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Schwarzschild black hole immersed in a homogeneous electromagnetic field

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An exact and simple enough solution of the Einstein-Maxwell field equations is presented. This electrovacuum static axisymmetric solution possesses a clear physical interpretation: it is an external field of nonrotating uncharged mass immersed in homogeneous external electromagnetic and gravitational fields—the Bertotti-Robinson universe. Unlike the well known Ernst solution for a black hole in the Melvin universe, in our solution the black hole is immersed in a space-time with a completely spatially homogeneous magnetic (or electric) field and with a different ($R^2 \times S^2$) topology. The influence of this specific background space-time topology, the structure of curvature singularities, the relation between the laws of motion, and the condition of the absence of any unphysical noncurvature singularities as well as some other questions are considered. A brief sketch of the solution construction method used here and of its various applications, a comparison with other methods, as well as a general discussion concerning the construction of solutions for interacting fields are presented in the Appendix.

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I. INTRODUCTION

The development in the previous two decades of a number of powerful solution-generating techniques and some direct methods for the construction of stationary axisymmetric solutions of the vacuum Einstein equations as well as of the electrovacuum Einstein-Maxwell field equations provided the derivation in a more or less explicit form of wide families of solutions with an arbitrary large number of free parameters. First of all, these are the solutions derived by Bäcklund transformations [1,2] (and for the Minkowski flat background in an even more explicit determinant form [3]), Belinskii-Zakharov vacuum solitons [4,5] and their determinant form [6], explicit form of vacuum solutions derived by symmetry, or Hoenselaers-Kinnersley-Xanthopoulos (HKX) transformations [7–10], electrovacuum N-soliton solutions [11,12] with the determinant form of these solutions given in [13,14], or in another Bäcklund transformation [15] or the potential space [16] contexts, as well as a very large class of solutions in which the Ernst potentials on the symmetry axes $\rho = 0$ are arbitrary *rational* functions of the z coordinate. Its derivation was described in detail in [14] and its general explicit determinant form has been presented in [17].

However, in spite of such a large variety of formally known stationary axisymmetric solutions, only a restricted number of them possess some interesting physical interpretations. An obvious majority of the solutions derived by these methods belongs to the most simply interpretable subclass of stationary axisymmetric solutions—asymptotically flat ones. A considerable number of physically interesting solutions are known which are not asymptotically flat. Among the last ones are various electromagnetic "universes," such as the Melvin solution [18] (earlier, however, this solution was derived by Bonnor [19]) and the Bertotti-Robinson solution [20,21] (with z homogeneous or completely spatially homogeneous magnetic fields, respectively) as well as the particlelike solutions with additional physically interesting parameters, the C metric [22] and the more general Plebański-Demiański solution [23], or the solutions which describe the interactions of black holes with various external fields, such as the Schwarzschild black hole on the Weyl static background [24], the Ernst "electrified C metric" [25], which demonstrated the disappearance of nodal singularity on the symmetry axis for the special choice of parameters which provides the balance between the electromagnetic forces and acceleration of each of the two black holes, or at last, the extremely elegant Ernst solution [26] for the Schwarzschild black hole in the Melvin magnetic universe as well as its generalization for the Kerr-Newman black hole immersed in this magnetic universe [27].

The main purpose of this paper is to present some three-parameter family of exact static axisymmetric electrovacuum solutions given in a simple form which possesses a clear physical interpretation: this is a solution for the external gravitational and electromagnetic fields outside an uncharged nonrotating black hole (i.e., Schwarzschild black hole) immersed in a background space-time with spatially homogeneous pure magnetic (or, after a dual rotation, pure electric) field—the Bertotti-Robinson electromagnetic universe.

A number of interesting properties of this solution and important differences with the Ernst solution for a black

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hole in the Melvin universe will be discussed in the subsequent sections. These are the physical interpretation of parameters, the connection between the law of motion and the conditions for the absence of nodal singularities, the existence of nonzero acceleration of the black hole at rest in the background field (whose value is dependent upon the position of a black hole in this field), the influence of the specific background topology (which admits the existence in this space-time of two parallel axes of symmetry), and some others.

Because of the simplicity of the solution presented here and its direct physical interpretation, as well as of its nontrivial physical, geometrical, and topological properties, we hope it could be found interesting and useful for various purposes.

At first we had not planned to include in this paper any details concerning the method used for the derivation of the solution as well as some other pure mathematical questions, because such descriptions could overshadow the discussion of various physical aspects. We then restricted our explanation to a short comment only that we use the integral equation method developed in [32] and described later in more detail in [14], or even more precisely, we had referred to, derived by this method and presented in [17], a general explicit form of the class of solutions, which Ernst potentials are defined by their boundary values on the axis of symmetry given as *arbitrary rational* functions of the Weyl coordinate along this axis.

Now, however, for benefit of the readers who are also interested in pure mathematical questions, we present an extended version of our paper supplied by the Appendix with rather detailed discussion of the method itself, its most important applications, and the comparison with other methods, reiterating (sometimes even in a more convenient form) some useful results.

II. FORMAL SOLUTION IN BIPOLAR COORDINATES

The most simple form (at external glance, at least) our solution assumes in bipolar coordinates $\{x_1, y_1, x_2, y_2\}$. This form is

$$ds^2 = \frac{(x_2^2+1)(x_1+y_2)^2}{(x_1^2-1)}dt^2 - f(d\alpha^2+d\beta^2) - \frac{b^2(1-y_2^2)(x_1^2-1)}{(x_1+y_2)^2}d\varphi^2 , \qquad (1)$$

where $\{\alpha, \beta\}$ are the usual Weyl coordinates (standard notations $\{\rho, z\}$ for these coordinates are not used here, and they are reserved for another more convenient pair of conformally flat coordinates). The Weyl coordinates $\{\alpha, \beta\}$ and every two of the four coordinates $\{x_1, y_1, x_2, y_2\}$ can be expressed in terms of any two others, due to the following relationship between them:

$$\alpha^2 = m^2(x_1^2-1)(1-y_1^2) = b^2(x_2^2+1)(1-y_2^2) ,$$

$$\beta = \beta_1 + mx_1y_1 = \beta_2 + bx_2y_2, \quad \beta_1 - \beta_2 = l .$$

The real parameters m, b, and l will be the only essential parameters of our solution. The conformal factor f in (1) is given by the expression

$$f = f_0 \frac{(x_1+y_2)^2}{(x_1^2-y_1^2)(x_2^2+y_2^2)} \left[\frac{x_1 - y_1 + \gamma x_2 - (\delta+1)y_2}{x_1+y_1+\gamma x_2 - (\delta-1)y_2}\right]^2 ,$$

where the constant parameters $\gamma = b/m$, $\delta = l/m$. The arbitrary real constant f_0 has to be chosen as $f_0 = 1$; this is one of the necessary conditions for the absence of nodal singularities on the axis of symmetry.

The two nonzero components of a complex electromagnetic potential Φ_i (whose real parts are the corresponding components of the usual four-vector electromagnetic potential A_i) can be expressed in the form of

$$\{\Phi_t, \Phi_\varphi\} = \left\{-i\left(x_2 + \frac{1}{\gamma}y_1\right), -b\frac{(x_1+1)(1+y_2)}{x_1+y_2}\right\} ,$$

which are the potentials of a pure magnetic field.

III. EXPLICIT FORM OF THE SOLUTION IN CYLINDRICAL COORDINATES

For the analysis of this solution it will be more convenient to use two other systems of coordinates: the "internal" or Schwarzschild-like spherical coordinates $\{r, \theta\}$ and the "external" cylindrical coordinates $\{\rho, z\}$ which are better adapted for the geometry of the background Bertotti-Robinson space-time. These coordinates are defined by the relations

$$x_1 = (r-m)/m, \quad x_2 = -\sinh(z/b) ,$$

$$y_1 = \cos\theta, \quad y_2 = -\cos(\rho/b) .$$

The coordinates $\{r, \theta\}$ can be expressed in terms of $\{\rho, z\}$ in the form

$$r - m = \tfrac{1}{2}(R_+ + R_-), \quad \cos\theta = \frac{1}{2m}(R_- - R_+) ,$$
$$R_+ = \sqrt{\left[l + m - b\sinh\frac{z}{b}\cos\frac{\rho}{b}\right]^2 + b^2\cosh^2\frac{z}{b}\sin^2\frac{\rho}{b}} , \qquad (2)$$
$$R_- = \sqrt{\left[l - m - b\sinh\frac{z}{b}\cos\frac{\rho}{b}\right]^2 + b^2\cosh^2\frac{z}{b}\sin^2\frac{\rho}{b}} .$$

Then the metric (1) takes the form

$$ds^2 = \left(\cosh^2 \frac{z}{b}\right) \frac{\left[r - m - m \cos \frac{\rho}{b}\right]^2}{r(r-2m)} dt^2 - \tilde{f}(d\rho^2 + dz^2) - \frac{b^2 r(r-2m) \sin^2 \frac{\rho}{b}}{\left[r - m - m \cos \frac{\rho}{b}\right]^2} d\varphi^2 \ , \tag{3}$$

where, in accordance with (2) $r = r(\rho, z)$, and the conformal factor is

$$\tilde{f} = \frac{\left[r - m - m \cos \frac{\rho}{b}\right]^2}{(r-m)^2 - m^2 \cos^2\theta} \left[\frac{r - m - m \cos\theta - b \sinh \frac{z}{b} + (l+m) \cos \frac{\rho}{b}}{r - m + m \cos\theta - b \sinh \frac{z}{b} + (l-m) \cos \frac{\rho}{b}}\right]^2 \ . \tag{4}$$

The nonzero components of the complex electromagnetic potential are

$$\{\Phi_t, \Phi_\varphi\} = \left\{ i\left(\sinh\frac{z}{b} - \frac{m}{b}\cos\theta\right), -\frac{br\left[1 - \cos\frac{\rho}{b}\right]}{r - m - m\cos\frac{\rho}{b}} \right\} . \tag{5}$$

The next step is to show that the background Bertotti-Robinson solution and Schwarzschild solution are limiting cases of the solution (2)–(5) and to clarify the physical interpretation of the parameters m, b, and l.

IV. THE BACKGROUND LIMIT: $m = 0$

By setting $m = 0$ in (2)–(5) the background Bertotti-Robinson electromagnetic universe is obtained immediately:

$$ds^2 = \cosh^2\left(\frac{z}{b}\right) dt^2 - d\rho^2 - dz^2 - b^2 \sin^2\left(\frac{\rho}{b}\right) d\varphi^2 \ , \tag{6}$$

with the complex electromagnetic potential in the form

$$\{\Phi_t, \Phi_\varphi\} = \left\{ i \sinh\left(\frac{z}{b}\right), -b\left[1 - \cos\left(\frac{\rho}{b}\right)\right]\right\} \ . \tag{7}$$

To understand the nature of the parameter b, we consider the solutions (6) and (7) in more detail. First of all, we can see that the inverse value of the parameter b determines the strength of the electromagnetic [pure magnetic, as in (7)] field; the only nonzero component of this field in the orthonormal frame is

$$H_{\hat{z}} = \frac{1}{b} \ .$$

This field completely vanishes in the limit $b \to \infty$.

From the expression (6) for the background metric follows another, pure geometrical interpretation of the parameter b: two-sections $\{t = \text{const}, z = \text{const}\}$ of this space-time possess a homogeneous internal geometry which coincides with the geometry of the usual two-sphere of radius b. Hence the space-time (6) is closed in ρ directions, and the radius of this closure is $\rho = \pi b$.

If we consider the points of the axis $\rho = 0$ on these spheres as the poles, then the coordinate lines ρ will be directed along the meridians and the opposite ("antipodal") poles will constitute another, "antipodal," axis of symmetry. The distance between these axes of symmetry is also πb; i.e., it coincides with the radius of closure of this space-time in ρ directions. When $b \to \infty$ (while the values of ρ and z remain finite), the curvature of the two-surfaces $\{t = \text{const}, z = \text{const}\}$ vanishes, and the radius of the closure becomes infinite; i.e., this closure disappears and the space-time metric becomes flat:

$$ds^2 = dt^2 - d\rho^2 - dz^2 - \rho^2 d\varphi^2 \ .$$

This simple analysis shows that, for any linear or exact nonlinear perturbations of this geometry (or of other fields in this space-time) produced by the sources placed on the axis $\rho = 0$, it would be natural to expect some "cylindrical mirror" effects, i.e., the appearance of focusing of the strength lines of these fields at the "antipodal" axis $\rho = \pi b$. This focusing can give rise to the presence of some singularities there, which can be interpreted as additional sources of these fields.

Another interesting property of this background space-time is that in an arbitrary static frame, with the metric components of the form (6), the world line of any test particle at rest will not be geodesic provided its location does not coincide with the origin of this static frame. The acceleration of any such resting particle is given by the expression

$$\mathbf{W} = \{0, 0, W^z, 0\}, \quad W^z = \frac{1}{b} \tanh\left(\frac{z}{b}\right) \tag{8}$$

(hence $\mathbf{W} = 0$ if and only if $z = 0$). At the same time it is useful to note that, because of the space-time symmetry, any space-time point can be chosen as the origin of some static frame with metric (6).

Referring to the equivalence principle we can conjecture that for a black hole (at least of small enough mass) the similar situation takes place; to be at rest anywhere in some static frame in the external field (6) a black hole must possess nonzero acceleration. This acceleration will be zero; i.e., the black hole will fall freely in this external gravitational field if and only if its location (in some sense) will coincide with the origin of the static frame. As we shall see from a further analysis of the presented exact solution, this is just the case, and this "application" of the equivalence principle to a black hole of a finite mass turns out to be quite correct.

V. THE SCHWARZSCHILD LIMIT: $b \to \infty$

The electromagnetic field in the solution (2)–(5) in the limit $b \to \infty$ (keeping finite the values of ρ and z) completely vanishes and metric (3) coincides with the Schwarzschild metric in its standard form

$$ds^2 = \frac{r-2m}{r}dt^2 - \frac{r}{r-2m}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) ,$$

where we have used the limiting relation between (ρ, z) and (r, θ) coordinates

$$\rho^2 = r(r-2m)\sin^2\theta, \quad z = z_1 + (r-m)\cos\theta .$$

We can conclude from this limiting case that the parameter m characterizes the mass of a black hole. Of course, in the presence of the external field, i.e., for finite values of b, the mass of the black hole can differ from the value of the parameter m, being a function of m and b (see the expression for the area of the horizon given below for the case $l = 0$).

VI. "BUFFER ZONE" STRUCTURE OF THE SOLUTION FOR $m \ll r \ll \pi b$

If the mass of a black hole is small enough in comparison with the characteristic scale of the external field, then the space-time outside the black hole splits on zones with different solution behavior. Thus, in the region nearest to a black hole $r \sim 2m$, the solution can be considered as the Schwarzschild solution slowly perturbed by the external field. At larger distances from the black hole ($m \ll r \ll \pi b$), we have a "buffer" zone where a proper field of the black hole decreased considerably but the "long-scale" influence of the external field is not important yet. In this zone, the metric approaches the Minkowski space-time metric, and only for $r \sim \pi b$ the geometry becomes the geometry of the Bertotti-Robinson space-time which is locally perturbed (considerably, however) by the existing distant black hole due to focusing effects caused by the topological properties of the background space-time.

VII. THE HORIZON

As could be expected, there is a regular horizon whose location is given by the conditions

$$H : \left\{\rho = 0, \quad l - m \le b \sinh\frac{z}{b} \le l + m\right\} . \qquad (9)$$

(The image of the horizon in these coordinates is simply a segment on the z axis, but this means the degeneration of these $\{\rho, z\}$ coordinates on the horizon. This situation is completely similar to that one on the horizon in the Schwarzschild solution, where the horizon is described by the segment $\{\rho = 0, l - m \le z \le l + m\}$.)

The metric components are regular functions of the coordinates near these points (other than the poles, which can be the points of local non-Euclidean structure of the geometry—see below about these singularities) and their behavior near the horizon can be determined in terms of the ρ^2 expansions [for $\rho \to 0$ and z satisfying (9)]:

$$g_{tt} = \frac{\left(b^2 - l^2 + m^2 + 2bl\sinh\frac{z}{b}\right)^2\rho^2}{4b^4\left[m^2 - \left(l - b\sinh\frac{z}{b}\right)^2\right]} + O(\rho^4) ,$$

$$f = \frac{4m^2\left(b^2 - l^2 + m^2 + 2bl\sinh\frac{z}{b}\right)^2\rho^2}{[b^2 + (l-m)^2]^2\left[m^2 - \left(l - b\sinh\frac{z}{b}\right)^2\right]} + O(\rho^4) ,$$

$$g_{\varphi\varphi} = -\frac{4b^4\cosh^2\left(\frac{z}{b}\right)\left[m^2 - \left(l - b\sinh\frac{z}{b}\right)^2\right]}{\left(b^2 - l^2 + m^2 + 2bl\sinh\frac{z}{b}\right)^2} + O(\rho^2) .$$

In the most physically interesting case $l = 0$, the area of the horizon of the free falling in the external field black hole is given by the nontrivial enough expression

$$A = 4\pi m^2\left\{\frac{2\gamma}{\sqrt{\gamma^2+1}} + \frac{\gamma^3}{\gamma^2+1}\ln\frac{\sqrt{\gamma^2+1}+1}{\sqrt{\gamma^2+1}-1}\right\} .$$

The nodal singularities in this case are absent.

VIII. CURVATURE SINGULARITIES

At the points of the segment $\tilde{H}$ on the "opposite" axis of symmetry $\rho = \pi b$, with the same values of z as for the points of the horizon H, other irregularities of the space-time geometry are located. These points are

$$\tilde{H} : \left\{\rho = \pi b, \quad l - m \le b \sinh\frac{z}{b} \le l + m\right\} .$$

These irregular points can be considered as additional field sources in the studied solution. To understand its nature in the most simple way, put our position on this second axis and "turn off" the external electromagnetic field; i.e., consider the transformation

$$\rho \to \rho' = \pi b - \rho$$

in the solution (2)–(5) and calculate the limiting form of this solution for $b \to \infty$. In this limit the external electromagnetic field vanishes, the horizon on the first axis will be "shifted" to an infinitely large distance from the points of $\tilde{H}$, and a pure vacuum solution will remain, which can be considered (and this appears quite natural) as describing the character of the proper external field of this additional source. This limiting vacuum solution in the $\{r, \theta\}$ coordinates, after the transformation

$$r \to r' = r - 2m ,$$

takes the easily recognizable form

$$ds^2 = \frac{r'+2m}{r'}dt^2 - \frac{r'}{r'+2m}dr'^2 - r'^2(d\theta^2 + \sin^2\theta\, d\varphi^2) .$$

This is nothing but the field of the negative mass $(-m)$

located there. This source does not possess a horizon. The points of $\tilde{H}$, which correspond to $r' = 0$, are the points of curvature singularity in our solution. It is clear that the appearance of this additional source is caused by the special kind of background space-time topology, i.e., by the closure of the physical three-space $t = \text{const}$ in ρ directions.

IX. THE LAWS OF MOTION AND EXISTENCE OF "NODAL" SINGULARITIES ON THE SYMMETRY AXES

The solution (2)–(5) depends on three real parameters. As we have shown before, two of them, m and b, are respectively the mass of a black hole on the first axis of symmetry (as well as the negative mass of "induced" source on the opposite axis) and the external homogeneous electromagnetic field. The third one, l, determines the location of the black hole on the symmetry axis $\rho = 0$ in the chosen static frame.

As was mentioned in Sec. IV, in the background Bertotti-Robinson universe any world line of a particle resting in some static frame (6) will not be geodesic provided its position does not coincide with the origin of this frame. A similar situation takes place for a black hole in this background field, and the parameter l describes not only the location of a black hole on the axis $\rho = 0$, but its acceleration as well.

However, it is clear that in the solution presented here there is no physical basis for nonzero black hole acceleration. As is well known now, such apparent contradiction with the laws of motion finds its solution in the appearance of additional singularities of a special kind. These are nodal singularities, which are not curvature singularities, but they consist of points of local non-Euclidean structure of the space-time geometry ("conical points"). These points can fill some finite or even infinite regions of the axis of symmetry.

The appearance of such nodal singularities in the space-times where the equilibrium of the field sources has not been provided by some forces (or some nongravitational interactions) has been observed in a number of other solutions of this type (C metric [28,29], "multi-Schwarzschild" solutions [30], "double-Kerr" solution [3], "double-Curzon" solution [31], Ernst solution for a charged black hole in electric field [25], and some others). These singularities can be interpreted as some concentrated sources, whose internal stresses provide the resulting equilibrium of different parts of the source or cause the accelerations possessed by each of these parts. In a more simple and perhaps more physical approach, instead of these singularities themselves, one can consider the conditions of the absence of such singularities as an important physical condition on the parameters of the solution which play the same role in general relativity as the well known laws of motion (or equilibrium conditions) in the Newtonian theory.

To find these conditions for the solution (2)–(5), consider the structure of the symmetry axes $\rho = 0$ and $\rho = \pi b$; each of these axes can be divided into three parts with different behavior of the fields nearby when $\rho \to 0$ and $\rho \to \pi b$, respectively:

$$(1),(1'):\quad -\infty < \sinh\frac{z}{b} < l - m\ ,$$

$$(2),(2'):\quad l - m \le \sinh\frac{z}{b} \le l + m\ ,$$

$$(3),(3'):\quad l + m < \sinh\frac{z}{b} < \infty\ ,$$

where the regions (1), (2), and (3) have $\rho = 0$, while for the points of the regions (1′), (2′), and (3′) the ρ coordinate is equal to $\rho = \pi b$. The regions (2) and (2′) have been considered before. These are the points corresponding to the horizon and curvature singularity, respectively. To establish the local Euclidean structure at the points of the axes of symmetry in the regions (1), (3) and (1′), (3′) we consider some small circles $\{t = \text{const}, \rho = \text{const}, z = \text{const}\}$ surrounding these points and calculate for each of them the limit for vanishing radius of the ratio P_0 of the radius $(\int_0^\rho \sqrt{f}d\rho)$ multiplied by 2π to the length $(\int_0^{2\pi}\sqrt{-g_{\varphi\varphi}}d\varphi)$. The values of P_0^2, for the different regions under consideration, are given by the expressions

$$P_0^2 = \lim_{\rho\to 0}\left(-\frac{f\rho^2}{g_{\varphi\varphi}}\right) \quad \text{for (1) and (3)}\ ,$$

$$P_0^2 = \lim_{\rho\to \pi b}\left(-\frac{f\rho^2}{g_{\varphi\varphi}}\right) \quad \text{for (1′) and (3′)}\ .$$

For regular points of each of the axes, where the local Euclidean structure of the space geometry takes place, the limit of this ratio must be equal to 1.

Direct calculations of these limits lead to the results

$$(1):\quad P_0^2 = 1,\quad (1'):\quad P_0^2 = 1\ ,$$

$$(3):\quad P_0^2 = \left[\frac{b^2 + (l+m)^2}{b^2 + (l-m)^2}\right]^2 , \tag{10}$$

$$(3'):\quad P_0^2 = \left[\frac{b^2 + (l+m)^2}{b^2 + (l-m)^2}\right]^2 .$$

One can see from these expressions that for general values of the parameters m, b, and l there are regions on each of the two symmetry axes where $P_0^2 \neq 1$. This means that at these points the local Euclidean structure is lost, and nodal singularities arise there. At the same time, it follows from (10) that for $m \neq 0$ all these singularities completely disappear, and the values of P_0^2 at these regions become equal to 1 if and only if we set $l = 0$.

X. FREELY FALLING BLACK HOLE IN A MAGNETIC UNIVERSE

The choice $l = 0$ corresponds to the equilibrium position of a black hole at the origin of a chosen static frame in the background field. To be at rest in this place for a test particle as well as for a black hole it is necessary to possess zero acceleration or, in other words, to be a freely falling body in the external gravitational field. Just this situation is described by the solution presented here with $l = 0$. Thus the most physically acceptable two-parametric subfamily of solutions (with free parameters m and b) takes the form

$$ds^2 = \cosh^2\left(\frac{z}{b}\right) \frac{\left[r - m - m\cos\frac{\rho}{b}\right]^2}{r(r-2m)} dt^2 - \tilde{f}(d\rho^2 + dz^2) - \frac{b^2 r(r-2m)\sin^2\frac{\rho}{b}}{\left[r - m - m\cos\frac{\rho}{b}\right]^2} d\varphi^2 , \tag{11}$$

where the conformal factor

$$\tilde{f} = \frac{\left[r - m - m\cos\frac{\rho}{b}\right]^2}{(r-m)^2 - m^2\cos^2\theta} \times \left[\frac{r - m - b\sinh\frac{z}{b} - m\left(\cos\theta - \cos\frac{\rho}{b}\right)}{r - m - b\sinh\frac{z}{b} + m\left(\cos\theta - \cos\frac{\rho}{b}\right)}\right]^2 , \tag{12}$$

the nonzero components of a complex electromagnetic potential are

$$\{\Phi_t, \Phi_\varphi\} = \left\{ i\left(\sinh\frac{z}{b} - \frac{m}{b}\cos\theta\right), -\frac{bf\left[1 - \cos\frac{\rho}{b}\right]}{r - m - m\cos\frac{\rho}{b}} \right\} , \tag{13}$$

and the function $r = r(\rho, z)$ is given by the expression

$$r = m + \frac{1}{2}\left\{ \sqrt{m^2 - 2mb\sinh\left(\frac{z}{b}\right)\cos\left(\frac{\rho}{b}\right) + b^2\left[\sinh^2\left(\frac{z}{b}\right) + \sin^2\left(\frac{\rho}{b}\right)\right]} + \sqrt{m^2 + 2mb\sinh\left(\frac{z}{b}\right)\cos\left(\frac{\rho}{b}\right) + b^2\left[\sinh^2\left(\frac{z}{b}\right) + \sin^2\left(\frac{\rho}{b}\right)\right]} \right\} . \tag{14}$$

XI. CONCLUDING REMARKS

The discussion given above on the physical and geometrical properties of the presented solution shows a number of interesting features of this solution.

Considered locally, in some finite region in the vicinity of a black hole horizon, this solution gives us a completely physically acceptable description of the interaction of the proper gravitational field of a black hole with the external homogeneous pure magnetic (or, after duality rotation, pure electric) field.

The global structure of this solution is determined by the specific topology of the background space-time; its closure in some directions originates certain focusing phenomena.

Our complete family of solutions confirms, in a realistic enough situation, the conjectured long ago connection (or even equivalence) between the laws of motion (equilibrium conditions) and the conditions of the absence of unphysical noncurvature singularities of the corresponding space-time geometry.

The proposed analysis of this solution showed its essential difference with the Ernst solution [26]; namely: completely spatially homogeneous structure of the external (background) electromagnetic and gravitational fields, the influence of the background space-time topology on the structure of the fields, explicit nontrivial relation between the laws of motion and the conditions of the absence of unphysical noncurvature singularities. Some of these properties are not possessed by the field configuration described by the Ernst solution, while the others do not appear so explicitly in his solution because the three-space closure in ρ directions (which also takes place in the Melvin and Ernst solutions) arises there at the infinite distance from the symmetry axis only, where the behavior of space-time metric, however, is even less regular than in the Bertotti-Robinson solution.

Following the main purpose of this paper, to present the considered above solution and to discuss some of its interesting properties, we decided to exclude from this paper the discussion of a more wide but more complicated family of solutions which depends upon other physically interesting parameters such as the angular momentum and the charge of a black hole as well as some variations of the properties of the background space-time geometry. We also have not considered here a time-dependent counterpart of our family of solutions which automatically arises as a result of application of the method used here. (Some wavelike or cosmological interpretation of this solution also could possess interesting features.) These considerations we would like to postpone for future publications.

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APPENDIX: ON THE CONSTRUCTION OF SOLUTIONS FOR INTERACTING FIELDS

The general method of analysis of Ernst equations which was used for the construction of the solution presented in this paper had been developed more than ten years ago and presented in a condensed form in [32]; later it was described with more detail also in [14]. This method is based on the "monodromy data transform" (i.e., on the representation of any local solution of the Ernst equations in terms of the monodromy data of the corresponding fundamental solution of the associated linear system), and on the construction of the linear singular integral equation which can be considered as an equivalent integral equation form of the Ernst equations.

Discussions of various applications of this method, besides [14], can be found in a number of proceedings of various conferences at those years. Thus a brief sketch of pure mathematical construction was given also in [33]; a general outlook concerning the properties of the monodromy data functions and the problem of construction of exact solutions with some *a priori* defined properties (including the construction of asymptotically flat solutions with given multipole structures or the solutions determined by some rational functions as the boundary values of the Ernst potentials on the axis of symmetry) have been presented in [34]; some new simple particular examples of static electrovacuum asymptotically flat solutions can be found in [35]; the properties of some wavelike exact solutions have been considered in [36,37]; the application of this method for the construction of a pure linear algorithm for the solution of various boundary value problems for the Ernst equations (the Cauchy problems and characteristic initial value problems for time-dependent fields as well as some boundary value problems for stationary fields) have been discussed in [38]; and, at last, the general explicit form of the whole class of solutions with arbitrary number of free parameters corresponding to arbitrary *rational* boundary values of the Ernst potentials on the axis of symmetry (or more generally, on some degenerate regular orbits of the space-time isometry group) was presented in [17]. Just this general form of solutions was used for the derivation of the so simple form of the nonlinear superposition of fields considered in this paper.

In the very short publications listed above various possible applications of this method have not been presented with enough completeness, but a more extended paper is expected to be published elsewhere in the near future. Hence some brief sketch of the method and of some of its applications probably could be found useful to be appended here. The present Appendix reiterates some of the results from the above-mentioned short communications (sometimes with some small generalizations or convenient simplifications), making to the end an emphasis on the construction of exact solutions for interacting fields produced by given sources.

1. The Ernst equations and the space of their local solutions

Using the differential form notations, where d means an exterior derivation on the "orbit space" $\{x^1, x^2\}$ of the two-dimensional Abelian space-time isometry group and * is a Hodge star operator, such that $^*dx^1 = -\epsilon dx^2$, $^*dx^2 = -dx^1$ with $\epsilon = 1$ for time-dependent fields and $\epsilon = -1$ for stationary fields, the Ernst equations may be written in a compact form:[1]

$$(\mathrm{Re}\mathcal{E} + \Phi\bar{\Phi})\left(d^*d\mathcal{E} + \frac{d\alpha}{\alpha}{}^*d\mathcal{E}\right) - (d\mathcal{E} + 2\bar{\Phi}d\Phi)^*d\mathcal{E} = 0\ ,$$
$$(\mathrm{A1})$$
$$(\mathrm{Re}\mathcal{E} + \Phi\bar{\Phi})\left(d^*d\Phi + \frac{d\alpha}{\alpha}{}^*d\Phi\right) - (d\mathcal{E} + 2\bar{\Phi}d\Phi)^*d\Phi = 0\ ,$$

where $\mathcal{E}$ and Φ are two complex Ernst potentials. The real function $\alpha(x^1, x^2)$ is a "harmonic" function, which satisfies a linear equation (d'Alembert equation for $\epsilon = 1$ or Laplace equation for $\epsilon = -1$), yielding the existence of another real "harmonically conjugated" function $\beta(x^1, x^2)$:

$$d^*d\alpha = 0, \quad d\beta = -\epsilon^*d\alpha\ . \qquad (\mathrm{A2})$$

The functions α and β, chosen as any solution of (A2), provided $^*d\alpha \wedge d\alpha \neq 0$, constitute a convenient set of local coordinates ("generalized Weyl coordinates") used further in special linear combinations $\xi = \beta + j\alpha, \eta = \beta - j\alpha$ where j is a "square root" of ϵ: $j = 1$ for $\epsilon = 1$ and $j = i$ for $\epsilon = -1$. The coordinates ξ, η are two real light-cone coordinates in the "hyperbolic" case ($\epsilon = 1$), or two complex conjugated to each other coordinates in the "elliptic" case ($\epsilon = -1$). For example, one may set $\xi = x + t, \eta = x - t$ or $\xi = z + i\rho, \eta = z - i\rho$ for time-dependent or stationary axisymmetric fields, respectively.

The gauge freedom existing in the definitions of the Ernst potentials enables one, without any loss of generality, to set for any solution of the Ernst equations at some chosen regular point $P_0(\xi_0, \eta_0)$:

$$\mathcal{E}(\xi_0, \eta_0) = \epsilon_0, \quad \Phi(\xi_0, \eta_0) = 0\ , \qquad (\mathrm{A3})$$

where $\epsilon_0 = -1$ for $\epsilon = 1$, and $\epsilon_0 = \pm 1$ for $\epsilon = -1$ providing the Lorentz signature of the corresponding space-time metric.

Everywhere below we consider the entire *space of local solutions* of the Ernst equations at some initial point $P_0(\xi_0, \eta_0)$, i.e., the whole set of pairs of complex potentials $\mathcal{E}, \Phi$ which are holomorphic functions of ξ and η in some local domains of the point P_0 and which satisfy the Ernst equations (A1) [together with (A2)] in these domains and fit the "normalization" conditions (A3) at the point P_0.

[1] For brevity we do not use a wedge symbol $\wedge$ in the products of differential forms.

2. Associated linear system and definition of monodromy data

The first step of our analysis of the Ernst equations is similar to that presented, more or less explicitly, in most of the known approaches for the construction of various solution generating methods for these equations. This is the construction of some overdetermined linear system of partial differential equations with a free complex ("spectral") parameter. The integrability conditions of this associated linear system together with some auxiliary conditions imposed on the structure of its general matrix integral are equivalent to the Ernst equations. A remarkable feature of the structure of this equivalent matrix problem[2] is that, despite a great variety of different solutions of the Ernst equations as functions of the spacetime coordinates ξ and η, the corresponding fundamental solutions Ψ of the associated linear system possess some universal analytical properties on the plane of the complex parameter w. At the same time, some details of the analytical structures of $\Psi(\xi,\eta,w)$ are different for different solutions and, moreover, they permit one to characterize uniquely any individual local solution.

The analysis accomplished in [32] allowed us to associate with any local solution of the Ernst equations a set of four functions depending upon the complex parameter w only:

$$\{\mathcal{E}(\xi,\eta),\Phi(\xi,\eta)\}\longleftrightarrow\{\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)\}\ , \tag{A4}$$

where the pairs of functions $\{\mathbf{u}_+(w),\mathbf{v}_+(w)\}$ and $\{\mathbf{u}_-(w),\mathbf{v}_-(w)\}$ are holomorphic in some local domains Ω_+ and Ω_- near the points $w=\xi_0$ and $w=\eta_0$, respectively. These domains are the "images" on the w plane of the local complex domains, which ξ and η run independently near $\xi=\xi_0$ and $\eta=\eta_0$, respectively: $\Omega_+=\{w,w=\xi\}$ and $\Omega_-=\{w,w=\eta\}$. Each of the domains $\Omega_\pm$ should be chosen symmetrically with respect to the real axis on the w plane for the case $\epsilon=1$, or they are symmetric to each other with respect to this axis for the case $\epsilon=-1$.

In general $\Omega_\pm$ do not overlap each other, and the functions $\mathbf{u}_+(w)$ and $\mathbf{u}_-(w)$ [as well as the functions $\mathbf{v}_+(w)$ and $\mathbf{v}_-(w)$] are *not* the analytical continuations of each other. However, it will be convenient to further consider instead of four functions $\mathbf{u}_\pm(w)$ and $\mathbf{v}_\pm(w)$ only two functions $\mathbf{u}(w)$ and $\mathbf{v}(w)$ each defined in the disconnected region $\Omega=\Omega_+\cup\Omega_-$. Each of these functions is represented in $\Omega_\pm$ by two (different in general) holomorphic functions:

$$\mathbf{u}(w)=\begin{cases}\mathbf{u}_+(w), & w\in\Omega_+,\\ \mathbf{u}_-(w), & w\in\Omega_-,\end{cases}$$

$$\mathbf{v}(w)=\begin{cases}\mathbf{v}_+(w), & w\in\Omega_+,\\ \mathbf{v}_-(w), & w\in\Omega_-.\end{cases} \tag{A5}$$

The complex conjugation of these functions, defined as

$$\mathbf{u}^\dagger(w)=\overline{\mathbf{u}(\bar{w})},\quad \mathbf{v}^\dagger(w)=\overline{\mathbf{v}(\bar{w})}\ , \tag{A6}$$

has to be understood taking into account that for any $w\in\Omega_+(\Omega_-)$ its complex conjugated point $\bar{w}\in\Omega_+(\Omega_-)$ for $\epsilon=1$, while $\bar{w}\in\Omega_-(\Omega_+)$ for $\epsilon=-1$. Then

$$\text{for } \epsilon=1:\quad \mathbf{u}^\dagger(w)=\begin{cases}\overline{\mathbf{u}_+(\bar{w})}, & w\in\Omega_+,\\ \overline{\mathbf{u}_-(\bar{w})}, & w\in\Omega_-,\end{cases}$$

$$\text{for } \epsilon=-1:\quad \mathbf{u}^\dagger(w)=\begin{cases}\overline{\mathbf{u}_-(\bar{w})}, & w\in\Omega_+,\\ \overline{\mathbf{u}_+(\bar{w})}, & w\in\Omega_-,\end{cases} \tag{A7}$$

and the function $\mathbf{v}^\dagger(w)$ has to be defined, in general, in a similar way.

The functions $\mathbf{u}(w),\mathbf{v}(w)$ [or their representatives $\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)$] can be called "monodromy data functions," because their definitions arose from the special structure of the monodromy matrices which characterize the behavior of $\Psi(\xi,\eta,w)$ at its singular points on the w plane. A simple definition of the monodromy data functions, which we are going to recall just below, formally differs from (but is equivalent to) that one given initially in [32] and [14], and it arises immediately from the generally analytical structure of $\Psi(\xi,\eta,w)$ described there.

For any local solution of the Ernst equations, a holomorphic branch of the corresponding fundamental solution $\Psi(\xi,\eta,w)$ of the associated linear system can be chosen so that it satisfies the following two "normalization" conditions, one at the initial point $P_0(\xi_0,\eta_0)$ for any $w\in\bar{\mathbb{C}}$ and the other at $w=\infty$ for any ξ,η near $\xi=\xi_0$ and $\eta=\eta_0$, respectively:[3]

[2]For electrovacuum fields this is a 3×3-matrix problem while the same matrix problem, but for 2×2 matrices, corresponds to a pure vacuum case. The linear system (which is a part of the matrix problem considered in [32]) for a pure vacuum case coincides with the linear equations discovered in [39] for the generating matrix function which produces an infinite hierarchy of matrix potentials for the Ernst equations. For the electrovacuum case, the same linear system in a little different form was used in the Hauser-Ernst integral equation method for effecting Kinnersley-Chitre symmetry transformations [40–42] and in the construction of electrovacuum N-soliton solutions suggested by one of the authors in [11,12]. The other conditions, constituting our 3×3-matrix problem also appeared in the papers mentioned above, but mostly as the necessary conditions which follow from the Ernst equations. It is important that the set of such necessary conditions, which were included in the formulation of our matrix problem, provides the equivalence of this problem to the Ernst equations.

[3]These conditions lead to the dependence of our construction upon the choice of the initial point $P_0(\xi_0,\eta_0)$; however, for brevity we shall not show further the dependence of various functions upon ξ_0 and η_0.

$$\mathbf{\Psi}(\xi_0,\eta_0,w)=\mathbf{I} \text{ and } \mathbf{\Psi}(\xi,\eta,w=\infty)=\mathbf{I}\,. \qquad \text{(A8)}$$

The matrix function $\mathbf{\Psi}(\xi,\eta,w)$ chosen in this manner and its inverse will be holomorphic everywhere on the w plane (including $w=\infty$) besides the four branch points $w=\xi_0, w=\eta_0, w=\xi, w=\eta$ and the points of the cut L joining them, where the components of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ exhibit jumps.[4]

We chose the cut L, consisting of two disconnected parts: L_+ with endpoints $w=\xi_0$ and $w=\xi$, and L_- with endpoints $w=\eta_0$ and $w=\eta$. The cuts L_+ and L_- are local in the sense that $L_+\subset\Omega_+$ and $L_-\subset\Omega_-$, and they are chosen as the segments of the real axis on the w plane for $\epsilon=1$ or symmetric to each other with respect to this axis for $\epsilon=-1$.

The local behavior of $\mathbf{\Psi}(\xi,\eta,w)$ and its inverse near the singular points and at the points of the local cuts $L_\pm$ can be characterized as follows. If one considers a path in the local region Ω_+ (or Ω_-), which starts from some point at one of the edges of the cut L_+ (or L_-), goes around one of the endpoints of $L_+(L_-)$, and ends at the corresponding point of $L_+(L_-)$ on its other edge, the analytical continuation of $\mathbf{\Psi}(\xi,\eta,w)$ along this path can be described by one of the linear transformations

$$\mathbf{\Psi}(\xi,\eta,w)\xrightarrow{L_\pm}\mathbf{\Psi}(\xi,\eta,w)\mathbf{C}_\pm(w)\,, \qquad \text{(A9)}$$

where w is any point of the corresponding cut L_+ or L_-. It can be shown that the monodromy matrices $\mathbf{C}_+$ and $\mathbf{C}_-$ are holomorphic functions of w in the local regions Ω_+ and Ω_-, respectively. The Ernst equations imply also a nice algebraic structure of these matrices:

$$\mathbf{C}_\pm(w)=\mathbf{I}-2\frac{\mathbf{l}_\pm(w)\otimes\mathbf{k}_\pm(w)}{\mathbf{k}_\pm(w)\cdot\mathbf{l}_\pm(w)} \quad \text{with } \mathbf{C}_+^2\equiv\mathbf{C}_-^2\equiv\mathbf{I}\,, \qquad \text{(A10)}$$

where $\mathbf{k}_+(w)$ and $\mathbf{l}_+(w)$ as well as $\mathbf{k}_-(w)$ and $\mathbf{l}_-(w)$ are complex three-dimensional row, and column, vector functions holomorphic in Ω_+ and Ω_-, respectively.

In addition to that, the existence and the Hermitian structure of the first integral of the associated linear system imply (and actually are equivalent to) the existence of explicit expressions for $\mathbf{l}_\pm(w)$ in terms of $\mathbf{k}_\pm(w)$ (see these expressions below). Hence all the components of $\mathbf{C}_\pm(w)$ are completely determined by $\mathbf{k}_\pm(w)$.

On the other hand, it is easy to see from (A10) that $\mathbf{C}_\pm(w)$ themselves determine the components of $\mathbf{k}_+(w)$ and $\mathbf{k}_-(w)$ only up to arbitrary common scalar multipliers holomorphic in Ω_+ and Ω_-, respectively, and, without loss of generality, we can set the first components of $\mathbf{k}_\pm(w)$ equal to 1. Denoting the other components as

$$\mathbf{k}_\pm(w)=\{1,\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)\}\,, \qquad \text{(A11)}$$

we can consider these expressions as definitions of the functions $\{\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)\}$ discussed above. By virtue of (A5) we can omit the indices "$\pm$" in this definition.

As one can see now, the functions $\{\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)\}$ are determined uniquely by the monodromy properties of the analytical function $\mathbf{\Psi}(\xi,\eta,w)$ near its singular points (i.e., in the local domains Ω_+ and Ω_-). At the same time, the matrix function $\mathbf{\Psi}(\xi,\eta,w)$ is uniquely determined for any given solution of the Ernst equations as the holomorphic branch of the fundamental solution of the associated linear system with the initial conditions (A8). This allows us to consider the sets of functions $\{\mathbf{u}(w),\mathbf{v}(w)\}$ [or equivalently $\{\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)\}$] as characterizing the individual local solutions and to study the properties of the arising mapping between the space of such functions and the space of local solutions of the Ernst equations. A further analysis showed that this is a one-to-one correspondence between these two spaces, and hence this mapping can be called "monodromy data transform."

3. Inverse problem of the monodromy data transform: The solution of the Ernst equations for given monodromy data

The general analytical properties of the fundamental solution $\mathbf{\Psi}(\xi,\eta,w)$ of the associated linear system enable one to express the functions $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$, satisfying the initial conditions (A8) for any $w\in\bar{\mathbb{C}}$, as the Cauchy integrals of their jumps over the cut L. In addition to that, there exist some linear relations between the limit values of these functions on the cuts $L_\pm$ with the monodromy data functions $\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)$ as the coefficients of these relations. The use of the known Sokhotski-Plemelj formulas allows us to reduce these linear relations to three decoupled scalar (i.e., nonmatrix) linear singular integral equations for the components of the jumps of $\mathbf{\Psi}$ on the cut L [32].[5]

The common scalar kernel of these integral equations as well as their right-hand sides are completely determined in terms of the functions $\mathbf{u}_\pm(w)$, $\mathbf{v}_\pm(w)$ and their complex conjugations. The important point is that the general structure of each of these integral equations provides the existence and uniqueness of the solution for any given set of monodromy data functions $\{\mathbf{u}_\pm(w),\mathbf{v}_\pm(w)\}$ (see [43]). It has been shown that the solution of only two of these equations is enough to determine the corresponding Ernst potentials $\mathcal{E}$ and Φ in quadratures. Therefore our monodromy data transform actually determines a

[4]These "global" analytical properties of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ follow from a very useful general theorem proved in [42] for such linear systems with a free complex parameter. However, its application in the present context needs some modification taking into account the specific structure of the cut L which in general consists of two disconnected cuts.

[5]Three other equations which can be derived for the jumps of components of $\mathbf{\Psi}^{-1}$ are equivalent to the previous ones. These equations possess an even more convenient structure and just these equations will be discussed below.

one-to-one mapping between the space of arbitrary chosen monodromy data and the entire space of local solutions of the Ernst equations, and this allows us to consider the derived equations (or even only two of them) as an equivalent integral equation form of the Ernst equations.

4. The linear integral equation form of the Ernst equations

We are ready now to discuss the structure of the above-mentioned linear singular integral equation. For brevity of notations we shall not write further the indices "±" for various functions, keeping in mind that all functions used below of the complex parameter w or of the points τ, ζ on the cut $L = L_+ + L_-$ are defined in the union of the two disconnected local regions $\Omega = \Omega_+ \cup \Omega_-$, where each of these functions is represented by two different holomorphic functions which may not be connected to each other by analytical continuation; we say in this case that these representative functions, in general, are *not analytically adjusted* to each other. Assuming these notions, the main linear integral equation can be written in the following general form, with the Cauchy principal value integral at the left-hand side:[6]

$$\frac{1}{\pi i} \fint_L \frac{\mathcal{K}(\tau,\zeta)}{\zeta-\tau} \varphi(\zeta) d\zeta = \mathbf{k}(\tau) \ , \qquad (A12)$$

where the cut $L = L_+ + L_-$, with the cuts $L_+ \subset \Omega_+$, and $L_- \subset \Omega_-$, where $\Omega_\pm$ has been defined earlier; $\tau, \zeta \in L$; the kernel $\mathcal{K}(\xi,\eta,\tau,\zeta)$ is a scalar function, while the unknown $\varphi(\xi,\eta,\zeta)$ and the right-hand side $\mathbf{k}(\tau)$ are three-component row-vector functions. Therefore (A12) is a set of three decoupled linear integral equations with the same scalar kernel but different right-hand sides.

The structure of $\mathcal{K}(\tau,\zeta)$ is determined by the expression[7]

$$\mathcal{K}(\tau,\zeta) = -[\lambda]_\zeta (\mathbf{k}(\tau) \cdot \mathbf{l}(\zeta)),$$

$$\lambda = \sqrt{(\zeta-\xi)(\zeta-\eta)/(\zeta-\xi_0)(\zeta-\eta_0)} \ , \qquad (A13)$$

where $[\lambda]_\zeta$ means the jump at the point $\zeta \in L$ of the function $\lambda(\xi,\eta,\zeta)$ which characterizes the branching of $\mathcal{K}(\tau,\zeta)$ at the four endpoints of L; $(\cdot)$ means a scalar product of the vector functions $\mathbf{k}(\tau)$ and $\mathbf{l}(\zeta)$, the components of which are determined in terms of the functions $\mathbf{u}(w), \mathbf{v}(w)$ and their complex conjugations:

$$\mathbf{k}(\tau) = \{1, \mathbf{u}(\tau), \mathbf{v}(\tau)\} \ ,$$

$$\mathbf{l}(\zeta) = \begin{pmatrix} 1 + i\epsilon_0(\zeta-\beta_0)\mathbf{u}^\dagger(\zeta) \\ -i\epsilon_0(\zeta-\beta_0) + \epsilon\alpha_0^2\mathbf{u}^\dagger(\zeta) \\ 4\epsilon_0(\zeta-\xi_0)(\zeta-\eta_0)\mathbf{v}^\dagger(\zeta) \end{pmatrix} , \qquad (A14)$$

where $\alpha_0 = (\xi_0 - \eta_0)/2j$, $\beta_0 = (\xi_0 + \eta_0)/2$; the functions $\mathbf{u}^\dagger(w)$, $\mathbf{v}^\dagger(w)$ have been defined in (A6) and (A7). For various classes of fields the sign symbols are $\epsilon = \pm 1$ and $\epsilon_0 = -1$ for $\epsilon = 1$ or $\epsilon_0 = \pm 1$ for $\epsilon = -1$.

At last, if the equations (A12) are solved for some given $\mathbf{u}(w)$ and $\mathbf{v}(w)$, then the components of a complex 3×3-matrix function $R(\xi,\eta)$ can be calculated by quadratures:

$$R = \frac{1}{\pi i} \int_L [\lambda]_\zeta \mathbf{l}(\zeta) \otimes \varphi(\zeta) d\zeta \ . \qquad (A15)$$

In this way, all the metric components (except only one of them, called the "conformal factor," which can be calculated by quadrature of another kind) and the components of the electromagnetic potential can be expressed algebraically in terms of the components of R and their complex conjugations. The Ernst potentials also can be expressed in these terms as

$$\mathcal{E} = \epsilon_0 - 2iR_1^2, \quad \Phi = 2iR_1^3 \ . \qquad (A16)$$

This shows that in order to find any local solution $\{\mathcal{E}(\xi,\eta), \Phi(\xi,\eta)\}$ of the Ernst equations one has to solve only two of the equations (A12) with the same scalar kernel $\mathcal{K}$ and with the functions $\mathbf{u}(\tau)$ and $\mathbf{v}(\tau)$ at the right-hand sides. Just these components of $\varphi(\zeta)$ take part in the expressions (A15) for the components of R present in (A16).

5. Applications: General analysis, exact solutions, and boundary value problems

The above-described definition of the monodromy data functions associated with any local solution of the Ernst equations, the construction of the corresponding monodromy data transform between the entire space of local solutions and the space of arbitrary chosen monodromy data, as well as the reformulation of the Ernst equation in the equivalent linear integral equation form provide us with some base to analyze various problems.

First of all, the constructed monodromy data transform allows us to analyze the structure of the space of local solutions of the Ernst equations: besides some formal classification of all possible solutions in accordance with the analytical structure of their monodromy data functions, various physical and geometrical properties of solutions can be expressed directly in terms of the analytical properties of these functions [34,35]. This suggests, in particular, some way to compare various solution gen-

[6]The coordinates ξ and η enter this equation as parameters: they define the location of the endpoints of $L_\pm$ on the w plane and enter explicitly into the kernel $\mathcal{K}$ through the function λ (see the expressions given just below). However, for simplicity, we omit here the dependence of $\mathcal{K}$ and of the unknown φ upon ξ and η.

[7]Instead of this function λ we can consider another (more simple perhaps) function defined as

$$\lambda(\xi,\eta,\zeta) = \begin{cases} \lambda_+(\xi,\zeta) = \sqrt{(\zeta-\xi)/(\zeta-\xi_0)}, & \zeta \in L_+, \\ \lambda_-(\eta,\zeta) = \sqrt{(\zeta-\eta)/(\zeta-\eta_0)}, & \zeta \in L_-. \end{cases}$$

This redefines the unknown functions φ_+ and φ_- by the multipliers $\lambda_-(\eta,\zeta)$ and $\lambda_+(\xi,\zeta)$ holomorphic on L_+ and L_-, respectively.

eration methods, to recognize among the generated solutions various known cases, and to clarify the physical interpretation of their parameters.

Moreover, large families of solutions with any finite number of free parameters can be calculated explicitly [14,34,17]. A great number of the known solutions can be recognized as particular cases among these explicitly calculated families. This opens the way for various generalizations of these known solutions and for consideration of their nonlinear superpositions (see the last subsections of this Appendix for more details and the main part of this paper for an example).

At last, another interesting question concerning the relation between the monodromy data and some complete set of the boundary data for the solutions could be mentioned here also. This question has been considered in [14,38], where it has been shown that the monodromy data transform provides us with some method for consideration of the initial value problems (Cauchy problem, characteristic initial value problem) for time-dependent fields, as well as some boundary problems for stationary fields. In all cases, a given complete set of boundary data functions permits one, in principle, to determine uniquely the monodromy data functions $\{\mathbf{u}_{\pm}(w), \mathbf{v}(w)\}$ through the solution of a certain system of linear ordinary differential equations with a complex parameter. Then, the solution of the boundary value problem could be constructed through the solution of the linear integral equation with these monodromy data functions in its kernel and at the right-hand side.

So, the linear integral equation method, whose key points were outlined in this Appendix, provides us with the most general base for various applications. It does not imply any restrictions on the class of fields under considerations, as well as on the space-time region where the fields are considered. The presented unified form of the method is valid for vacuum as well as electrovacuum fields in both stationary or time-dependent cases. [It would be useful perhaps to note that this approach admits a nontrivial generalization, which introduces also "into the game" a massless two-component spinor ("neutrino") field.] The Einstein-Maxwell-Weyl field equations with the same two-dimensional space-time isometry group (the integrability of which was proved in [44]) also can be reduced to a scalar linear integral equation form [32].

6. Comparison with other approaches

We postpone a detailed comparison of the present approach with various earlier developed solutions generating algorithms and with the known integral equation methods to some more enlarged publication, restricting ourselves here to some short comments only.

Thus the first sketch of the construction of some Riemann problem and of the corresponding linear integral equation (but in the matrix form and for the vacuum case only) have been presented by Belinskii and Zakharov [4] (see also Cosgrove [45] for more details).

For vacuum and electrovacuum cases, an elaborated and effective approach, the construction of some homogeneous Hilbert problem and of the corresponding matrix linear integral equation for effecting the Kinnersley-Chitre symmetry transformations, has been developed by Hauser and Ernst [40–42]. Later some technical improvement of the Hauser-Ernst approach for the case of the Minkowski seed metric was suggested by Sibgatullin [46], who reduced the Hauser-Ernst matrix integral equation for this case to a much more simple one-component integral equation. The Hauser-Ernst analysis includes many nice and useful auxiliary results, however, it was based on the essential restriction on the class of solutions under consideration given by the regularity axis condition. This condition [which is equivalent in terms of the above-defined monodromy data functions to the constraints $\mathbf{u}_+(w) = \mathbf{u}_-(w)$ and $\mathbf{v}_+(w) = \mathbf{v}_-(w)$,[8] with a necessary location of the initial point P_0 on the axis of symmetry] is very reasonable physically for stationary axisymmetric fields but it loses this physical motivation for nonaxisymmetric and time-dependent fields (waves, cosmological solutions). In addition to that, a characterization of the solutions in terms of their boundary data on the axis of symmetry makes very problematic a consideration (even in principle) of the boundary value problems even for the fields satisfying the regularity axis condition but on the boundaries which do not coincide with the axis of symmetry.

Another approach to the analysis of pure vacuum field equations, without any additional restrictions on the class of solutions under consideration, within the inverse scattering context was proposed by Neugebauer [47]. It was based on a different idea which allowed one to reduce the vacuum field equations (equivalent to the corresponding Ernst equation) to a one-component linear singular integral equation as well as to derive some analog of the Gelfand-Levitan-Marchenko equation. At first glance, it appears (purely subjectively, perhaps) that this singular equation possesses a slightly more complicated structure than the similar equation arising from the construction presented above. However, it seems that a more careful analysis could relate directly Neugebauer's scattering data functions $a(k)$ and $b(k)$ with the monodromy data functions $\mathbf{u}_{\pm}(w)$ for the vacuum case where $\mathbf{v}_{\pm}(w) \equiv 0$.

A brilliant application of the integral equation methods was found recently by Neugebauer and Meinel [48]. Their analysis of a complicated and very interesting physical problem—the structure of a dust disk rigidly rotating in its own gravitational field—provided not only an example of an effective solution of the corresponding boundary value problem, but it was the first example of a simultaneous solution of the internal and external problems: the derived density distribution in the disk provides the regularity of the asymptotically flat external field.

As was mentioned earlier, the class of solutions of the equation (A12) constructed explicitly [14,17] (see also the last subsections of this Appendix) corresponds to *arbi-*

[8] Further on we refer to this case as analytically adjusted monodromy data.

trary analytically adjusted rational monodromy data functions and includes the solutions with an arbitrary large but finite number of free parameters. Concerning the relationship between these solutions and various families constructed with the use of different solution generating approaches or integral equation methods, the following things should be noticed.

(1) In the vacuum stationary axisymmetric case, the simple poles in the monodromy data function $\mathbf{u}(w) \equiv \mathbf{u}_+(w) \equiv \mathbf{u}_-(w)$ [with $\mathbf{v}_\pm(w) = 0$] correspond to Belinskii-Zakharov solitons or Neugebauer's Bäcklund transformations with the Minkowski or some other background (or seed) solution, where Ernst potential on the axis $\rho = 0$ is a rational function of the Weyl coordinate z. In the electrovacuum case, the simple complex poles of the functions $\mathbf{u}(w) \equiv \mathbf{u}_+(w) \equiv \mathbf{u}_-(w)$ and $\mathbf{v}(w) \equiv \mathbf{v}_+(w) \equiv \mathbf{v}_-(w)$ correspond to electrovacuum solitons [11,12] on the electrovacuum background with similar properties. The multiple poles obviously correspond to some limit cases of soliton solutions also. However, the presence of real poles of the functions $\mathbf{u}(w)$ and $\mathbf{v}(w)$ in the electrovacuum case, as well as of their polynomial parts, generates new solutions which do not arise as the result of application of the above-mentioned soliton generating techniques [14]. (Some of the polynomial solutions can be generated however by other solution generating methods— by Harrison transformation, for example, or by another (a little more general) nonsoliton solution generating method [49].)

(2) The solutions, with analytically adjusted monodromy data functions $\mathbf{u}_+(w) = \mathbf{u}_-(w) \equiv \mathbf{u}(w)$, $\mathbf{v}_+(w) = \mathbf{v}_-(w) \equiv \mathbf{v}(w)$ in the stationary axisymmetric case, possess (locally) a regular axis of symmetry and therefore, in principle, these solutions with rational $\mathbf{u}(w), \mathbf{v}(w)$ can be calculated also using the Hauser-Ernst integral equation method or its, appropriated for the Minkowski seed metric, more simple Sibgatullin's form. Eventually, a number of asymptotically flat solutions of this kind have been published recently by Manko and Sibgatullin (see for example [50] and the papers cited there). However, all these formally calculated solutions obviously are very specialized particular cases of the general and compact form of the whole class of such solutions presented in [17], and at least some of them simply have to coincide with solitons or their limit cases. In addition to that, some technical problem which has not been avoided in [50] and related papers can be noted here. This is the dependence of the constructed solutions upon the specific set of parameters determined implicitly—the roots of certain algebraic equations, whose order depends on the complexity of the chosen rational boundary values of the Ernst potentials on the axis. The matter is that a trivial change of the system of independent parameters together with the use of the existing identities between any two pairs of bipolar coordinates enable one to simplify (sometimes considerably) the constructed solution (see the next subsection for more details and the solution presented in the main part of this paper for an example).

(3) The analyses of the structure of the monodromy data functions and the corresponding solutions of the singular integral equation for some already known essentially different solutions (such as some singular wave-like or cosmological-like solutions), as well as of some links with other solution generating procedures (such as the construction of finite-gap solutions suggested by Korotkin and Matveev [51]) could show some new ways for a systematical construction of new types of solutions characterized, for example, by analytically not adjusted [$\mathbf{u}_+(w) \neq \mathbf{u}_-(w)$ and $\mathbf{v}_+(w) \neq \mathbf{v}_-(w)$], but rational, perhaps, or some algebraic, respectively to the above-mentioned cases, monodromy data functions.

7. Calculation of exact solutions

Despite the rather simple form of the linear integral equation (A12) and the existence of the solution for any given functions $\mathbf{u}_\pm(w)$ and $\mathbf{v}_\pm(w)$, the explicit calculation of solutions can be performed for some special choices of the functions $\mathbf{u}_\pm(w), \mathbf{v}_\pm(w)$ only. A very large class of explicitly derivable solutions arises if we suppose that the following three conditions are satisfied [14,34,17].

(1) The functions $\mathbf{u}_\pm(w)$, as well as the functions $\mathbf{v}_\pm(w)$, are analytically "adjusted," i.e., they are the analytical continuations of each other:

$$\mathbf{u}_+(w) = \mathbf{u}_-(w) \equiv \mathbf{u}(w), \quad \mathbf{v}_+(w) = \mathbf{v}_-(w) \equiv \mathbf{v}(w) \ . \tag{A17}$$

Then, the region of holomorphicity of $\mathbf{u}(w)$ and $\mathbf{v}(w)$ covers both cuts $L_\pm$ and only one vector function $\varphi(\xi,\eta,\zeta) \equiv \varphi_+(\xi,\eta,\zeta) \equiv \varphi_-(\xi,\eta,\zeta)$ should be determined from (A12) and used in (A15), (A16) for the calculation of the solution.

(2) The initial point P_0 is located on the curve (or surface) $\alpha = 0$, i.e., we set $\alpha_0 = 0$.[9] The condition $\alpha_0 = 0$ is not necessary for the explicit calculation of the solutions but it simplifies the structure of the kernel of the integral equation, as well as the final expressions.[10]

(3) The functions $\mathbf{u}(w), \mathbf{v}(w)$ are arbitrary rational:

$$\mathbf{u}(w) = \frac{U(w)}{Q(w)} \text{ and } \mathbf{v}(w) = \frac{V(w)}{Q(w)} \ , \tag{A18}$$

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[9]The condition (A17) provides (locally) the regularity of the space-time geometry at the points with $\alpha = 0$, and the initial point P_0 can be conveniently located there. For this choice of P_0 the functions $\mathbf{u}(w)$ and $\mathbf{v}(w)$ can be simply related with the boundary values of the Ernst potentials on the boundary $\alpha = 0$:

$$\mathcal{E}(0,\beta) = \epsilon_0 - 2i(\beta - \beta_0)\mathbf{u}(\beta), \quad \mathbf{\Phi}(0,\beta) = 2i(\beta - \beta_0)\mathbf{v}(\beta) \ .$$

[10]For more special classes of fields, such as the asymptotically flat stationary axisymmetric fields (with $\alpha \equiv \rho, \beta \equiv z$), for example, an additional simplification of the structure of the main linear integral equation can be achieved if we shift the initial point P_0 along the z axis to infinity using the substitution $\lim_{\beta_0 \to \infty} \mathbf{u}(w)/\beta_0 \to \mathbf{u}(w)$ and $\lim_{\beta_0 \to \infty} \mathbf{v}(w)/\beta_0 \to \mathbf{v}(w)$.

where for the polynomials $U(w), V(w)$, and $Q(w)$ we use the notation

$$U(w)=\sum_{k=0}^{N_u} u_k w^k,\quad V(w)=\sum_{k=0}^{N_v} v_k w^k, \quad \text{and} \quad \begin{cases} Q(w)=1, & N_q=0,\\ Q(w)=\prod_{k=1}^{N_q}(w-h_k), & N_q>0. \end{cases} \tag{A19}$$

Here N_u, N_v, N_q are arbitrary non-negative integers; u_k, v_k, and h_k, for each k, are arbitrary real or complex parameters. Some of the constants h_k may coincide providing $Q(\zeta)$ with roots of any multiplicity.

The method for the construction of explicit solutions of the equation (A12) and the calculation of the corresponding Ernst potentials for the monodromy data functions defined by (A17)–(A19) have been explained in [14] and discussed in [34]. The explicit compact form of the general solution for this class of monodromy data functions was presented in [17]. We repeat this form here with a slight generalization due to the appearance of an additional sign symbol ϵ_0 and with β_0 not necessarily equal to zero as it was in [17].

In order to construct the general form of the solution for the case (A17)–(A19) we define, first, a set of auxiliary polynomials $P(\zeta), R_k(\zeta)$ $(k=0,1,\ldots,N_0-1)$:

$$\begin{aligned} P(\zeta)&=Q(\zeta)Q^\dagger(\zeta)+i\epsilon_0(\zeta-\beta_0)[U^\dagger(\zeta)Q(\zeta)-U(\zeta)Q^\dagger(\zeta)]+4\epsilon_0(\zeta-\beta_0)^2V(\zeta)V^\dagger(\zeta)\ ,\\ \sum_{k=0}^{N_0-1}R_k(\zeta)\tau^k&=i\epsilon_0(\zeta-\beta_0)Q^\dagger(\zeta)\left[\frac{U(\zeta)Q(\tau)-U(\tau)Q(\zeta)}{\zeta-\tau}\right]\\ &\quad-4\epsilon_0(\zeta-\beta_0)^2V^\dagger(\zeta)\left[\frac{V(\zeta)Q(\tau)-V(\tau)Q(\zeta)}{\zeta-\tau}\right]\ , \end{aligned} \tag{A20}$$

where $N_0=\max\{N_u,N_v,N_q\}$, and a dagger denotes a complex conjugation of the polynomial coefficients. We denote the degree of $P(\zeta)$ by N_p and represent this polynomial in a factorized form

$$P(\zeta)=p_0\prod_{k=1}^{N_p}(\zeta-w_k)\ , \tag{A21}$$

where the new set of parameters $\{w_k\}$ $(k=1,2,\ldots,N_p)$ may consist of the real parameters as well as of complex conjugated pairs. These parameters as the roots of $P(\zeta)$ are obviously dependent upon the coefficients of polynomials in (A20), but the factorization (A21) does not mean at all that we need to solve the algebraic equation $P(\zeta)=0$ explicitly. Instead, we use $\{w_k\}$ (together with $\{h_k\}$ perhaps) as a new set of parameters, expressing the coefficients of $P(\zeta)$ and hence, the coefficients in (A20) (or some of them) in terms of independent real and imaginary parts of $\{w_k\}$.

Now we proceed to the calculations with the parameters u_k $(k=1,2,\ldots,N_u)$, v_k $(k=1,2,\ldots,N_v)$, h_k $(k=1,2,\ldots,N_q)$, w_k $(k=1,2,\ldots,N_p)$, and p_0, keeping in mind their mutual dependence through the equation $P(\zeta)=p_0\prod_{k=1}^{N_p}(\zeta-w_k)$.

As the next step of our calculations, we have to evaluate a set of quadratures—the Cauchy principal value and proper integrals over the cut $L=L_+ + L_-$—whose integrands are the products of the jumps on L of the functions λ [defined in (A13)] or $1/\lambda$ and some rational functions of ζ:

$$\begin{aligned} L_k(\xi,\eta,\tau)&=\frac{1}{\pi i}\,⨍_L\frac{[\lambda^{-1}]_\zeta\zeta^k}{(\zeta-\tau)Q(\zeta)}d\zeta\ ,\\ M_k(\xi,\eta)&=\frac{1}{\pi i}\int_L[\lambda^{-1}]_\zeta[Q^\dagger(\zeta)\\ &\quad+i\epsilon_0(\zeta-\beta_0)U^\dagger(\zeta)]\frac{Q(\zeta)}{P(\zeta)}L_k(\zeta)d\zeta\ ,\\ D_{kl}(\xi,\eta)&=\delta_{kl}+\frac{1}{\pi i}\int_L[\lambda]_\zeta\frac{R_k(\zeta)}{P(\zeta)}L_l(\zeta)d\zeta\ , \end{aligned} \tag{A22}$$

where $k,l=0,1,\ldots,N_0$. It is important to note that all these integrals are only abbreviations for the explicit expressions which are trivially calculated as the sums of residues of their integrals at the zeros of $Q(\zeta)$, the poles of $L_k(\zeta)$ [which coincide with the zeros of $Q(\zeta)$], the zeros of $P(\zeta)$, and at infinity.[11] We keep the integral form of the expressions (A22) for brevity only, because all expressions in this form remain the same independently on the multiplicity of the roots of $Q(\zeta)$ and $P(\zeta)$.

At last, we define the components of two $N_0\times N_0$ matrices $\|G\|\|F\|$:

$$G_{kl}(\xi,\eta)=D_{kl}(\xi,\eta)+2i\epsilon_0 u_k M_l(\xi,\eta),$$

$$F_{kl}(\xi,\eta)=D_{kl}(\xi,\eta)-2iv_k M_l(\xi,\eta)\ .$$

Then the explicit determinant expressions for the Ernst potentials are[12]

$$\mathcal{E}=\epsilon_0\frac{\det\|G\|}{\det\|D\|},\quad \Phi=\frac{\det\|F\|}{\det\|D\|}\ . \tag{A23}$$

These expressions give us the general solution of (A1) for arbitrary analytically adjusted rational monodromy

[11] For example, for $\tau\in L$ and $h\bar{\in}L$ we have

$$\frac{1}{\pi i}⨍_L\frac{[\lambda]_\zeta d\zeta}{\zeta-\tau}=-1,\quad \frac{1}{\pi i}\int_L\frac{[\lambda]_\zeta d\zeta}{\zeta-h}=\lambda(h)-1,$$

$$\frac{1}{\pi i}⨍_L\frac{[\lambda]_\zeta d\zeta}{(\zeta-\tau)(\zeta-h)^2}=\frac{d}{d\zeta}\left[\frac{\lambda(\zeta)}{\zeta-\tau}\right]_{\zeta=h}\ .$$

[12] At the initial point P_0 these potentials satisfy the conditions which differ insignificantly from (A3): $\mathcal{E}(\xi_0,\eta_0)=\epsilon_0$, $\Phi(\xi_0,\eta_0)=1$.

data functions (A17)–(A19) or, equivalently, for arbitrary given rational boundary values of the Ernst potentials (as the functions of β) on the regular boundary $\alpha = 0$ near the initial point P_0.

8. On the nonlinear superposition of fields

Though the nonlinearity of the Ernst equations does not allow one to superpose any two given field configurations, this can be done (formally at least) within some restricted but very large classes of fields. Thus, besides vacuum static axisymmetric (Weyl) fields or linear polarized Einstein-Rosen waves, when the superposition facilities come from the reduction of the Ernst equations for these fields to the linear three-dimensional axisymmetric Laplace equation or Euler-Poisson-Darboux equation, respectively, the application of various solution generating methods to the Ernst equations (Bäcklund or symmetry transformations, soliton generating techniques) permits one to superpose the fields of some special kinds of sources—the asymptotically flat external fields of Kerr- or Kerr-Newman-like sources or their coaxial combinations or limiting cases as well as special kinds of incident waves with arbitrary chosen external (background) fields.

The integral equation method based on the monodromy data transform provides us with some additional facilities. A large number of the already known solutions for the fields, whose interactions would be of some interest, possess rational analytically adjusted structure of the monodromy data functions (or, equivalently for this case, rational structure of the Ernst potentials at the boundary $\alpha = 0$ as functions of another Weyl coordinate β). Below we present a short list of examples (discussed before in [14,35,34]) which shows the simplest cases of monodromy data functions and the corresponding well known solutions, or their generalizations, which can be derived as specifications of the general form (A23) (u_0, u_1, v_0, v_1, h_0 are arbitrary complex constants):

$\mathbf{u}(w) = 0$, $\mathbf{v}(w) = 0$ — Minkowski space-time ,

$\mathbf{u}(w) = u_0$, $\mathbf{v}(w) = 0$ — Rindler flat metric (with the reference frame acceleration determined by Im $u_0 \neq 0$) ,

$\mathbf{u}(w) = 0$, $\mathbf{v}(w) = v_0$ — Bertotti-Robinson electromagnetic universe,

$\mathbf{u}(w) = u_0 + u_1 w$, $\mathbf{v}(w) = v_0$ — Melvin electromagnetic universe ,

$\mathbf{u}(w) = \frac{u_0}{w-h_0}$, $\mathbf{v}(w) = \frac{v_0}{w-h_0}$ — Kerr-Newman solution ,

$\mathbf{u}(w) = u_0 + \frac{u_1}{w-h_0}$, $\mathbf{v}(w) = \frac{v_1}{w-h_0}$ — electrovacuum subfamily of the Plebański-Demiański solution ,

$\mathbf{u}(w) = u_0 + \frac{u_1}{w-h_0}$, $\mathbf{v}(w) = v_0 + \frac{v_1}{w-h_0}$ — the solution for a black hole immersed in a homogeneous electromagnetic field (a particular case was considered in this paper) .

In addition to that, the same calculations for different signs $\epsilon = \pm 1, \epsilon_0 = \pm 1$ lead us to the time-dependent analogs of various stationary solutions and vice versa.

The existence of an explicit form (A23) for any solution of the class (A17)–(A19) allows one to calculate the solutions corresponding to any superpositions (linear, for example), or, even more generally, to any *rational* combination of their monodromy data (or boundary value) functions. The calculated superposed families of solutions will include the original solutions as the particular cases continuously connected by additionally arising parameters. However, for the correct physical interpretation of the new solutions one has to investigate, in detail, their physical and geometrical properties.

The proposed method avoids the appearance of pure gauge parameters during the calculation. It allows one also to eliminate the parameters which cause the presence in the solution of any unphysical singularities, as well as singularities, which can be considered as some additional field sources, changing the expected physical interpretation of the constructed solution.

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