2)^2p_1}{V}$, $\dot{\tau}^2 \propto \frac{(\tau^2)^2p_2}{V}$, and τ^2 becomes a strong suppression [stronger than $\exp(\sigma t)$], while $1/V$ is at most $\sim \sigma$. Thus the adiabatic treatment for τ^1 and τ^2 becomes better and better as $\tau^2 \to 0$: $\dot{\omega}_A/\omega_A^2 \sim \dot{\lambda}_A/\lambda_A^{3/2} \sim (\tau^2)^{3/2}\frac{\dot{\tau}^2}{(\tau^2)^2} = (\tau^2)^{1/2}\left(\frac{\dot{\tau}^2}{\tau^2}\right) \to 0$ [see Eq. (5)]. Furthermore, $\dot{V}$ does not harm the adiabatic treatment because of the conformal invariance of the matter field, as has already been discussed previously [Sec. IIB, after Eq. (9)]. See Fig. 3. [On the other hand, we should also note that the special class of trajectories characterized by $\tau^1 \equiv n/2$ (n an integer), $\tau^2 \to \infty$, is not appropriate for the adiabatic treatment: By (iv), τ^2 tends to infinity even stronger than $\exp(\sigma t)$. However, because of the modular invariance, the trajectories for which $\tau^1 \equiv$ const and $\tau^2 \downarrow 0$ give the good information of the class of these trajectories.]

Another support for the present result comes from the consideration of the case of the negative cosmological constant without matter field. It is straightforward to introduce the Λ term [24] [see Eq. (16)]:

$$\alpha H = \frac{\alpha^2(\tau^2)^2}{2V}(p_1^2+p_2^2) - \frac{1}{2}\alpha^2\sigma^2V - \alpha\Lambda V = 0 , \qquad (21\Lambda)$$

$$\dot{V} = -\alpha\sigma V , \qquad (22\Lambda a)$$

$$\dot{\sigma} = \frac{\alpha}{2}\sigma^2 + \frac{\alpha(\tau^2)^2}{2V^2}(p_1^2+p_2^2) + \Lambda , \qquad (22\Lambda b)$$

$$\dot{\tau}^1 = \frac{\alpha}{V}(\tau^2)^2p_1 , \qquad (23\Lambda a)$$

$$\dot{p}_1 = 0 , \qquad (23\Lambda b)$$

$$\dot{\tau}^2 = \frac{\alpha}{V}(\tau^2)^2p_2 , \qquad (24\Lambda a)$$

$$\dot{p}_2 = -\frac{\alpha}{V}\tau^2(p_1^2+p_2^2) . \qquad (24\Lambda b)$$

Here, $-\Lambda$ corresponds to the cosmological constant ($\Lambda > 0$). Because of the negativity of the last term in Eq. (21Λ), the same kind of evolution for (τ^1,τ^2) as in the case of the matter field is observed. [It is also notable that (21Λ)–(24Λb) can be solved analytically [24].]

[12]The remarks for the last paragraph of Sec. III B apply here, too. See the footnote there.

It strongly suggests that the instability is independent of the adiabatic treatment. At the same time, we should note the essential difference between our case and the case of the negative cosmological constant. Especially, the difference between (23b) and (23Λb) is prominent. Furthermore, $\hbar(\tau^2)^{3/2}f(\tau)V^{-3/2}$ [which corresponds to Λ comparing (21) with (21Λ)] depends on (τ^1,τ^2) and V, which causes a highly nontrivial evolution.

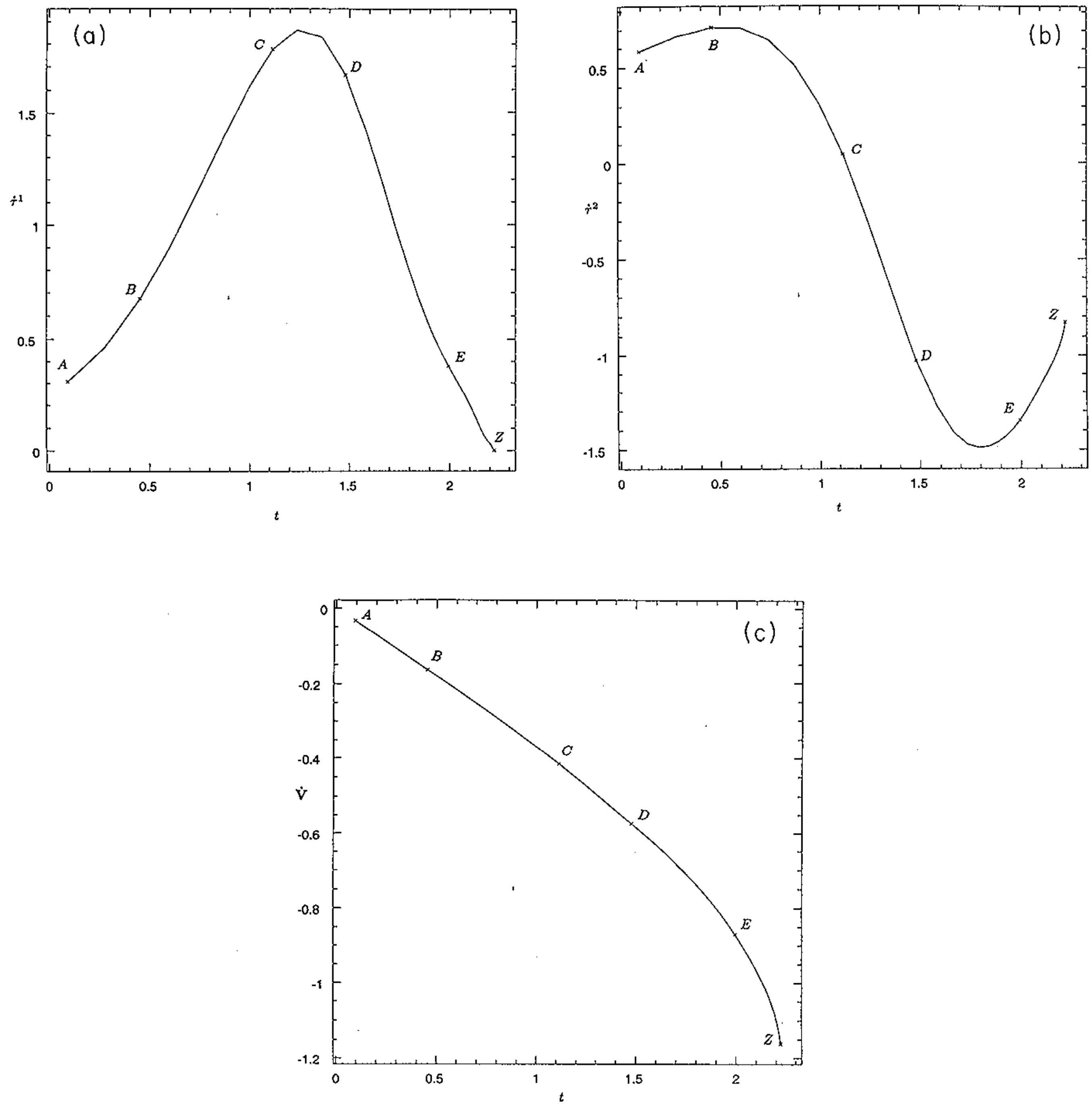

FIG. 3. (a) Value of $\dot{\tau}^1$ during the evolution shown in Figs. 2(a)–2(c). A, 0.250 at $t = 0.000$; B, 0.674 at $T = 0.454$; C, 1.776 at $T = 1.116$; D, 1.662 at $t = 1.480$; E, 0.378 at $t = 1.995$; and Z, 6.683×10^{-4} at $t = 2.223$. (b) Value of $\dot{\tau}^2$ during the evolution shown in Figs. 2(a)–2(c). A, 0.545 at $t = 0.000$; B, 0.716 at $t = 0.454$; C, 4.871×10^{-2} at $t = 1.116$; D, -1.029 at $t = 1.480$; E, -1.347 at $t = 1.995$; Z, -0.828 at $t = 2.223$. (c) Value of $\dot{V}$ during the evolution shown in Figs. 2(a)–2(c). A, 0.000 at $t = 0.000$; B, -0.164 at $t = 0.454$; C, -0.417 at $t = 1.116$; D, -0.574 at $t = 1.480$; E, -0.870 at $t = 1.995$; Z, -1.161 at $t = 2.223$.

IV. EFFECTIVE ACTION FOR THE MODULAR DEGREES OF FREEDOM

A. Partition function

We have treated so far the back reaction of the quantum field on the modular degrees of freedom, in the sense that the semiclassical Einstein equation (1) has been solved, with $\langle T_{\alpha\beta}\rangle$ on the right-hand side being calculated in the background spacetime. We can handle the same problem in a more systematic manner by the path-integral approach. The significance of this investigation is as follows.

First, we know that we can derive Eq. (1) formally by taking the first variation of the phase with respect to $g_{\alpha\beta}$ in the in-in path-integral expression for $g_{\alpha\beta}$ and ψ [5–7]. However, when we discuss the semiclassical gravity in more detail, it is preferable to take into account the effects coming from the path-integral measure of $g_{\alpha\beta}$. Since we cannot fix the measure in a reasonable manner, we usually do not discuss much this effect. Fortunately, our model is simple enough to investigate the measure to a great extent, by making use of the techniques developed in string theories [15,16].

Second, regarding this problem, we expect that the measure in the original phase space, $\int[dh_{ab}d\pi^{ab}dN\,dN_a]$, should reduce to the standard canonical measure in terms of the reduced phase space variables, $\int[d\tau^A dp_A][dV\,d\sigma][dN]$, after gauge fixing, according to the general theory of the path integral for the first-class constrained systems [25]. Analyzing this reduction process in detail for the case of our model is highly nontrivial and helpful for deeper understanding of the path-integral approach to quantum gravity [26].

Third, furthermore, our model also becomes a test candidate for another fundamental problem: the validity of the minisuperspace approach in quantum cosmology. It is essential in our reduction procedure that the condition of $N=$ const on Σ is compatible with the equations of motion (Sec. III A). In the context of quantum cosmology, it can correspond to the minisuperspace approach: We often impose the special form on metrics, which is compatible with the equations of motion, and quantize them within this subclass of metrics, for tractability. Then a fundamental question arises as to whether this approximate treatment reflects faithfully the main features of the full-quantized system. The results may depend on which space is chosen as the starting whole phase space, viz., whether we start from the full phase space (full quantization) or from its subspace (minisuperspace quantization). In the former case, it is expected that some extra factor emerges in the measure, since in this case the condition $N=$ const on Σ should be treated as an extra constraint, rather than just an *Ansatz*. If so, this extra factor can have some influence on the semiclassical evolution of the system. A similar effect can arise from our assumption of the spatial homogeneity of our torus model (Sec. II A). Our model is suitable for the detailed analysis of this fundamental problem. In the present paper, however, we restrict ourselves to the treatment in the manner of minisuperspace models, which itself is one consistent treatment.

Fourth, when we need to investigate validity conditions of the semiclassical treatment described by Eq. (1), then we have to study the second variation of the effective action $W[V_+,\tau_+;V_-,\tau_-]$ [7]. Thus we need to estimate $W[V_+,\tau_+;V_-,\tau_-]$ using the in-in path-integral formalism.

We first discuss within the framework of the standard in-out path-integral formalism [8] and later generalize it to the in-in formalism. In this subsection, we shall derive the expression for the partition function Z in terms of the reduced phase space variables. In the next subsection, we shall estimate the effective action for matter, $W[V_+,\tau_+;V_-,\tau_-]$.

The partition function in our case is given by

$$
\begin{aligned}
Z &= \mathcal{N}\int[dh_{ab}d\pi^{ab}dN\,dN_a][d\psi]\exp\left(i\int(\pi^{ab}\dot{h}_{ab}+p_\psi\dot{\psi}-N\mathcal{H}-N_a\mathcal{H}^a)\right)\\
&= \mathcal{N}\int[dh_{ab}d\pi^{ab}dN\,dN_a]\exp\left(i\int(\pi^{ab}\dot{h}_{ab}-N\mathcal{H}-N_a\mathcal{H}^a)\right)\ , \qquad (25)
\end{aligned}
$$

where, in the last line, we understand that the matter degrees ψ have been integrated out and suitable vacuum expectation values have appeared in $\mathcal{H}$ and $\mathcal{H}^a$ (e.g., $T_{\alpha\beta}n^\alpha n^\beta\to\langle T_{\alpha\beta}\rangle n^\alpha n^\beta$). (See the next subsection for a more explicit discussion.)

Integrating over the multiplier N_a is equivalent to inserting $\delta(\mathcal{H}^a)$ and setting N_a to be an arbitrary value if needed.[13] Let us set $N_a=0$:

$$
Z=\mathcal{N}\int[dh_{ab}d\pi^{ab}dN]\delta(\mathcal{H}^a)\exp\left(i\int(\pi^{ab}\dot{h}_{ab}-N\mathcal{H})\right). \qquad (26)
$$

The action is invariant under the time reparametrization and Diff(Σ) (the diffeomorphism on the spatial surface Σ). Now gauge fixing is needed to make this expression meaningful. The gauge-fixing condition for Diff(Σ), which is directly connected to our classical treatment in Sec. III, is

$$
h_{ab}-V\hat{h}_{ab}=0\ , \qquad (27)
$$

where $\hat{h}_{ab}$ is given by (3).

[13]This situation is parallel to the case of QED. In the latter case, $A_0\mathrm{div}\vec{E}$ appears in the action. One can set $A_0=0$ if needed, provided that $\delta(\mathrm{div}\vec{E})$ is inserted in the integrand.

At this stage, we need to fix our general attitude for the treatment of our model. Any two-dimensional metric h_{ab} is conformally flat [15,16], and the conformal factor V is a function of spatial coordinates (as well as a time parameter t) in general, $V = V(t, \xi^1, \xi^2)$. Here, furthermore, we set a further restriction to construct a tractable model, which we have investigated in the previous sections: We restrict the class of spatial metrics h_{ab} to the one in which V becomes spatially constant, $V = V(t)$. At the same time, the lapse function N is restricted to $N = N(t)$. Both of these *Ansätze* are compatible with the classical equations of motion. Such restrictions on the class of the path-integral variables correspond to the minisuperspace models in quantum cosmology. (See Sec. V for more discussions on this point.)

The treatment for the time-reparametrization invariance is well investigated [27]. The final result is neat: Introducing the physical time $T = \int^t dt\, N(t)$, one computes a transition amplitude from time 0 to time T. Then integrate over the result with respect to T [27]. Here we shall not do it explicitly, since we are mainly interested in the semiclassical evolution of the system. We understand that we follow the above procedure whenever needed.

Then

$$\begin{aligned} Z &= \mathcal{N} \int [dV\, dv'^a d^2\tau][dh_{ab} d\pi^{ab} dN]\delta(h_{ab} - V\hat{h}_{ab})\Delta_{\rm FP}\delta((P_1^\dagger \pi)^a) \exp(iS) \\ &= \mathcal{N} \int [dV\, dv'^a d^2\tau][d\tilde{\pi}^{ab} d\sigma] J [dN] \Delta_{{\rm FP}|h_{ab}=V\hat{h}_{ab}} \delta((P_1^\dagger \tilde{\pi})^a) \exp(iS_{|h_{ab}=V\hat{h}_{ab}}) \ , \end{aligned}$$

where $S = \int \pi^{ab}\dot{h}_{ab} - N\mathcal{H}$. Note that $h_{ab} = V(t)\hat{h}_{ab}$ corresponds to choosing the York's time slicing $K = \pi^a{}_a/V =$ const with respect to the spatial coordinates [21] (see Sec. III A). Thus only the traceless part of π^{ab}, $\tilde{\pi}^{ab} = \pi^{ab} - \frac{1}{2}\pi h^{ab} = \tilde{K}^{ab}V$, has remained in the argument of the δ function in the last line above. Accordingly, the change of the integral variables $\pi^{ab} \to (\tilde{\pi}^{ab}, \sigma)$ has been performed and J is the Jacobian factor associated with this change. Employing the method in Appendix B, J can be determined as follows: A natural diffeo-invariant inner product[14] for $\delta\pi^{ab}$ is $(\delta\pi, \delta\pi) = \int d^2\xi \sqrt{h} h_{ac} h_{bd} \delta\pi^{ab}\delta\pi^{cd}$. Substituting $\pi^{ab} = \tilde{\pi}^{ab} - \frac{1}{2}h^{ab}\sigma V$, we get $(\delta\pi, \delta\pi) = (\delta\tilde{\pi}, \delta\tilde{\pi}) + \frac{1}{2}V^2(\delta\sigma, \delta\sigma)$, where $(\delta\sigma, \delta\sigma) = \int d^2\xi\sqrt{h}(\delta\sigma)^2$. Thus $1 = J\int d\tilde{\pi}^{ab} d\sigma \exp[-(\delta\pi, \delta\pi)]$, so that $J = V$ up to an unimportant numerical factor.

The Faddeev-Popov determinant $\Delta_{\rm FP}$ in our case is equivalent to the Jacobian associated with the change of the integral variables from h_{ab} to $(V, v'^a, (\tau^1, \tau^2))$, where $v'^a \notin {\rm Ker}P_1$. Thus we can employ the method in Appendix B again to determine $\Delta_{\rm FP}$: From Eq. (A1),

$$\begin{aligned} \|\delta h_{ab}\|^2 &= \|\delta\phi h_{ab} + (P_1 v')_{ab} + \mathcal{T}_{Aab}\delta\tau^A\|^2 \\ &= 4(\delta\phi, \delta\phi) + (v', P_1^\dagger P_1 v') + (\mathcal{T}_A, \mathcal{T}_B)\delta\tau^A\delta\tau^B \ . \end{aligned}$$

Then (note that $d\phi = dV/V$)

$$1 = \Delta_{\rm FP}\int dV\, dv'\, d^2\tau \exp(-\|\delta h\|^2) = \Delta_{\rm FP}({\rm det}' P_1^\dagger P_1)^{-1/2} {\rm det}^{-1/2}(\mathcal{T}_A, \mathcal{T}_B)V^{-1} \ .$$

Thus

$$\Delta_{\rm FP} = ({\rm det}' P_1^\dagger P_1)^{1/2} {\rm det}^{1/2}(\mathcal{T}_A, \mathcal{T}_B)V \ .$$

Thus

$$Z = {\rm Vol}_{{\rm Diff}_0}\mathcal{N}\int [dV\, d^2\tau][d\tilde{\pi}^{ab} d\sigma][dN] \left(\frac{{\rm det}' P_1^\dagger P_1}{{\rm det}(\chi^\alpha, \chi^\beta)}\right)^{1/2} {\rm det}^{1/2}(\mathcal{T}_A, \mathcal{T}_B)V^2 \delta(\,(P_1^\dagger\tilde{\pi})^a) \exp(iS_{|h_{ab}=V\hat{h}_{ab}}) \ , \qquad (28)$$

where $\{\chi^\alpha\}_{\alpha=1,2}$ is the basis for ${\rm Ker}P_1$, a space of conformal Killing vectors.[15]

[14]An appropriate power of $\alpha := l_{\rm Planck}$ should be multiplied to the formulas in order to adjust physical dimensions like Eq. (A3). It is easy and not significant for the present discussions, and so we omit the factor.

[15]Any element in ${\rm Diff}_0$ (diffeomorphism on Σ homotopic to 1) is associated with a vector v^a, which can be decomposed as $v^a = v'^a + \lambda_\alpha\chi^\alpha$, where $v'^a \notin {\rm Ker}P_1$. Noting the argument in Appendix A, $\int [dv^a] = \int [dv'^a] d^2\lambda\, {\rm det}^{1/2}(\chi^\alpha, \chi^\beta)$, which means ${\rm Vol}_{{\rm Diff}_0} = (\int [dv'^a]){\rm Vol}_{{\rm Ker}P_1}$. Thus, by factorizing $(\int [dv'^a]) = {\rm Vol}_{{\rm Diff}_0}/{\rm Vol}_{{\rm Ker}P_1}$ from the path-integral, the factor ${\rm det}^{-1/2}(\chi^\alpha, \chi^\beta)$ appears. Here the factor $(\int d^2\lambda)^{-1}$ is absorbed into the normalization $\mathcal{N}$.

Let us investigate the factor $\delta((P_1^\dagger\tilde{\pi})^a)$. According to Eq. (C1) in Appendix C [$A = P_1^\dagger$, $\vec{x} = \tilde{\pi}^{ab}$, $f(\vec{x}) = \exp(iS)$, and $\{\Psi^A\}_{A=1,2}$ are the zero modes for $P_1^\dagger$],

$$\int [d\tilde{\pi}^{ab}]\delta((P_1^\dagger\tilde{\pi})^a)\exp(iS[h_{ab} = V\hat{h}_{ab}, \tilde{\pi}^{ab}, \sigma, N]) = \int [d^2p]\frac{\det^{1/2}(\Psi^A,\Psi^B)}{\det' P_1^\dagger}\exp\left(i\int dt(p_A\dot{\tau}^A + \sigma\dot{V} - N\mathcal{H})\right) . \quad (29)$$

Here, in the last line, the nonzero-mode components of $\tilde{\pi}^{ab}$ have been set to be zero according to the formula (C1). This is equivalent to substituting $\tilde{\pi}^{ab} = \tilde{K}^{ab}V = \sum_A p_A\Psi^{Aab}V$ into the action. Therefore this is the path-integral version of the procedure of solving the momentum constraint in Sec. III A.

Thus

$$Z = \mathcal{N}\int [d\tau^A dp_A][dV\,d\sigma][dN]\frac{\det'^{1/2}P_1^\dagger P_1}{\det'^{1/2}P_1P_1^\dagger}\frac{\det^{1/2}(\mathcal{T}_A,\mathcal{T}_B)}{\det^{1/2}(\chi^\alpha,\chi^\beta)}\det^{1/2}(\Psi^A,\Psi^B)V^2\exp\left(i\int dt(p_A\dot{\tau}^A + \sigma\dot{V} - N\mathcal{H})\right) . \quad (30)$$

Here $\det' P_1^\dagger = (\det' P_1P_1^\dagger)^{1/2}$ has been used.[16]

Now we choose $\{\mathcal{T}_A\}_{A=1,2}$ and $\{\Psi^A\}_{A=1,2}$ as $(\mathcal{T}_A,\Psi^B) = \delta_A^B$ (see Sec. III B), so that $\det^{1/2}(\mathcal{T}_A,\mathcal{T}_B)\det^{1/2}(\Psi^A,\Psi^B) = 1$. For our case of a locally flat torus, $\{\chi^\alpha\}_{\alpha=1,2}$ can be chosen as $\chi^{1a} = (1,0)$ and $\chi^{2a} = (0,1)$, without inducing any critical point as vector fields. Then $\det^{-1/2}(\chi^\alpha,\chi^\beta)\cdot V^2 = 1$.

Finally, $\det'^{1/2}P_1^\dagger P_1$ and $\det'^{1/2}P_1P_1^\dagger$ should be estimated. The map P_1 is a map from a space of two-vector fields to a space of second rank, symmetric and traceless tensor fields, and the map $P_1^\dagger$ is a map from the latter space to the former space. Note that each of the spaces can be represented as a two-component vector fields. Now it is convenient to use the complex coordinates $(z,\bar{z})$, with respect to which both P_1 and $P_1^\dagger$ become diagonal [15]. Let $z = x + iy$, $\bar{z} = x - iy$. Then the line element becomes ($e^\phi := V$), $ds^2 = e^\phi\hat{h}_{ab}d\xi^1 d\xi^2 = e^\phi(dx^2 + dy^2) = e^\phi dz\,d\bar{z}$, so that

$$h_{ab} = \begin{pmatrix} 0 & \frac{1}{2}e^\phi \\ \frac{1}{2}e^\phi & 0 \end{pmatrix}_{(z,\bar{z})} .$$

[The suffix $(z,\bar{z})$ is for the explicit indication of the coordinates employed.[17]] The following arguments are valid for a general spatial metric h_{ab} on a torus, so that we shall discuss in general terms. Only at the final stage [Eq. (34) below], we set the condition that $\phi = \ln V$ = spatially constant.

Now both P_1 and $P_1^\dagger$ can be regarded as a map from a two-component field to another two-component field:

$$P_1 : {}^t(v^z, v^{\bar{z}}) \mapsto {}^t((P_1v)^{zz}, (P_1v)^{\bar{z}\bar{z}}) ,$$

$$P_1^\dagger : {}^t(w^{zz}, w^{\bar{z}\bar{z}}) \mapsto {}^t((P_1^\dagger w)^z, (P_1^\dagger w)^{\bar{z}}) ,$$

where w^{ab} is a symmetric, traceless tensor field and ${}^t(\cdot,\cdot)$ indicates the transposition. In this sense, P_1 and $P_1^\dagger$ are represented as

$$P_1 = \begin{pmatrix} 4e^{-\phi}\bar{\partial} & 0 \\ 0 & 4e^{-\phi}\partial \end{pmatrix} ,$$

$$P_1^\dagger = \begin{pmatrix} -2e^{-2\phi}\partial e^{2\phi} & 0 \\ 0 & -2e^{-2\phi}\bar{\partial}e^{2\phi} \end{pmatrix} . \quad (31)$$

Thus $P_1^\dagger P_1 : {}^t(v^z, v^{\bar{z}}) \mapsto {}^t((P_1^\dagger P_1 v)^z, (P_1^\dagger P_1)^{\bar{z}})$ is represented as

$$P_1^\dagger P_1 = \begin{pmatrix} -8e^{-2\phi}\partial e^\phi\bar{\partial} & 0 \\ 0 & -8e^{-2\phi}\bar{\partial}e^\phi\partial \end{pmatrix}$$

$$= \begin{pmatrix} 2\Delta +^{(2)} R & 0 \\ 0 & 2\Delta +^{(2)} R \end{pmatrix} , \quad (32)$$

where $\Delta = -D_aD^a$ and ${}^{(2)}R$ are the Laplacian and the scalar curvature, respectively, defined by the covariant derivative (D_a) with respect to $e^\phi\hat{h}_{ab}$. Similarly, $P_1P_1^\dagger$: ${}^t(w^{zz}, w^{\bar{z}\bar{z}}) \mapsto {}^t((P_1P_1^\dagger w)^{zz}, (P_1P_1^\dagger w)^{\bar{z}\bar{z}})$ is represented as

$$P_1P_1^\dagger = \begin{pmatrix} -8e^{-\phi}\bar{\partial}e^{-2\phi}\partial & 0 \\ 0 & -8e^{-\phi}\partial e^{-2\phi}\bar{\partial} \end{pmatrix}$$

$$= \begin{pmatrix} 2\Delta - 2^{(2)}R & 0 \\ 0 & 2\Delta - 2^{(2)}R \end{pmatrix} . \quad (33)$$

Therefore

[16] This equality can be shown by estimating an integral $I = \int dw'^{ab}\exp[-(P_1^\dagger w', P_1^\dagger w')]$ in two different manners (here w'^{ab} is symmetric, traceless and $\notin \mathrm{Ker}P_1^\dagger$): One way is $I = \int dw'\exp[-(w', P_1P_1^\dagger w')] = (\det' P_1P_1^\dagger)^{-1/2}$, and the other way is $I = \int d(P_1^\dagger w')(\det' P_1^\dagger)^{-1}\exp[-(P_1^\dagger w', P_1^\dagger w')] = (\det' P_1^\dagger)^{-1}$. This change of the integral variables in the latter estimation is valid since the space of the original variables (w'^{ab}) is isomorphic as a vector space to the space of the new variables $[(P_1^\dagger w')^a]$ by the map $P_1^\dagger$. See below, in the text.

[17] We shall use the following facts: $\partial := \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$; $\vec{v} = (v^1, v^2)_{(x,y)} = (v^1 + iv^2, v^1 - iv^2)_{(z,\bar{z})}$, i.e., $v^z = v^1 + iv^2$, $v^{\bar{z}} = v^1 - iv^2 = \overline{v^z}(v^1, v^2 \in \mathbf{R})$. Let T^{ab} be symmetric and traceless and let its components in(x,y) coordinates, T^{11}, etc., are real, then $(T^{ab})_{(z,\bar{z})} = \mathrm{diag}(2(T^{11} + iT^{12}), 2(T^{11} - iT^{12}))$, i.e., $T^{zz} = 2(T^{11} + iT^{12})$, $T^{\bar{z}\bar{z}} = 2(T^{11} - iT^{12}) = \overline{T^{zz}}$, and the other components vanish; the Christoffel symbols become $\Gamma^z_{zz} = \partial\phi$, $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \bar{\partial}\phi = \overline{\Gamma^z_{zz}}$ and the others vanish.

$$\det{}'^{1/2} P_1^\dagger P_1 = \det{}'(2\Delta + ({}^{(2)}R) ,$$

$$\det{}'^{1/2} P_1 P_1^\dagger = \det{}'(2\Delta - 2{}^{(2)}R) .$$

In our model of locally flat tori ($\phi = \ln V$=spatially constant), thus,

$$\det{}'^{1/2} P_1^\dagger P_1 = \det{}'^{1/2} P_1 P_1^\dagger = \det{}'(2\Delta) .$$

Finally, we obtain

$$Z = \mathcal{N} \int [d\tau^A dp_A][dV\, d\sigma][dN] \times \exp\left(i \int (p_A \dot{\tau}^A + \sigma \dot{V} - N\mathcal{H})\right) . \quad (34)$$

The integral region for τ^1, τ^2 should be understood as on the moduli space, $\mathcal{M}_{g=1}$: As is indicated in Eq. (28), $\mathrm{Diff}_0(\Sigma)$ (the diffeomorphism group on Σ homotopic to 1) has been factorized from the path integral. What is really needed to be factorized is the whole diffeomorphism group on Σ, $\mathrm{Diff}(\Sigma)$. Note that [15,16]

$$\begin{aligned}\mathcal{M}_g &\simeq \mathrm{Riem}(\Sigma)/G_{\mathrm{Weyl}} \times \mathrm{Diff}(\Sigma) \\ &\simeq [\mathrm{Riem}(\Sigma)/G_{\mathrm{Weyl}} \times \mathrm{Diff}_0(\Sigma)]/G_{\mathrm{MC}} \\ &\simeq H_+/\mathrm{PSL}(2,\mathbf{Z}) \simeq D(H_+)/\sim .\end{aligned}$$

Here $G_{\mathrm{MC}} := \mathrm{Diff}(\Sigma)/\mathrm{Diff}_0(\Sigma)$ is the mapping-class group for Σ and $G_{\mathrm{MC}} \simeq \mathrm{PSL}(2,\mathbf{Z})$ for $\Sigma \simeq T^2$ (i.e., a group of 2×2 unimodular matrices with integer elements, modulo sign). $D(H_+)$ is the fundamental region in H_+ (upper half-plane) with respect to the action of $\mathrm{PSL}(2,\mathbf{Z})$ (e.g., the Dirichlet region $D = \{z \in H_+ | |\mathrm{Re} z| \leq \frac{1}{2}, |z| \geq 1\}$) and $/\sim$ indicates the identification $(\tau^1, \tau^2) \sim -(\tau^1, \tau^2)$ on the boundary of D [15,16]. Thus the integral region for (τ^1, τ^2) in Eq. (34) should be understood as over $\mathcal{M}_{g=1}$ rather than over H_+, considering that we have factorized the volume of the mapping-class group $G_{\mathrm{MC}} \simeq \mathrm{PSL}(2,\mathbf{Z})$ as well $\mathrm{Diff}_0(\Sigma)$ from the path integral.

If we integrate out the momenta p_A and σ in Eq. (34), we get

$$Z = \mathcal{N} \int \left[\frac{d\tau^A}{(\tau^2)^2}\right] \left[\frac{dV}{\sqrt{V}}\right] [dN] \exp\left(i \int dt\, N(t) \left\{\frac{V}{2\alpha(\tau^2)^2} \frac{1}{N^2}[(\dot{\tau}^1)^2 + (\dot{\tau}^1)^2] - \frac{1}{2\alpha N^2 V}\dot{V}^2 + \hbar(\tau^2)^{3/2} f(\tau V^{-1/2}\right\}\right) . \quad (35)$$

Note that the kinetic term for (τ^1, τ^2) in the action is proportional to $\mathcal{G}_{AB}\dot{\tau}^A\dot{\tau}^B$ and the last term in the action is proportional to the nonholomorphic Eisenstein series $G(\tau, \frac{3}{2})$ [see below Eq. (18)]. Thus Z is modular invariant since both the measure $\frac{d^2\tau}{(\tau^2)^2}$ and the action are modular invariant.

B. Estimation of the functional determinant for the matter

Now we estimate the path integral for the matter ψ in Eq. (25). Our aim is to obtain the effective action of the form $W[V, \tau^1(\cdot), \tau^2(\cdot)]$ by integrating out quantum fluctuations of the matter. Generalizing the framework to the in-in formalism and getting $W[V_+, \tau_+; V_-, \tau_-]$, one can discuss the validity conditions for the semiclassical treatment [7], Eq. (1). At this stage, the peculiarity of the system including gravity is prominent. In the standard treatment of a dissipative system, like a quantum Brownian motion [17], the interaction between the subsystem and the environment is described by a weak, linear coupling. In our case, however, there is no such interaction term between gravity (analogous to the subsystem) and matter (analogous to the environment). Rather, the interaction is bilinear in ψ and nonlinear in (τ^1, τ^2) and V, as is seen from Eq. (7). Thus it requires a new treatment for a deeper analysis. Here we should be content with only a rough estimation of the effect of the nonlinear coupling. We want to estimate the partition function for the matter:

$$\begin{aligned} Z_\psi &= \int [d\psi] \exp\left\{-\frac{1}{2\hbar} \int \psi(-\tilde{\partial}^2 + \tfrac{1}{8}\tilde{R})\psi\sqrt{\tilde{g}}\right. \\ &= \mathrm{Det}^{-1/2}\left.\left[\frac{V}{2\pi\hbar}(-\tilde{\partial}^2 + \tfrac{1}{8}\tilde{R})\right]\right\} \\ &= \exp\left(-\frac{1}{\hbar}\tilde{W}[\tau(\cdot)]\right) . \end{aligned}$$

Here "$\tilde{\ }$" denotes the Riemannian signature quantity. We calculate using the metric $\tilde{g}_{\alpha\beta} = (1, V\hat{h}_{ab})$ with Eq. (3). It is difficult to estimate the above functional determinant exactly for a general function $(\tau^1(\cdot), \tau^2(\cdot))$ and $V(\cdot)$. From the viewpoint of the quantum dissipative system, this difficulty comes from the peculiarity of the interaction between gravity and matter. As discussed in the beginning of Sec. II B, we treat the back-reaction problem in the sense that we investigate the modification of the background geometry due to matter, i.e., due to $\langle T_{\alpha\beta}\rangle$ calculated on the background spacetime. We have chosen as a background a flat spacetime. Thus, for the lowest order approximation, we treat τ^1, τ^2, and V as constants, so that we can set $\tilde{R} = 0$. This treatment corresponds to the lowest order estimation of the functional form of the effective potential in standard quantum field theory [28].

Thus we need to estimate the determinant of the operator

$$\hat{A} := -\frac{\alpha^2 V}{2\pi\hbar}\tilde{\partial}^2 = -\frac{\alpha^2 V}{2\pi\hbar}(\partial_0^2 + V^{-1}\hat{h}^{ab}\partial_a\partial_b) ,$$

where $\hbar$ and α^2 have been inserted for the convenience of recovering a formula for pseudo-Riemannian signature. Now we need to solve the heat equation [28]

$$\hat{A}\rho = -\frac{\partial}{\partial s}\rho ,$$

$$\lim_{s\downarrow 0}\rho(x,y,s) = \delta^{(3)}(x-y) .$$

Here $x := (x^0 = t, \xi^1, \xi^2)$. Taking care of the periodicity in space, the solution is given by

$$\rho(x,x',s) = \left(\frac{\hbar}{2\alpha^2 V s}\right)^{3/2}\sum_{n_1,n_2}\exp\left(-\frac{\pi\hbar}{2\alpha^2 V s}\{(x^0-x^{0'})^2 + V\hat{h}_{ab}(\xi-\xi'+n)^a(\xi-\xi'+n)^b\}\right) ,$$

especially,

$$\rho(x,x,s) = \left(\frac{\hbar}{2\alpha^2 V s}\right)^{3/2}\sum_{n_1,n_2}\exp\left(-\frac{\pi\hbar}{2\alpha^2 s}(n,n)\right) ,$$

where $(n,n) := \hat{h}_{ab}n^a n^b = \frac{1}{\tau^2}(n_1^2 + 2\tau^1 n_1 n_2 + |\tau|^2 n_2^2)$. Thus the ζ function associated with $\hat{A}$ is [28]

$$\begin{aligned}\zeta_A(z) &= \frac{1}{\Gamma(z)}\int_0^\infty ds\, s^{z-1}\mathrm{Tr}\rho(s) \\ &= \frac{\Omega}{\Gamma(z)}\sum_{n_1 n_2}\int_0^\infty ds\, s^{z-1}\left(\frac{\hbar}{2\alpha^2 V s}\right)^{3/2}\exp\left(-\frac{\pi\hbar}{2\alpha^2 s}(n,n)\right) \\ &= \left(\frac{\hbar}{2\alpha^2 V}\right)^{3/2}\Omega\frac{z\Gamma(\frac{3}{2}-z)}{\Gamma(z+1)}\sum_{n_1 n_2}\left(\frac{\pi\hbar}{2\alpha^2}(n,n)\right)^{z-3/2} ,\end{aligned} \tag{36}$$

where $\Omega = \int d^3x\sqrt{g}$ and a transformation of variable s $[x := \frac{\pi\hbar}{2\alpha^2}(n,n)s^{-1}]$ has been done to get the formula in the last line from the middle line. Noting that $\frac{d}{dz}|_{z=0}\left(\frac{z\Gamma(\frac{3}{2}-z)}{\Gamma(z+1)}C^{z-l}\right) = \frac{\sqrt{\pi}}{2}C^{-l}$ for $\forall C$ when C is independent of z, we get

$$\zeta_A'(0) = \frac{\Omega}{2\pi\alpha^3 V^{3/2}}\sum_{n_1 n_2}{}'(n,n)^{-3/2} .$$

Thus we get

$$\begin{aligned}\tilde{W} &= \frac{\hbar}{2}\ln \mathrm{Det}\hat{A} = -\frac{\hbar}{2}\zeta_A'(0) \\ &= -\frac{\hbar\Omega}{4\pi\alpha^3 V^{3/2}}(\tau^2)^{3/2} \\ &\quad\times\sum_{n_1 n_2}{}'\frac{1}{(n_1^2 + 2\tau^1 n_1 n_2 + |\tau|^2 n_2^2)^{3/2}} .\end{aligned} \tag{37a}$$

To recover W for the pseudo-Riemannian signature, we replace $\hbar \to i\hbar$, $\alpha \to i\alpha$ (no change in Ω, $d\tilde{x}^0dx^1dx^2dx^3 \leftrightarrow dx^0dx^1dx^2dx^3$). This replacement comes from the comparison between $W = i\frac{\hbar}{2}\ln \mathrm{Det}\left[\frac{i\alpha^2}{2\pi\hbar}(-\partial^2)\right]$ and $\tilde{W} = \frac{\hbar}{2}\ln \mathrm{Det}\left[\frac{\alpha^2}{2\pi\hbar}(-\tilde{\partial}^2)\right]$. Thus

$$\begin{aligned}W[\tau^1,\tau^2] &= \frac{\hbar\Omega}{4\pi\alpha^3 V^{3/2}}(\tau^2)^{3/2} \\ &\quad\times\sum_{n_1 n_2}{}'\frac{1}{(n_1^2 + 2\tau^1 n_1 n_2 + |\tau|^2 n_2^2)^{3/2}} .\end{aligned} \tag{37b}$$

Since we have used the expectation value of the energy-momentum tensor for the matter, $\langle T_{\alpha\beta}\rangle$, to couple with gravity [Eq. (13b) or (16)], we need to use the in-in path-integral formalism, rather than the standard in-out formalism [5–7]. Then the matter part of the action (pseudo-Riemannian) [see Eq. (7)] should be reinterpreted as

$$\begin{aligned}S_\psi &= -\frac{1}{2}\int_c (g^{\alpha\beta}\partial_\alpha\psi\,\partial_\beta\psi + \tfrac{1}{8}\bar{R}\psi^2)\sqrt{-g} \\ &= -\frac{1}{2}\int_+ (g^{\alpha\beta}\partial_\alpha\psi\,\partial_\beta\psi + \tfrac{1}{8}R\psi^2)\sqrt{-g} + \tfrac{1}{2}\int_- (g^{\alpha\beta}\partial_\alpha\psi\,\partial_\beta\psi + \tfrac{1}{8}R\psi^2)\sqrt{-g} ,\end{aligned}$$

where c stands for the closed-time contour and $+$ and $-$ stand for, respectively, the $+$ branch and the $-$ branch of the time contour. Then

$$S_\psi = \int_+ \sqrt{-g_+}\psi \frac{1}{2\hbar}(\partial^2 - \tfrac{1}{8}R)\psi - \int_- \sqrt{g_-}\psi \frac{1}{2\hbar}(\partial^2 - \tfrac{1}{8}R)\psi$$
$$= \int (\psi_+ \psi_-) \begin{pmatrix} \frac{1}{2\hbar}(\partial_+^2 - \frac{1}{8}R)\sqrt{-g_+} & 0 \\ 0 & -\frac{1}{2\hbar}(\partial_-^2 - \frac{1}{8}R)\sqrt{-g_-} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} .$$

Since $+$ and $-$ components are separated completely, it is enough to look at only the $+$ sector (or $-$ sector).

Now let us investigate the effective action $S[\tau^1,\tau^2,V;N] = S_g[\tau^1,\tau^2,V;N] + W[\tau^1,\tau^2,V;N]$, where $S_g[\tau^1,V;N]$ is the reduced action for gravity in terms of the configuration variables and $W[\tau^1,\tau^2,V;N]$ is given by Eq. (37b). The effective action $S[\tau^1,\tau^2,V;N]$ is what has appeared in the exponent in Eq. (35). It should be noted that the first variations of $S[\phi,\tau^1,\tau^2]$ with respect to N, V, and (τ^1,τ^2) reproduce exactly Eqs. (21)–(24). This result shows the following two points.

First, our approximation for the estimation of $\text{Det}^{-1/2}\left[\frac{1}{\hbar}(-\tilde{\partial}^2 + \frac{1}{8}\tilde{R})\right]$, treating τ^1, τ^2, and V as if they were constants so that $\tilde{R}=0$, corresponds to the approximation used to solve the semiclassical Einstein equation (1). Namely, $\langle T_{\alpha\beta}\rangle$, calculated on a flat background, is used in Eq. (1) to estimate the deviation from the original background geometry. As is discussed at the beginning of Sec. II B, the latter approximation has been implemented for the tractability of the problem, at the expense of the self-consistency of Eq. (1). Such an approximation is what is usually meant by the term "back reaction," and this may be the best we can do in practice.

Second, regarding the path-integral expressions in the Lagrangian formalism, like Eq. (35): We can reproduce Eq. (1) [or, equivalently, Eqs. (22)–(24)] from the phase part $S_g + W$ in the partition function Z with the matter part integrated and without taking care of the contributions from the measure for V and (τ^1,τ^2) [see Eq. (35)]. However, we now know explicitly the nontrivial path-integral measure for V and (τ^1,τ^2) as is shown in Eq. (35). There should be an $O(\hbar)$ correction to Eq. (1) coming from the path-integral measure for $g_{\alpha\beta}$, and this correction will cause a nontrivial correction to the dynamics of $g_{\alpha\beta}$. We shall come back to this point in the next section.

V. DISCUSSION

In this paper, we have investigated the semiclassical dynamics of the topological degrees of freedom, (τ^1,τ^2), which has been seldom discussed so far. By reducing the spacetime dimension to 3, we could concentrate on the study of a finite number of topological modes and we could describe the back-reaction effect from matter to topological modes, explicitly. We observed a nontrivial dynamics caused by the back reaction. The back reaction makes the toroidal universe unstable: The shape of the torus becomes thinner and thinner, while its total two-volume becomes smaller and smaller. These are universal behaviors of the system independent of the initial conditions, which is justified by the asymptotic analysis of the set of dynamical equations. This observation implies the importance of the investigation of topological aspects for a deeper understanding of quantum gravity. Moreover, we could fix the path-integral measure for (τ^1,τ^2) and V and observe that the partition function is expressed in terms of the canonical variables for the reduced phase space with the standard Liouville measure.

Let us note a few points regarding the path-integral measure.

We obtained the path-integral expression on the reduced phase space with the Liouville measure [Eq. (34)], while the path integral on the configuration space requires a nontrivial measure [Eq. (35)]. Indeed, the combination $\frac{d\tau^A}{(\tau^2)^2}$ is essential to make the partition function modular invariant. In our model, the semiclassical Einstein equation (1) corresponds to Eqs. (21)–(24) and they are derived from the variation of the exponent in Eqs. (34) or (35). It means that, from the viewpoint of the Lagrangian formalism, the semiclassical Einstein equation is derived from the variation of the phase part in the partition function, with the measure factor untouched. Thus the measure factor gives the $O(\hbar)$ correction to Eq. (1). In our model, the term $\int dt\, N(t)\hbar(2\ln\tau^2 + \frac{1}{2}\ln V)$ can be added to the action as a correction. [Note the time reparametrization invariance implied in Eq. (34).] Then it is a nontrivial question worthwhile to investigate which is better as the semiclassical description, the semiclassical Einstein equation in terms of the canonical variables [Eqs. (21)–(24) in our case], or the same in terms of the configuration variables with suitable corrections originating from the measure. If we perform the path integral exactly, both the canonical and Lagrangian formalisms will give equivalent results, but they will not be equivalent within the accuracy of the semiclassical approximation.

Another important problem is linked with the validity of the minisuperspace treatment. We have investigated the homogeneous model, which is equivalent to assuming $N = N(t)$, $V = V(t)$ [see the discussion in Sec. IV A, below Eq. (27)]. We can set this *Ansatz* since it is compatible with the dynamics. This treatment corresponds to the minisuperspace approach in quantum cosmology. Though such a treatment is completely self-consistent, it is important to question to what extent such a treatment reflects the original full quantum theory faithfully. From the viewpoint of the original full system, the restrictions are regarded as extra constraints on the phase space. These constraints can modify the path integral measure for the reduced variables (minisuperspace variables). Since this problem is a fundamental one, it should be investigated separately. Our model may be a good test candidate to investigate this point in detail.

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APPENDIX A: BRIEF SUMMARY ON THE MODULI SPACE

We give here a concise summary on the moduli space just for fixing the terminology and notations used in Secs. III and IV. See, e.g., [15,16] for more detailed information.

Let Σ be a two-dimensional, compact, closed, orientable manifold with genus g. The moduli space $\mathcal{M}_g$ of Σ is defined as $\mathcal{M}_g \simeq \mathrm{Riem}(\Sigma)/G_{\mathrm{Weyl}} \times \mathrm{Diff}(\Sigma)$, where $\mathrm{Riem}(\Sigma)$ is a space of all Riemannian metrics on Σ, G_{Weyl} is for the Weyl group, and $\mathrm{Diff}(\Sigma)$ is for the diffeomorphism group on Σ. The universal covering space of $\mathcal{M}_g$ is called the Teichmüller space. Now the tangent space of $\mathcal{M}_g$, $T(\mathcal{M}_g)$, can be investigated as follows: Any variation of the spatial metric $\delta h_{ab} \in T(\mathrm{Riem}(\Sigma))$ can be decomposed into the trace part and the traceless part; the latter is furthermore decomposed into the diffeomorphism $\delta_D h_{ab}$ and the moduli deformation $\delta_M h_{ab}$:

$$\begin{aligned}\delta h_{ab} &= \delta_W h_{ab} + \delta_D h_{ab} + \delta_M h_{ab} \\ &= \delta_W h_{ab} + (P_1 v)_{ab} + \mathcal{T}_{Aab}\delta\tau^A \ , \end{aligned} \tag{A1}$$

where

$$\delta_W h_{ab} = \delta\phi h_{ab} \quad \text{for } \exists \delta\phi \ , \tag{A2a}$$

$$(P_1 v)_{ab} = D_a v_b + D_b v_a - D_c v^c h_{ab} \quad \text{for } \exists v^a \ , \tag{A2b}$$

$$\mathcal{T}_{Aab} = \frac{\partial h_{ab}}{\partial \tau^A} - \frac{1}{2} h^{cd} \frac{\partial h_{cd}}{\partial \tau^A} h_{ab} \ . \tag{A2c}$$

Here $\{\tau^A\}_{A=1,\ldots,\dim_{\mathbf{R}}\mathcal{M}_g}$ are the Teichmüller parameters specifying a point in $\mathcal{M}_g$. A natural inner product on $T(\mathrm{Riem}(\Sigma))$ is introduced as

$$(A,B) := \frac{1}{\alpha^2}\int_\Sigma d^2x\sqrt{h}h^{ac}h^{bd}A_{ab}B_{cd} \quad \text{for } \forall A_{ab}, \forall B_{ab} \in T(\mathrm{Riem}(\Sigma)) \ , \tag{A3}$$

where α is the Planck length, inserted to adjust the physical dimension. Then the tangent space of the moduli space, $T(\mathcal{M}_g)$, can be characterized by the set of all symmetric, traceless (covariant) tensors which are perpendicular to $T(\mathrm{Diff}(\Sigma))$ with respect to the inner product (A3), the latter condition being equivalent to the condition

$$(P_1^\dagger w)^a = -2D_b w^{ab} = 0 \tag{A4}$$

for $w \in T^*(\mathcal{M}_g)$. Thus $\dim_{\mathbf{R}}\mathcal{M}_g = \dim_{\mathbf{R}}T^*(\mathcal{M}_g) = \dim_{\mathbf{R}}\mathrm{Ker}P_1^\dagger$, which is known as $=0$, $=2$, and $=6g-6$ for $g=0$, $g=1$, and $g\geq 2$, respectively. It is also known that $\dim_{\mathbf{R}}\mathrm{Ker}P_1 - \dim_{\mathbf{R}}\mathrm{Ker}P_1^\dagger = 6-6g$ (Riemann-Roch theorem). For the case of a torus ($g=1$), then, $\dim_{\mathbf{R}}\mathcal{M} = 2$ and $\dim_{\mathbf{R}}\mathrm{Ker}P_1 = 2$. Thus two Teichmüller parameters (τ^1, τ^2) are needed to describe the modular deformations $\delta_M h_{ab} \in T(\mathcal{M}_{g=1})$ and two independent vectors $\{\chi^\alpha\}_{\alpha=1,2}$ are needed as the basis of $\mathrm{Ker}P_1$.

Let $\{\mathcal{T}_{Aab}\}_{A=1,2,\ldots,\dim_{\mathbf{R}}\mathcal{M}_g}$ be the basis of $T(\mathcal{M}_g)$ and $\{\Psi^{Aab}\}_{A=1,2,\ldots,\dim_{\mathbf{R}}\mathcal{M}_g}$ be the basis of $T^*(\mathcal{M}_g)$. They can be chosen to satisfy $(\Psi^A, \mathcal{T}_B) = \delta^A{}_B$. Then they define a metric on $\mathcal{M}_{g=1}$ (the Weil-Peterson metric), induced from the inner product, Eq. (A3), on $T(\mathrm{Riem}(\Sigma))$:

$$\begin{aligned}\mathcal{G}_{AB} &= (\mathcal{T}_A, \mathcal{T}_B) \ , \\ \mathcal{G}^{AB} &= (\Psi^A, \Psi^B) = \text{inverse matrix of } \mathcal{G}_{AB} \ . \end{aligned} \tag{A5}$$

APPENDIX B: THE JACOBIAN ASSOCIATED WITH A CHANGE OF INTEGRAL VARIABLE

Let us note a convenient method to specify the Jacobian associated with a change of integral variables. (See, e.g., [15].)

If a line element ds is given on a space of integral variables (X^A, $A=1,2,\ldots,n$) as $ds^2 = G_{AB}dX^AdX^B =: (dX,dX)$, then $d^nX\sqrt{\det G}$ is a natural integral measure, where $\sqrt{\det G}$ takes care of the Jacobian factor. Suppose we change the variables from X^A to $X^{A'}$; then, $d^nX'\sqrt{\det G'}$ is the corresponding integral measure for the new variables. Now a convenient way to find out the expression for $\sqrt{\det G'}$ is (1) express δX^A in terms of $\delta X^{A'}$, $\delta X^A = \frac{\partial X^A}{\partial X^{A'}}\delta X^{A'}$, (2) then express $(\delta X, \delta X)$ in terms of $\delta X'$, $(\delta X,\delta X) = \frac{\partial X^A}{\partial X^{A'}}\frac{\partial X^B}{\partial X^{B'}}(\delta X^{A'},\delta X^{B'})$ (this should be equivalent to $G_{A'B'}\delta X^{A'}\delta X^{B'}$), and (3) then determine the Jacobian J by setting $1 = J\int d^n\delta X' \exp[-(\delta X,\delta X)]$, since this should be equivalent to $1 = J(\det G'/\pi)^{-1/2}$. (The factor π is usually unimportant and omitted.)

APPENDIX C: A FORMULA FOR THE δ FUNCTION

Let us derive a formula which modifies an integral including $\delta(A(\vec{x}))$ into a more practical form. Here A is a linear operator possibly with zero modes.

Let us consider an integral, $I = \int d\vec{x}\, \delta(A\vec{x}) f(\vec{x})$. Let $\{\Psi^A\}$ $(A = 1, 2, \ldots, m = \dim \mathrm{Ker} A)$ be the zero modes for A. Then any element $\vec{x}$ of a vector space $\mathcal{V}$ can be decomposed as $\vec{x} = \vec{X} + \sum_A p_A \vec{\Psi}^A$, where $\vec{X} \in \mathcal{V}/\mathrm{Ker}A$. Now we change the integral variables from $\vec{x}$ to $(\vec{X}, p_A)$. Then $(\vec{x}, \vec{x}) = (\vec{X}, \vec{X}) + (\vec{\Psi}^A, \vec{\Psi}^B) p_A p_B$, where $(\cdot, \cdot)$ is a suitable inner product, which is assumed to be given. Thus according to Appendix B, the associated Jacobian J becomes $J = \det^{1/2}(\Psi^A, \Psi^B)$. Thus

$$
\begin{aligned}
I &= \int d\vec{X}\, d\vec{p} \det{}^{1/2}(\Psi^A, \Psi^B) \delta(A\vec{X}) f(\vec{X}, \vec{p}) \\
&= \int d\vec{p} \det{}^{1/2}(\Psi^A, \Psi^B) (\det{}' A)^{-1} f(\vec{X} = \vec{0}, \vec{p}) ,
\end{aligned}
$$

where an equality $\delta(A\vec{X}) = (\det' A)^{-1} \delta(\vec{X})$ when $\vec{X} \in \mathcal{V}/\mathrm{Ker}A$ has been used in the last line. (This equality can be shown easily by the variable change from $\vec{X}$ to $\vec{Y} = A\vec{X}$.) Therefore we have obtained a formula

$$
\int d\vec{x}\, \delta(A\vec{x}) f(\vec{x}) = \int d\vec{p} \det{}^{1/2}(\Psi^A, \Psi^B) (\det{}' A)^{-1} f(\vec{X} = \vec{0}, \vec{p}) . \tag{C1}
$$

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