

## Relativistic material reference systems

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This work closes certain gaps in the literature on material reference systems in general relativity. It is shown that perfect fluids are a special case of DeWitt's relativistic elastic media and that the velocity-potential formalism for perfect fluids can be interpreted as describing a perfect fluid coupled to a fleet of clocks. A Hamiltonian analysis of the elastic media with clocks is carried out and the constraints that arise when the system is coupled to gravity are studied. When the Hamiltonian constraint is resolved with respect to the clock momentum, the resulting true Hamiltonian is found to be a functional only of the gravitational variables. The true Hamiltonian is explicitly displayed when the medium is dust, and is shown to depend on the detailed construction of the clocks.

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### I. INTRODUCTION

The use of material reference systems in general relativity has a long and noble history. Beginning with the systems of rods and clocks conceived by Einstein [1] and Hilbert [2], material systems have been used as a physical means of specifying events in spacetime and for addressing conceptual questions in classical gravity. That such systems also provide important tools for *quantum* gravity was pointed out by DeWitt [3], who used them to analyze the implications of the uncertainty principle for measurements of the gravitational field.

The original systems of rigid rods and massless clocks discussed by Einstein and Hilbert represent unphysical idealizations. Since their time, attempts have been made to remedy this shortcoming by developing a more physically realistic description of the reference medium. While still a phenomenological description, a dynamical reference system consistent with relativity may be found in the clocks and elastic media studied by DeWitt [3]. In the same spirit, perfect fluids [4] have been employed as reference systems in the quantization of various model problems in gravity [5,6]. More recently, Kuchař *et al.* [7] have developed a scheme for incorporating reference systems in general relativity through the introduction of coordinate conditions [8]. For the cases examined thus far, the reference materials that arise through this approach have certain unphysical properties. This has motivated the investigation of pressureless perfect fluid (dust), which is unrelated to any obvious coordinate condition, as a phenomenological but physically realistic reference system [9].

Here, we pursue the modest goal of closing certain gaps in the literature on reference materials. First, we establish in Sec. II the connection between the elastic media of DeWitt [3] and the relativistic perfect fluids of Refs. [10,11], which include dust as a special case. In particular, we show that perfect fluids are equivalent to elastic media when the latter are homogeneous and isotropic.

We then turn to the study of reference *systems* defined by the coupling of clocks to reference *materials*. In Sec. III A we demonstrate that a perfect fluid, as described by the action  $S$  of Refs. [11,12] for the isentropic case,<sup>1</sup> can be interpreted as a perfect fluid *coupled to a fleet of clocks* (with the details presented in Appendix A). We show the inequivalence of this clock coupling to the coupling used by DeWitt [3], and discuss the advantages of DeWitt's method. We also point out that for a nonisentropic perfect fluid  $S$ , the thermasy [14] can be reinterpreted as a clock variable with DeWitt-type coupling. Then, in preparation for Sec. IV, we perform in Sec. III B a Hamiltonian analysis of the reference system formed by adding clocks (in the manner of DeWitt) to the elastic medium. This proceeds along the lines of that done in Refs. [11,12] (see also [15,6]) for perfect fluids.

In Sec. IV we study the coupling of the reference system to gravity and the resulting canonical constraints. A canonical formalism for dust and matter clocks has proven useful in the study of homogeneous quantum cosmologies [5] and quantum gravity in general [9]; the results of Sec. IV set the stage for similar investigations using other reference systems. When the Hamiltonian constraint is resolved with respect to the clock momenta,

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<sup>1</sup>This action is a modification of the action of Ref. [4], which in turn is a relativistic generalization of the nonrelativistic perfect fluid action of Ref. [13].

we find that the true Hamiltonian depends only on the gravitational variables, not on the clock or particle variables. (Some details are derived in Appendix B.) The case of dust is particularly simple, and we explore the use of more general clocks than those used in Ref. [9]. For this medium and for appropriately designed clocks, the constraints can be resolved with respect to the clock momenta using only analytic operations (i.e., without taking a square root). This allows us to bypass a technical difficulty when defining the quantum theory [9] although, as we point out, interpretational difficulties appear in its place.

We use the following notation, which is consistent with those of Refs. [11,12]. The action for a reference *system*, which includes both matter and clocks, will be denoted by  $S$ , while the action for the corresponding reference material alone, without clocks, will be denoted by  $\bar{S}$ . Thus, the addition of clocks changes a “barred” system into an “unbarred” system.

## II. ELASTIC MEDIA AND PERFECT FLUIDS

In this section we address the relation between DeWitt’s elastic media and the perfect fluids of Refs. [10–12]. In fact, the main difference is that the elastic medium is presented in the Lagrangian picture while the perfect fluids are typically presented in the Eulerian picture. Thus, we first review the elastic medium in the original Lagrangian description, and then rewrite it in the Eulerian picture. The perfect fluid action is then recognized to be a special case of the action for an elastic medium.

We use the terms Lagrangian and Eulerian in the following way. The term “Lagrangian picture” refers to the description in which the basic variables tell the spatial location of a given particle, or the spacetime location of a given event on a given particle world line. The term “Eulerian picture” refers to the description in which the basic variables tell which particle resides at a given spatial location, or which particle passes through a given spacetime event.

### A. A review of elastic media

We now begin with a summary of DeWitt’s elastic media [3]. A single free relativistic particle moving in a spacetime  $\mathcal{M} = \Sigma \times \mathbf{R}$  with metric  $\gamma_{\alpha\beta}$  can be described by the action

$$\bar{S}_1[\Upsilon^\alpha; \gamma_{\alpha\beta}] = - \int d\sigma m \sqrt{-\dot{\Upsilon}^\alpha \dot{\Upsilon}^\beta \gamma_{\alpha\beta}(\Upsilon)}. \quad (2.1)$$

Here,  $\sigma$  is an arbitrary parameter along the particle world line and  $y^\alpha = \Upsilon^\alpha(\sigma)$  are the spacetime coordinates of the particle. Also,  $m > 0$  is the mass of the particle and the dot denotes a derivative with respect to  $\sigma$ . The semi-colon notation in  $\bar{S}_1[\Upsilon^\alpha; \gamma_{\alpha\beta}]$  indicates that this action is to be varied with respect to  $\Upsilon^\alpha$ , with  $\gamma_{\alpha\beta}$  treated as a background field.

As in Ref. [3], we may also consider fleets of such particles and we may add local interactions. If the particles are labeled by a set of Lagrangian coordinates  $\zeta^i$ ,  $i \in \{1, 2, 3\}$ , then such a system can be described by the action

$$\begin{aligned} \bar{S}_{LL}[\Upsilon^\alpha; \gamma_{\alpha\beta}] \\ = \int d\sigma \int_S d^3\zeta \left\{ -(\underline{n}m + \underline{w}) \sqrt{-\dot{\Upsilon}^\alpha \dot{\Upsilon}^\beta \gamma_{\alpha\beta}(\Upsilon)} \right\}, \end{aligned} \quad (2.2)$$

where  $S$  is the “matter space” manifold [10,11] whose points  $\zeta \in S$  label the particle world lines. In Eq. (2.2), the dynamical variables  $\Upsilon^\alpha$  are functions of  $\sigma$  and  $\zeta^i$ . The quantity  $\underline{n}$  is the particle number density, so that  $\underline{n} d^3\zeta$  is the number of particles in the coordinate cell  $d^3\zeta$ . The quantity  $\underline{w}$  is the interaction energy density in the comoving frame, so that  $\underline{w} d^3\zeta$  is the interaction energy in the coordinate cell  $d^3\zeta$  as measured in the rest frame of the particles. In order to make the transformation properties of these expressions clear, we explicitly indicate density weights with respect to changes of the coordinates  $\zeta^i$  on  $S$  with underlines. Thus,  $\underline{n}$  and  $\underline{w}$  are densities in the matter space  $S$ .

The functions  $m$ ,  $\underline{n}$ , and  $\underline{w}$  can depend explicitly on  $\zeta^i$  and, in addition, the interaction energy  $\underline{w}$  can have an ultralocal dependence on the matter space metric (“fleet metric”)  $h_{ij}$ . The fleet metric is defined such that  $ds = \sqrt{h_{ij} d\zeta^i d\zeta^j}$  measures the orthogonal distance  $ds$  between neighboring world lines with Lagrangian coordinates  $\zeta^i$  and  $\zeta^i + d\zeta^i$ . By including an explicit dependence on  $\zeta^i$  in the mass  $m$  and interaction energy  $\underline{w}$ , we allow for the possibility that the particles are not identical. In terms of the matter four-velocity

$$U^\alpha = \frac{\dot{\Upsilon}^\alpha}{\sqrt{-\dot{\Upsilon}^\beta \dot{\Upsilon}^\delta \gamma_{\beta\delta}}}, \quad (2.3)$$

the matter space metric takes the form

$$h_{ij} = \Upsilon^\alpha_{,i} (\gamma_{\alpha\beta} + U_\alpha U_\beta) \Upsilon^\beta_{,j}, \quad (2.4)$$

where the commas denote derivatives with respect to  $\zeta^i$ . In general,  $h_{ij}$  will not be the metric of any hypersurface of the spacetime manifold  $\mathcal{M}$ .

The system described by the action  $\bar{S}_{LL}$  of Eq. (2.2) is referred to as an *elastic medium*. The subscript  $LL$  indicates that it is the Lagrangian form (as opposed to the Hamiltonian form) of the action in the Lagrangian picture. Observe that this system is reparametrization invariant. That is,  $\bar{S}_{LL}$  is invariant under the transformation  $\delta\Upsilon^\alpha = -\dot{\Upsilon}^\alpha \epsilon$  induced by a reparametrization  $\sigma \rightarrow \sigma + \epsilon(\sigma, \zeta)$  of the particle world lines, where  $\epsilon$  vanishes at the end points in  $\sigma$ .

### B. The connection to fluids

$$\sigma = Z^0(y), \quad \zeta^i = Z^i(y), \quad (2.5)$$

We will now derive the Eulerian description of the elastic medium, in which the action is written as an integral over arbitrary spacetime coordinates  $y^\alpha$  on  $\mathcal{M}$ , and show its relations to the perfect fluids of Refs. [10–12].

To begin, let us assume that in the region of spacetime described, one and only one particle of the medium passes through each event. Then the Lagrangian coordinates  $\zeta^i$  along with the world line parameters  $\sigma$  form a set of coordinates in the spacetime region. These coordinates are related to the coordinates  $y^\alpha$  by the mappings  $y^\alpha = \Upsilon^\alpha(\sigma, \zeta)$ . We can introduce the inverse mappings

such that  $y^\alpha = \Upsilon^\alpha(Z^0(y), Z^i(y))$  is an identity. Note that  $Z^i(y)$  give the labels  $\zeta^i$  of the particle present at the event  $y^\alpha$ . The transition from the Lagrangian picture to the Eulerian picture is obtained by a change of dynamical variables in which  $\Upsilon^\alpha(\sigma, \zeta)$  is replaced by  $Z^0(y)$  and  $Z^i(y)$ .

To perform this change of variables, we first calculate the Jacobian  $|\partial(\sigma, \zeta)/\partial y|$ . Let us assume that the coordinate system  $\sigma, \zeta^i$  has the orientation of  $\mathcal{M}$ . Then, we have the identity

$$\begin{aligned} d\sigma \wedge d\zeta^1 \wedge d\zeta^2 \wedge d\zeta^3 &= Z^0_{,\alpha} Z^1_{,\beta} Z^2_{,\gamma} Z^3_{,\delta} dy^\alpha \wedge dy^\beta \wedge dy^\gamma \wedge dy^\delta \\ &= -\frac{1}{3!} \frac{\sqrt{-\gamma}}{\sqrt{h}} \epsilon^{\alpha\beta\gamma\delta} \epsilon_{ijk} Z^0_{,\alpha} Z^i_{,\beta} Z^j_{,\gamma} Z^k_{,\delta} dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3, \end{aligned} \quad (2.6)$$

where the commas followed by greek letters denote derivatives with respect to  $y^\alpha$ . Here,  $\epsilon^{\alpha\beta\gamma\delta}$  is the totally antisymmetric contravariant tensor on  $\mathcal{M}$  with  $\epsilon^{0123} = -1/\sqrt{-\gamma}$ , and  $\gamma$  is the determinant of the spacetime metric  $\gamma_{\alpha\beta}$ . Similarly,  $\epsilon_{ijk}$  is the totally antisymmetric covariant tensor on  $\mathcal{S}$  with  $\epsilon_{123} = \sqrt{h}$ . Observe that the inverse fleet metric can be written as

$$h^{ij} = Z^i_{,\alpha} \gamma^{\alpha\beta} Z^j_{,\beta}, \quad (2.7)$$

so that  $h$  can be expressed in terms of the Eulerian variables as  $h = 1/\det(h^{ij})$ . It will also be convenient to express the particle four-velocity  $U^\alpha$  in terms of the new variables:

$$U^\alpha = -(1/3!) \epsilon^{\alpha\beta\gamma\delta} Z^i_{,\beta} Z^j_{,\gamma} Z^k_{,\delta} \epsilon_{ijk}. \quad (2.8)$$

One can see that this expression is indeed the particle velocity by verifying that  $U^\alpha Z^i_{,\alpha} = 0$  and  $U^\alpha U_\alpha = -1$ . This allows us to rewrite the measure (2.6) as

$$\begin{aligned} d\sigma \wedge d\zeta^1 \wedge d\zeta^2 \wedge d\zeta^3 \\ = \sqrt{-\gamma/h} Z^0_{,\alpha} U^\alpha dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3. \end{aligned} \quad (2.9)$$

Finally, note that  $\dot{\Upsilon}^\alpha$  is proportional to  $U^\alpha$  and  $\dot{\Upsilon}^\alpha Z^0_{,\alpha} = 1$ , so that  $\dot{\Upsilon}^\alpha = U^\alpha / (U^\beta Z^0_{,\beta})$ . Combining this with the above results, we find that the action (2.2) takes the form

$$\bar{S}_{LE}[Z^i; \gamma_{\alpha\beta}] = - \int_{\mathcal{M}} d^4y \sqrt{-\gamma/h} (\underline{n}m + \underline{w}) \quad (2.10)$$

in the Eulerian picture. Here, the mass  $m$  and number density  $\underline{n}$  are fixed functions of  $Z^i$  while the interaction energy  $\underline{w}$  is a fixed function of  $h_{ij}$  and  $Z^k$ . The fleet metric  $h_{ij}$  is taken to be a function of the spacetime metric  $\gamma_{\alpha\beta}$  and the variables  $Z^i$  through Eq. (2.7).

Observe that the Eulerian form (2.10) of the action does not depend on the variable  $Z^0$ . This is a consequence of reparametrization invariance: A reparametrization of the world lines induces the transformation  $\delta Z^0 = \epsilon$  while leaving the other variables  $Z^i$

alone. The action (2.10) is invariant precisely because it is independent of  $Z^0$ .<sup>2</sup> This gauge freedom is removed simply by dropping  $Z^0$  from the list of dynamical variables. Thus, we view the action (2.10) as a functional of  $Z^i$  (and  $\gamma_{\alpha\beta}$ ) only.

From the action (2.10), it is straightforward to show that the isentropic perfect fluid action  $\bar{S}$  given by Eq. (6.15) of Ref. [11] (or the isentropic case of Eq. (4.20) of Ref. [12]) is equivalent to a ‘‘homogeneous and isotropic elastic medium.’’ To do so, consider the case in which the mass  $m$  is independent of the particle labels  $Z^i$ , so the particles are identical. Also assume that the *proper* interaction energy density  $w = \underline{w}/\sqrt{h}$  (which is the interaction energy per unit proper spatial volume as measured in the rest frame of the matter) depends only on the *proper* particle number density  $n = \underline{n}/\sqrt{h}$  (which is the number of particles per unit proper spatial volume as measured in the rest frame of the matter). That is,  $\underline{w}$  depends on  $Z^i$  and  $h_{ij}$  only through the combination  $\underline{w} = \sqrt{h}w(\underline{n}/\sqrt{h})$  for some function  $w(n)$ .

The factor  $(\underline{n}m + \underline{w})/\sqrt{h}$  that appears in the integrand of the action  $\bar{S}_{LE}$  is the *proper* energy density of the medium, which we will denote by  $\rho$ . Our restriction to a homogeneous and isotropic medium implies that  $\rho$  only depends on the proper number density,  $\rho = \rho(n)$ . Thus,

<sup>2</sup>There is a subtle point here. The action (2.2) is defined for a fixed integration region in  $\mathcal{S} \times \mathcal{R}$ ; that is, for fixed ranges of the integration parameters  $\sigma$  and  $\zeta^i$ . The integration region in  $\mathcal{M}$  is determined by the integration region in  $\mathcal{S} \times \mathcal{R}$  only if we fix  $\Upsilon^\alpha$  at the boundaries. Then, in particular, the end points in  $\sigma$  determine initial and final hypersurfaces in  $\mathcal{M}$  which we assume to be spacelike. Since the range of  $\sigma$  in (2.2) is fixed, the action functional (2.10) is defined for the class of variables  $Z^0(y)$  with fixed values on the initial and final hypersurfaces in  $\mathcal{M}$ . Therefore the gauge freedom in (2.10) consists of variations  $\delta Z^0 = \epsilon$  for which  $\epsilon$  vanishes on the initial and final hypersurfaces. In this way, we see that the gauge freedoms for the actions (2.2) and (2.10) coincide.

the action (2.10) in the homogeneous, isotropic case may be written as

$$\bar{S}_{LE}[Z^i; \gamma_{\alpha\beta}] = - \int_{\mathcal{M}} d^4y \sqrt{-\gamma} \rho(n), \quad (2.11)$$

$$n = \underline{n}/\sqrt{h} = \left[ \frac{1}{3!} \underline{n}^2 \bar{\epsilon}_{ijk} \bar{\epsilon}_{i'j'k'} Z^i{}_{,\alpha} Z^{i'}{}_{,\alpha'} \gamma^{\alpha\alpha'} Z^j{}_{,\beta} Z^{j'}{}_{,\beta'} \gamma^{\beta\beta'} Z^k{}_{,\delta} Z^{k'}{}_{,\delta'} \gamma^{\delta\delta'} \right]^{1/2}. \quad (2.12)$$

Here,  $\bar{\epsilon}_{ijk}$  is the antisymmetric tensor density (of weight  $-1$ ) on  $\mathcal{S}$  with  $\bar{\epsilon}_{123} = 1$ . If we identify the tensor  $\eta_{ijk}(\zeta)$  of Refs. [11,12] with  $\underline{n}(\zeta)\bar{\epsilon}_{ijk}$ , then the expression (2.12) exactly matches the definition of  $n$  given through Eq. (5.1) of [11] (and through Eqs. (4.23) and (4.26) of [12]). As a result, the (“barred”) one component isentropic perfect fluid of Refs. [11,12] is seen to be a special case of the relativistic elastic medium of DeWitt [3].

### III. CLOCKS AND REFERENCE SYSTEMS

Reference materials such as elastic media and perfect fluids can be used to provide a physical system of coordinates in space. However, by itself, a reference material does not provide a complete coordinate system in *space-time*, as all points along a given particle world line are labeled by the same coordinates  $\zeta^i$ . This can be remedied by adding an additional degree of freedom to the particles whose value changes along the world lines. Such a degree of freedom may be called a “fleet of clocks” and a reference medium coupled to a fleet of clocks is said to constitute a *reference system*. Note that, so far, we have not distinguished between “good clocks” which accurately measure proper time and “bad clocks” whose readings vary along the world lines in a more complicated way.

#### A. Coupling clocks

The literature contains two different mechanisms for coupling additional “clock” degrees of freedom to a reference material. One of these was used by DeWitt [3] and was explicitly described as a coupling of clocks to an elastic medium. The other is implicitly contained in the literature on isentropic (single component) perfect fluids [4,11,12], although the word “clock” does not appear in any of these works. Moreover, with a reinterpretation of variables and a suitable choice of equation of state, a *nonisentropic* perfect fluid is actually equivalent to a homogeneous, isotropic elastic medium with the DeWitt-type clock coupling. The following summary along with the results in Appendix A should clarify this situation.

We first restrict our attention to isentropic perfect fluids. Historically, two different action principles were developed for relativistic perfect fluids which both used

where  $n = \underline{n}/\sqrt{h}$ . This is precisely the perfect fluid action  $\bar{S}$  given in Refs. [11,12] for the isentropic case, although in those references the number density  $n$  was expressed somewhat differently. To see that the definition of  $n$  in Refs. [11,12] agrees with the present definition, we use Eq. (2.7) to write the number density explicitly as

scalar fields (“velocity potentials”) as the basic variables. These actions were later shown to be equivalent [12]. In particular, the action of Ref. [4], which we will denote by  $S$ , was shown to differ from the action of Ref. [10], which is denoted by  $\bar{S}$  in Eq. (2.11), by the addition of one degree of freedom per space point. This additional degree of freedom is cyclic, so the action  $\bar{S}$  can be derived from  $S$  by removing the extra degree of freedom through Routh’s procedure [16].

Since the “unbarred” description of perfect fluids contains an extra degree of freedom per space point, and since this degree of freedom changes along the world lines, the philosophy stated at the beginning of this section allows us to interpret this degree of freedom as representing a fleet of clocks. For this reason, we refer to the “unbarred” fluid as a fluid *coupled to a fleet of clocks*.

For both the clock coupling discussed by DeWitt and the one implicit in the perfect fluid literature, the essential idea is to add a pair  $(\Theta, J)$  of first order degrees of freedom for each particle in the medium. Thus, we consider two fields,  $\Theta(\sigma, \zeta)$  and  $J(\sigma, \zeta)$ , and add the first order kinetic term

$$\int d\sigma \int_{\mathcal{S}} d^3\zeta \underline{n} J \dot{\Theta} \quad (3.1)$$

to the Lagrangian picture action (2.2). The Eulerian form of the kinetic term may be obtained from Eq. (2.9) and the relation  $\delta_{\beta}^{\alpha} = \dot{\Upsilon}^{\alpha} Z^0{}_{,\beta} + \Upsilon^{\alpha}{}_{,i} Z^i{}_{,\beta}$ , and is given by

$$\int_{\mathcal{M}} d^4y \underline{n} J \Theta_{,\alpha} U^{\alpha} \sqrt{-\gamma/h}. \quad (3.2)$$

The clocks are then coupled to the particles by letting either  $\underline{n}(\zeta)$  or  $m(\zeta)$  depend on  $J$  in the original action.

To produce the “unbarred” fluid action of Refs. [4,11,12], the clocks are coupled by letting  $\underline{n}(\zeta) \rightarrow J \underline{n}(\zeta)$  in the action (2.11) and adding the kinetic term (3.2). A detailed explanation of how this produces the “unbarred” action is given in Appendix A. This method of clock coupling does not, in general, lead to a “good” set of clocks. This can be seen from the equation of motion for  $J$ , which is

$$\Theta_{,\alpha} U^{\alpha} = \rho'(J \underline{n}/\sqrt{h}), \quad (3.3)$$

where  $\rho'(n) = \partial\rho(n)/\partial n$ . The relation (3.3) shows that the rate of advance of the clock variable  $\Theta$  relative to

the flow of proper time may depend on the particle density  $n$ . Thus, for clocks coupled in this way, the internal clock mechanism is not shielded from the external pressures and forces between particles. For some equations of state  $\rho(n)$ , such clocks may not even run monotonically in proper time.

Because the method  $\underline{n} \rightarrow J\underline{n}$  of clock coupling does not, in general, yield a good set of clocks, we will focus on the method described by DeWitt [3]. In this method, the mass  $m$  is allowed to depend on  $J$ . Thus, the mass of each particle is no longer a fixed constant and, in fact, it acts as a Hamiltonian  $m(\zeta^i, J)$  that drives the motion of the clock  $\Theta(\zeta^i)$  attached to particle  $\zeta^i$ . In the Lagrangian picture, the action  $S_{LL}[\Upsilon^\alpha, J, \Theta; \gamma_{\alpha\beta}]$  is obtained by adding Eqs. (2.2) and (3.1) and taking the number density, mass, and internal energy density to depend on  $\zeta^i$ ,  $J$ , and  $h_{jk}$  through  $\underline{n} = \underline{n}(\zeta^i)$ ,  $m = m(\zeta^i, J)$ , and  $\underline{w} = \underline{w}(\zeta^i, h_{jk})$ . The action also takes a simple form in the Eulerian picture:

$$S_{LE}[Z^i, J, \Theta; \gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4y \sqrt{-\gamma/h} \{ \underline{n} J \Theta_{,\alpha} U^\alpha - (\underline{n} m + \underline{w}) \}, \quad (3.4)$$

where  $\underline{n} = \underline{n}(Z^i)$ ,  $m = m(Z^i, J)$ , and  $\underline{w} = \underline{w}(Z^i, h_{jk})$ . Note that the equation of motion for  $J$  shows that the clock satisfies

$$\Theta_{,\alpha} U^\alpha = \partial m / \partial J, \quad (3.5)$$

while the equation of motion for  $\Theta$  shows that  $J$  (and therefore  $\partial m / \partial J$ ) is a constant along each particle world line. Thus, for the DeWitt-type coupling,  $\Theta$  increases in direct proportion to proper time along the world line and always defines a good clock. We see that (3.3) and (3.5) coincide when the medium is dust, since in that case  $\rho'$  is a constant.

Also note that many different clock Hamiltonians  $m(\zeta, J)$  lead to equivalent results. This is because, for any invertible function  $f(J)$ , the replacement of  $J$  by  $\tilde{J} = f(J)$  and  $\Theta$  by  $\tilde{\Theta} = \Theta / f'(J)$  changes the kinetic term (3.1) only by a boundary term but changes the clock Hamiltonian from  $m(\zeta, J)$  to  $m(\zeta, f(J))$ . Thus, any two clock Hamiltonians  $m_1$  and  $m_2$  related by  $m_1(\zeta, J) = m_2(\zeta, f(J))$  for invertible  $f$  are equivalent.

Finally, consider a homogeneous and isotropic reference system with DeWitt-type clock coupling. As before, the internal energy density has the form  $\underline{w} = \sqrt{hw}(\underline{n}/\sqrt{h})$ , and now the mass is a function of  $J$  only:  $m = m(J)$ . From Eq. (3.4), the action becomes

$$S_{LE}[Z^i, J, \Theta; \gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4y \sqrt{-\gamma} \{ \underline{n} J \Theta_{,\alpha} U^\alpha / \sqrt{h} - \rho(\underline{n}/\sqrt{h}, J) \}, \quad (3.6)$$

where  $\rho(n, J) = nm(J) + w(n)$ . This action is equiva-

lent to the action (referred to as the “hybrid action” in Ref. [12]) for a nonisentropic perfect fluid with equation of state  $\rho(n, J) = nm(J) + w(n)$ . In the perfect fluid literature, the variable  $J$  is interpreted as the entropy per particle and the variable  $\Theta$  is interpreted as the thermasy [14]. (The thermasy is a variable whose gradient along the particle world lines is proportional to the local temperature.) The connection between the action (3.6) above and the action of Ref. [12] can be established easily. One simply uses Eqs. (2.7) and (2.8) to show that the quantity  $\sqrt{-\gamma/h} \underline{n} U^\alpha$  that appears in the action (3.6) is the same function of  $Z^i$  and  $\gamma_{\alpha\beta}$  as the quantity  $J^\alpha$  that appears in the “hybrid action” of Ref. [12].

## B. Hamiltonian formulation of reference systems

We would like to study the diffeomorphism invariance of general relativity by using the reference system defined by DeWitt’s method of coupling clocks to an elastic medium. Since we will examine this issue from the canonical perspective, we first perform a Hamiltonian analysis of the reference system. In order to make contact with the usual Hamiltonian description of gravity, we will work initially in the Eulerian picture.

We begin by foliating spacetime  $\mathcal{M}$  with a family of hypersurfaces  $\Sigma_t$  labeled by the parameter  $t$ . If  $x^\alpha$  are coordinates on a hypersurface, then the spacetime coordinates  $y^\alpha$  are related to  $(t, x^\alpha)$  through mappings  $y^\alpha = Y^\alpha(t, x)$ . As usual, the lapse function  $N^\perp$ , shift vector  $N^\alpha$ , and spatial metric  $g_{ab}$  are related to the spacetime metric by

$$\dot{Y}^\alpha = N^\perp n^\alpha + N^\alpha Y^\alpha_{,a}, \quad (3.7)$$

$$g_{ab} = Y^\alpha_{,a} \gamma_{\alpha\beta} Y^\beta_{,b}, \quad (3.8)$$

$$\gamma^{\alpha\beta} = -n^\alpha n^\beta + Y^\alpha_{,a} g^{ab} Y^\beta_{,b}, \quad (3.9)$$

where  $n^\alpha$  is the unit normal of the hypersurfaces. The quantities  $N^\perp$ ,  $N^\alpha$ ,  $g_{ab}$ , and  $n^\alpha$  are functions of  $t$  and  $x^\alpha$ . The spacetime metric depends on  $t$  and  $x^\alpha$  through the mapping  $Y^\alpha$ ; that is,  $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(Y(t, x))$ . Note that we are now using the overdot to denote  $\partial/\partial t$ .

The variables  $Z^i(y)$ ,  $\Theta$ , and  $J$  are spacetime scalars. They can be pulled back from  $\mathcal{M}$  to  $\Sigma \times \mathbb{R}$  by the mapping  $Y^\alpha(t, x)$  to yield  $t$ -dependent scalars on  $\Sigma$  which we denote by the same kernel letter: for example,  $Z^i(y) \rightarrow Z^i(t, x)$ . With the definitions above, it is not difficult to show that (with a slight abuse of notation)

$$Z^i_{,\alpha} n^\alpha = (\dot{Z}^i - N^\alpha Z^i_{,\alpha}) / N^\perp, \quad (3.10)$$

$$Z^i_{,\alpha} Y^\alpha_{,a} = Z^i_{,a}, \quad (3.11)$$

$$Z^i_{,\alpha} n^\alpha = -n^\alpha Z^i_{,\beta} n^\beta + Y^\alpha_{,a} g^{ab} Z^i_{,b}. \quad (3.12)$$

Similar relations hold for  $\Theta$  and  $J$ .

The function  $Z^i(t, x)$  is a  $t$ -dependent mapping from the space  $\Sigma$  to the matter space  $\mathcal{S}$ . It is useful to consider the inverse mappings  $X^\alpha(t, \zeta)$ , defined by  $\zeta^i = Z^i(t, X(t, \zeta))$ . The mappings  $Z^i$  and  $X^\alpha$  are related by the identities  $\delta_j^i = Z^i_{,\alpha} X^\alpha_{,j}$ ,  $\delta_b^a = X^\alpha_{,i} Z^i_{,b}$ ,  $\dot{Z}^i = -Z^i_{,\alpha} X^\alpha_{,t}$ , and  $\dot{X}^\alpha = -X^\alpha_{,i} Z^i_{,t}$ . The notation used

here is somewhat abbreviated in that we have omitted any explicit specification of the functional dependences. We will continue this practice below.

Now consider the clock kinetic term (3.2). From Eq. (2.8) we have

$$\Theta_{,\alpha} U^\alpha = -\frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta} \Theta_{,\alpha} Z^{\beta i} Z^{\gamma j} Z^{\delta k} \epsilon_{ijk} \quad (3.13)$$

which, using Eq. (3.12) and the identity  $n^\alpha \epsilon_{\alpha\beta\gamma\delta} Y^\beta_{,a} Y^\gamma_{,b} Y^\delta_{,c} = \epsilon_{abc}$ , takes the form

$$\begin{aligned} \Theta_{,\alpha} U^\alpha &= \frac{1}{N^\perp} (\dot{\Theta} - N^\alpha \Theta_{,\alpha}) \frac{1}{3!} \epsilon^{abc} Z^i_{,a} Z^j_{,b} Z^k_{,c} \epsilon_{ijk} \\ &\quad - \frac{1}{2N^\perp} (\dot{Z}^i - N^\alpha Z^i_{,\alpha}) \epsilon^{abc} \Theta_{,\alpha} Z^j_{,b} Z^k_{,c} \epsilon_{ijk}. \end{aligned} \quad (3.14)$$

It is convenient to write this result as

$$\Theta_{,\alpha} U^\alpha = \Gamma \dot{\Theta} / N^\perp + \Gamma (V^\alpha - N^\alpha / N^\perp) \Theta_{,\alpha}, \quad (3.15)$$

where

$$S[Z^i, J, \Theta; N^\perp, N^\alpha, g_{ab}] = \int dt \int_\Sigma d^3x (\det Z^i_{,\alpha}) \left\{ \underline{n} J \dot{\Theta} + \underline{n} J (N^\perp V^\alpha - N^\alpha) \Theta_{,\alpha} - N^\perp (\underline{n} m + \underline{w}) / \Gamma \right\}. \quad (3.20)$$

It is now straightforward to proceed with the canonical analysis. The momentum conjugate to  $\Theta$  is  $(\det Z^i_{,\alpha}) \underline{n} J$ , which we will denote by  $\Pi$ . The momentum conjugate to  $Z^i$  is

$$P_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{Z}^i} = -(\det Z^j_{,\alpha}) \left[ \Gamma (\underline{n} m + \underline{w}) V_b + \underline{t}_{k\ell} Z^k_{,b} Z^\ell_{,c} V^c / \Gamma + \underline{n} J \Theta_{,b} \right] X^b_{,i}, \quad (3.21)$$

where  $\underline{t}_{ij} = 2(\partial \underline{w} / \partial h^{ij})$  is the Lagrangian frame stress tensor [3]. Note that  $\dot{Z}^i$  appears in this expression only in the combination  $V^\alpha = -(\dot{Z}^i X^\alpha_{,i} - N^\alpha) / N^\perp$ . Thus, the solution for  $\dot{Z}^i$  as a function of the canonical variables, lapse, and shift has the form

$$\dot{Z}^i = (N^\alpha - N^\perp V^\alpha) Z^i_{,\alpha}, \quad (3.22)$$

where we view  $V^\alpha$  as a function of the canonical variables through Eqs. (3.21) and (3.18) for  $P_i$  and  $h^{ij}$ .

Collecting together the above results, we find the Hamiltonian form of the action for an elastic medium coupled to clocks in the Eulerian picture:

$$\begin{aligned} S_{HE}[Z^i, P_i, \Theta, \Pi; N^\perp, N^\alpha, g_{ab}] \\ = \int dt \int_\Sigma d^3x \left\{ P_i \dot{Z}^i + \Pi \dot{\Theta} - N^\perp \mathcal{H}_\perp^m - N^\alpha \mathcal{H}_\alpha^m \right\}, \end{aligned} \quad (3.23)$$

where

$$\mathcal{H}_\alpha^m = P_i Z^i_{,\alpha} + \Pi \Theta_{,\alpha}, \quad (3.24)$$

$$\mathcal{H}_\perp^m = (\det Z^i_{,\alpha}) [\Gamma (\underline{n} m + \underline{w}) + \underline{t}_{ij} Z^i_{,\alpha} V^\alpha Z^j_{,b} V^b / \Gamma]. \quad (3.25)$$

$$\Gamma \equiv -n_\alpha U^\alpha = \frac{1}{3!} \epsilon^{abc} Z^i_{,a} Z^j_{,b} Z^k_{,c} \epsilon_{ijk}, \quad (3.16)$$

$$\begin{aligned} V^\alpha &\equiv \frac{1}{N^\perp} (\dot{X}^\alpha + N^\alpha) \\ &= -\frac{1}{2N^\perp \Gamma} \epsilon^{abc} \epsilon_{ijk} Z^j_{,b} Z^k_{,c} (\dot{Z}^i - N^\alpha Z^i_{,\alpha}). \end{aligned} \quad (3.17)$$

Here,  $V^\alpha$  is the spatial velocity of the particles as measured by the observers who are at rest in  $\Sigma_t$  and  $\Gamma = 1/\sqrt{1 - V^\alpha g_{ab} V^b}$  is the relativistic "gamma" factor that characterizes the boost between these observers and the comoving reference frame.

Using Eqs. (2.7) and (3.16), we find that the fleet metric and its determinant can be expressed as

$$h^{ij} = Z^i_{,a} Z^j_{,b} (g^{ab} - V^\alpha V^\beta), \quad (3.18)$$

$$\sqrt{h} = \sqrt{g} \Gamma / (\det Z^i_{,\alpha}). \quad (3.19)$$

These results, along with Eq. (3.15), show that the action (3.4) can be written as

Here,  $\Gamma$  is defined in Eq. (3.16) and the matter spatial velocity  $V^\alpha$  and fleet metric  $h_{ij}$  are defined as functions of the canonical variables through Eqs. (3.18) and (3.21). The clock Hamiltonian, number density, and interaction energy density have dependences  $m = m(\Pi / \underline{n} (\det Z^i_{,\alpha}), Z^j)$ ,  $\underline{n} = \underline{n}(Z^i)$ , and  $\underline{w} = \underline{w}(h_{ij}, Z^k)$ .

#### IV. REFERENCE SYSTEMS IN GENERAL RELATIVITY

Thus far we have treated the gravitational field as an external background. In this section, we shall include gravity as a dynamical field and study our reference system as coupled to general relativity.

##### A. Coupling clocks to the gravitational field

Since the action for the matter-clock system contains no derivatives of the spacetime metric, the action for that system coupled to gravity is obtained by adding its action to the gravitational action. In the Eulerian setting, we have

$$\begin{aligned}
S_{HE}[Z^i, P_i, \Theta, \Pi, g_{ab}, p^{ab}, N^\perp, N^a] \\
= \int dt \int_\Sigma d^3x \{ P_i \dot{Z}^i + \Pi \dot{\Theta} + p^{ab} \dot{g}_{ab} \\
- N^\perp (\mathcal{H}_\perp^m + \mathcal{H}_\perp^g) - N^a (\mathcal{H}_a^m + \mathcal{H}_a^g) \}, \quad (4.1)
\end{aligned}$$

where  $\mathcal{H}_\perp^g$  and  $\mathcal{H}_a^g$  are the familiar constraints of vacuum general relativity [17]. The coupled system is subject to the constraints

$$\mathcal{H}_\perp \equiv \mathcal{H}_\perp^m + \mathcal{H}_\perp^g = 0, \quad (4.2)$$

$$\mathcal{H}_a \equiv \mathcal{H}_a^m + \mathcal{H}_a^g = 0, \quad (4.3)$$

since the lapse function  $N^\perp$  and shift vector  $N^a$  are varied in the action principle.

It is also useful to consider the coupled system in the Lagrangian picture. The Hamiltonian form of the action, namely  $S_{HL}$ , can be obtained from a 3+1 decomposition of the Lagrangian form of the action in the Lagrangian picture, namely  $S_{LL}$ . Alternatively,  $S_{HL}$  can be obtained from the Eulerian picture action  $S_{HE}$  by performing a point canonical transformation from Eulerian to Lagrangian variables. The details of this transformation are described in Ref. [12] for the perfect fluid case and are essentially the same here. Thus, we define the new matter variables  $X^a(\zeta)$  such that  $Z^i(X(\zeta)) = \zeta^i$ . We also define the new clock variable  $\Theta(\zeta) = \Theta(X(\zeta))$  and the new gravitational field variable

$$g_{ij}(\zeta) = X^a{}_{,i}(\zeta) X^b{}_{,j}(\zeta) g_{ab}(X(\zeta)). \quad (4.4)$$

This is the spatial metric expressed in the Lagrangian system of coordinates induced on the slices  $\Sigma$  by the matter. The new conjugate momenta are defined by [12]

$$\underline{P}_a(\zeta) = -(\det X^b{}_{,j}) \mathcal{H}_a(X(\zeta)), \quad (4.5)$$

$$\underline{\Pi}(\zeta) = (\det X^b{}_{,j}) \Pi(X(\zeta)), \quad (4.6)$$

$$\begin{aligned}
\underline{p}^{ij}(\zeta) &= (\det X^c{}_{,k}) Z^i{}_{,a}(X(\zeta)) Z^j{}_{,b}(X(\zeta)) \\
&\quad \times p^{ab}(X(\zeta)). \quad (4.7)
\end{aligned}$$

With this transformation, the action (4.1) becomes

$$\begin{aligned}
S_{HL}[X^a, \underline{P}_a, \Theta, \underline{\Pi}, g_{ij}, \underline{p}^{ij}, N^\perp, N^a] \\
= \int dt \int_S d^3\zeta \{ \underline{P}_a \dot{X}^a + \underline{\Pi} \dot{\Theta} + \underline{p}^{ij} \dot{g}_{ij} \\
- N^\perp (X) \underline{\mathcal{H}}_\perp + N^a (X) \underline{\mathcal{P}}_a \}, \quad (4.8)
\end{aligned}$$

where the lapse and shift are functions of  $X^a(\zeta)$  and the Hamiltonian constraint is  $\underline{\mathcal{H}}_\perp = (\det X^a{}_{,i}) \mathcal{H}_\perp$ . In terms of the new canonical variables, we have

$$\underline{\mathcal{H}}_\perp = \Gamma(\underline{nm} + \underline{w}) + \underline{t}_{ij} Z^i{}_{,a} V^a Z^j{}_{,b} V^b / \Gamma + \underline{\mathcal{H}}_\perp^g. \quad (4.9)$$

Here, we have used  $\underline{\mathcal{H}}_\perp^g(\zeta) = (\det X^a{}_{,i}) \mathcal{H}_\perp^g(X(\zeta))$  to denote the pullback [by the mapping  $X(\zeta)$ ] of the gravitational contribution to the Hamiltonian constraint from  $\Sigma$  to  $\mathcal{S}$ . Since the Lagrangian gravitational variables  $g_{ij}(\zeta)$  and  $\underline{p}^{ij}(\zeta)$  are obtained from the Eulerian gravitational

variables  $g_{ab}(x)$  and  $p^{ab}(x)$  by the pullback mapping from  $\Sigma$  to  $\mathcal{S}$ , the term  $\underline{\mathcal{H}}_\perp^g$  is constructed by replacing the Eulerian variables in  $\mathcal{H}_\perp^g$  with the corresponding Lagrangian variables. Likewise, we can define the pullback of the gravitational contribution to the momentum constraint by  $\underline{\mathcal{H}}_i^g(\zeta) = (\det X^b{}_{,j}) X^a{}_{,i} \mathcal{H}_a^g(X(\zeta))$ .

The velocity  $V^a$  and fleet metric  $h_{ij}$  that appear in the Hamiltonian constraint (4.9) must, of course, be expressed in terms of the new canonical variables. It turns out to be convenient to work with  $V_a X^a{}_{,i}$ , which is the velocity expressed in the Lagrangian coordinate system, in place of  $V^a$ . In Appendix B we show that  $h_{ij}$  and  $V_a X^a{}_{,i}$  can be expressed in terms of  $\underline{\Pi}$ ,  $\underline{P}_a X^a{}_{,i}$ ,  $g_{ij}$ , and  $\underline{p}^{ij}$ .

In the action (4.8), the lapse function and shift vector are functions of  $X^a(\zeta)$ . We can define the new variables  $N^\perp(\zeta)$  and  $N^i(\zeta) = Z^i{}_{,a} N^a(X(\zeta))$  by mapping  $N^\perp(x)$  and  $N^a(x)$  from  $\Sigma$  to  $\mathcal{S}$ . The action then reads

$$\begin{aligned}
S_{HL}[X^a, \underline{P}_a, \Theta, \underline{\Pi}, g_{ij}, \underline{p}^{ij}, N^\perp, N^i] \\
= \int dt \int_S d^3\zeta \{ \underline{P}_a \dot{X}^a + \underline{\Pi} \dot{\Theta} + \underline{p}^{ij} \dot{g}_{ij} \\
- N^\perp \underline{\mathcal{H}}_\perp - N^i \underline{\mathcal{H}}_i \}, \quad (4.10)
\end{aligned}$$

where the Hamiltonian and momentum constraints are

$$\underline{\mathcal{H}}_\perp = \Gamma(\underline{nm} + \underline{w}) + \Gamma^3 \underline{t}^{ij} V_a X^a{}_{,i} V_b X^b{}_{,j} + \underline{\mathcal{H}}_\perp^g, \quad (4.11)$$

$$\underline{\mathcal{H}}_i = -\underline{P}_a X^a{}_{,i}. \quad (4.12)$$

Here, the Hamiltonian constraint  $\underline{\mathcal{H}}_\perp$  is obtained from Eq. (4.9) and the identity (B2) of Appendix B. The momentum constraint  $\underline{\mathcal{H}}_i$  equals the result obtained by mapping  $\mathcal{H}_a$  to the matter space  $\mathcal{S}$ ; see Eq. (4.5).

As a final remark, let us point out that there are two symmetries that play important roles in the description of reference systems [9,11,12]. The first is the freedom to reset the clocks. That is, the action is invariant under a transformation in which the zero point of each clock is changed. The infinitesimal version of this transformation is generated through the Poisson brackets by the functional

$$Q[\vartheta] = \int_S d^3\zeta \vartheta \underline{\Pi} = \int_\Sigma d^3x \vartheta(Z(x)) \Pi(x). \quad (4.13)$$

Thus, the clock at  $\zeta^i$  is transformed according to  $\delta\Theta(\zeta) \equiv \{\Theta(\zeta), Q[\vartheta]\} = \vartheta(\zeta)$ . The canonical variables in the Lagrangian picture, other than  $\Theta(\zeta)$ , remain unchanged under this transformation. In the Eulerian picture, we have  $\delta\Theta(x) = \vartheta(Z(x))$  and  $\delta P_i(x) = -\vartheta_{,i}(Z(x)) \Pi(x)$ , with the other canonical variables unchanged. The second symmetry is an invariance with respect to changes in the Lagrangian coordinate labels. That is, the action is invariant under diffeomorphisms of the matter space  $\mathcal{S}$ . The infinitesimal version of this transformation is generated by

$$\begin{aligned}
 Q[\xi^i] &= \int_S d^3\zeta \xi^i (\underline{P}_a X^a{}_{,i} + \underline{\Pi} \Theta_{,i} + \underline{\mathcal{H}}_i) \\
 &= - \int_\Sigma d^3x \xi^i (Z(x)) P_i(x), \tag{4.14}
 \end{aligned}$$

where the vector field  $\xi^i(\zeta)$  is the infinitesimal generator of the diffeomorphism. The canonical variables in the Lagrangian picture just transform according to their  $\mathcal{S}$ -tensor character. For the Eulerian picture variables, we find  $\delta Z^i(x) = -\xi^i(Z(x))$  and  $\delta P_i(x) = \xi^j{}_{,i}(Z(x)) P_j(x)$ , while the remaining variables (which are tensors on  $\Sigma$ ) are invariant.

**B. New constraints and the gravitational Hamiltonian**

When the Hamiltonian constraint can be resolved for the clock momenta, a new set of constraints may be introduced that allows the system to be deparametrized. We begin by recalling that, as shown in Appendix B, the fleet metric  $h_{ij}$  and velocity  $V_a X^a{}_{,i}$  are functions of  $\underline{\Pi}$ ,  $g_{ij}$ ,  $p^{ij}$ , and  $\underline{\mathcal{H}}_i = -\underline{P}_a X^a{}_{,i}$ . It follows that the Hamiltonian constraint (4.11) depends on the gravitational canonical variables  $g_{ij}$  and  $p^{ij}$ , the clock momentum  $\underline{\Pi}$ , and also the particle canonical variables in the combinations given by the momentum constraints  $\underline{\mathcal{H}}_i$ . We can set  $\underline{\mathcal{H}}_i$  equal to zero in  $\underline{\mathcal{H}}_\perp$  without changing the content of the constraints  $\underline{\mathcal{H}}_\perp = 0$ ,  $\underline{\mathcal{H}}_i = 0$ . Then the Hamiltonian constraint  $\underline{\mathcal{H}}_\perp$  depends only on  $g_{ij}$ ,  $p^{ij}$ , and  $\underline{\Pi}$ .<sup>3</sup>

Let us assume that we can resolve the constraint  $\underline{\mathcal{H}}_\perp = 0$  with respect to  $\underline{\Pi}$ . We would then obtain a new constraint that has the form

$$\underline{\mathcal{H}}_\perp = \underline{\Pi} + \underline{h}[g_{ij}, p^{ij}], \tag{4.15}$$

where the true (gravitational) Hamiltonian  $\underline{h}$  is a functional of the gravitational variables only. The constraints  $\underline{\mathcal{H}}_\perp = 0$  and  $\underline{\mathcal{H}}_i = 0$  constitute a complete set of constraints for the system which are equivalent to the original Hamiltonian and momentum constraints. We can smear  $\underline{\mathcal{H}}_\perp$  with a prescribed function  $N^\dagger(\zeta)$  on  $\mathcal{S}$  to form the functional  $H[N^\dagger]$ . The smeared constraint  $H[N^\dagger]$  generates, through the Poisson brackets, changes in the canonical variables that result when the hypersurface  $\Sigma$  is displaced along the particle world lines by the clock time  $N^\dagger(\zeta)$ .

The constraints  $\underline{\mathcal{H}}_\perp$  have vanishing Poisson brackets among themselves and with the momentum constraints  $\underline{\mathcal{H}}_i$ . As usual, the momentum constraints form a representation of the Lie algebra of spatial diffeomorphisms.

<sup>3</sup>This perhaps surprising result can be understood from the following simple observations. The clock variable  $\Theta$  does not appear in  $\underline{\mathcal{H}}_\perp$  since  $\Theta$  is cyclic. In addition, the particle variables  $X^a$  and  $\underline{P}_a$  must appear in  $\underline{\mathcal{H}}_\perp$  in the combination  $\underline{P}_a X^a{}_{,i}$ , since this is the only combination of canonical variables that involves  $X^a$  or  $\underline{P}_a$  and transforms as a tensor on  $\mathcal{S}$ . By setting  $\underline{P}_a X^a{}_{,i}$  equal to zero, we find that  $\underline{\mathcal{H}}_\perp$  depends only on  $g_{ij}$ ,  $p^{ij}$ , and  $\underline{\Pi}$ .

(Note that, since the gravitational variables  $g_{ij}$ ,  $p^{ij}$  and clock variables  $\Theta$ ,  $\underline{\Pi}$  are independent of  $\Sigma$ , they have vanishing Poisson brackets with  $\underline{\mathcal{H}}_i$ .) Since there are no nonvanishing cross terms in the brackets of  $\underline{\mathcal{H}}_\perp$  with itself, the gravitational Hamiltonian  $\underline{h}$  must have vanishing brackets with itself:  $\{\underline{h}(\zeta), \underline{h}(\zeta')\} = 0$ . We should emphasize that this property holds for any gravitational Hamiltonian  $\underline{h}$  that can be derived through the coupling to an elastic medium with clocks; that is, for any choice of internal energy density  $\underline{w} = \underline{w}(\zeta^i, h_{jk})$ . In practice, the equations that determine  $\underline{h}$  are not solvable by analytical means except for the simplest of cases, such as dust ( $\underline{w} = 0$ ).<sup>4</sup>

Finally, note that the constraint (4.15) allows the system to be deparametrized. That is, consider the action  $S_{HL}$  with  $N^\perp \underline{\mathcal{H}}_\perp$  replaced by  $N^\dagger \underline{\mathcal{H}}_\perp$ . The particle variables  $X^a$  and  $\underline{P}_a$  are completely decoupled from the gravitational and clock variables and can be dropped from the action. The variables  $N^\dagger$  and  $\underline{\Pi}$  can be eliminated through the solution of their equations of motion, and the parameter  $t$  can be replaced by the clock time  $\theta = \Theta(t)$ . This leads to the action

$$S[g_{ij}, p^{ij}] = \int d\theta \int_S d^3\zeta \{ \dot{p}^{ij} g_{ij} - \underline{h}[g_{ij}, p^{ij}] \}, \tag{4.16}$$

where the overdot denotes  $\partial/\partial\theta$ . This action for the gravitational field contains no gauge (diffeomorphism) invariance.

**C. The case of dust**

We will now explicitly display the results for dust, where the interaction energy density vanishes ( $\underline{w} = 0$ ). In this case, the Hamiltonian constraint (4.11) becomes

$$\underline{\mathcal{H}}_\perp = \sqrt{(\underline{n}m)^2 + \underline{\mathcal{H}}_i^g g^{ij} \underline{\mathcal{H}}_j^g} + \underline{\mathcal{H}}_\perp^g. \tag{4.17}$$

Recall that  $\underline{n} = \underline{n}(\zeta)$  and  $m = m(\zeta, \underline{\Pi}(\zeta)/\underline{n}(\zeta))$ . Let us consider various possible choices for  $\underline{n}m$ . With the clock Hamiltonian  $m(\zeta, J) = J/k$ , where  $k$  is a positive constant, then  $\underline{n}m = \underline{\Pi}/k$  and the new constraint takes the form  $\underline{\mathcal{H}}_\perp = \underline{\Pi} + \underline{h}$  with the gravitational Hamiltonian

$$\underline{h} = \mp k \sqrt{(\underline{\mathcal{H}}_\perp^g)^2 - \underline{\mathcal{H}}_i^g g^{ij} \underline{\mathcal{H}}_j^g}. \tag{4.18}$$

This result can be found in the work of Demaret and Moncrief [6] [one must specialize their Eq. (51) to the case of dust] and is also derived in Ref. [9].

As described in Ref. [9], the presence of the square root in  $\underline{h}$  creates a serious difficulty for defining a quan-

<sup>4</sup>Some other matter couplings also generate true Hamiltonians that depend only on the gravitational variables. Examples that can be solved analytically include the massless scalar field [18] and certain perfect fluids with "bad" clock couplings [6,19].



tum theory. In particular, it implies that the physical Hilbert space must be restricted to those states for which the operator  $(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g$  is non-negative, and the physical observables must be restricted to those operators that keep the states in the physical Hilbert space. The difficulty is that the obvious candidates for physical observables, the metric  $\hat{g}_{ij} = g_{ij} \times$  and its conjugate  $\hat{p}^{ij} = -i\delta/\delta g_{ij}$  acting by multiplication and differentiation, do not satisfy this criterion.

While different choices for the clock Hamiltonian are classically equivalent, as long as they are defined for the same range of  $J$ , they do not necessarily lead to the same quantum theory. Thus, we now investigate other possibilities. With the clock Hamiltonian  $m(\zeta, J) = \sqrt{J/k}$  (where we assume  $J \geq 0$ ), we find that  $\underline{n}m = \sqrt{\underline{n}\Pi/k}$  and the gravitational Hamiltonian is

$$\underline{h} = -k[(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g]/\underline{n}. \quad (4.19)$$

This result suffers from the same difficulty as the square root Hamiltonian (4.18) if we insist that the momentum  $\underline{\Pi}$  should be interpreted in terms of a real, non negative clock Hamiltonian  $m$ . In that case,  $\underline{\Pi}$  must be non negative (assuming the number density  $\underline{n}$  is positive), and the new constraint  $\hat{\mathcal{H}}_{\perp} = \underline{\Pi} + \underline{h} = 0$  implies that  $\underline{h}$  of Eq. (4.19) must be nonpositive; we again have the requirement  $(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g \geq 0$  on the quantum theory. On the other hand, it is not obvious that the condition  $\underline{\Pi} \geq 0$  must be kept, so one might argue that it should be dropped. This leads to a quantum theory that is free from the difficulties of the condition  $(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g \geq 0$ , but there might arise a problem of interpretation for the theory since the variables  $\Theta$  and  $\underline{\Pi}$  would no longer have a simple interpretation in terms of a real clock Hamiltonian.

Clearly, by choosing the clock Hamiltonian  $m(\zeta, J)$  appropriately, we can arrange for the gravitational Hamiltonian  $\underline{h}$  to be given by any invertible function of  $(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g$  (with  $\underline{n}$  appearing as necessary to keep the density weights balanced). In particular, for the clock Hamiltonian  $m(\zeta, J) = (J/k)^{1/4}$  (again with  $J \geq 0$ ), we have  $\underline{n}m = \underline{n}^{3/4}(\underline{\Pi}/k)^{1/4}$  and

$$\underline{h} = -k[(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g]/(\underline{n})^3. \quad (4.20)$$

This choice for the gravitational Hamiltonian appears to avoid the problems encountered in the other cases. In particular, the clock momentum remains non negative,  $\underline{\Pi} \geq 0$ , for all solutions of the constraint  $\hat{\mathcal{H}}_{\perp} = 0$ , even on states for which  $(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g$  is a negative operator. However, some difficulties with interpretation do arise if we attempt to take seriously the constraint in its original form  $\hat{\mathcal{H}}_{\perp} = 0$ . In that case, for states satisfying the constraints and on which  $(\hat{\mathcal{H}}_{\perp}^g)^2 - \hat{\mathcal{H}}_i^g \hat{g}^{ij} \hat{\mathcal{H}}_j^g$  is negative, we find that  $m^2 < 0$ . Thus, for such states the clocks are tachyonic.<sup>5</sup> One might also be concerned that there

<sup>5</sup>Said differently, if the mass is still defined by the real branch of  $(J/k)^{1/4}$ , it will not satisfy  $\hat{\mathcal{H}}_{\perp} = 0$  for all solutions of the new constraint  $\hat{\mathcal{H}}_{\perp} = 0$ .

is something pathological about a Hamiltonian  $\underline{h}$  that is eighth order in the gravitational momenta.

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## APPENDIX A: PERFECT FLUIDS AS FLUIDS COUPLED TO CLOCKS

By making the substitution  $\underline{n} \rightarrow J\underline{n}$  in the action (2.11) and adding the kinetic term (3.2), we obtain an action for an isentropic perfect fluid coupled to clocks:

$$S[Z^i, J, \Theta; \gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4y [-\sqrt{-\gamma}\rho(J\underline{n}/\sqrt{h}) + \underline{n}J\Theta_{,\alpha}U^{\alpha}\sqrt{-\gamma/h}]. \quad (A1)$$

Here,  $h$  and  $U^{\alpha}$  are expressed in terms of the label fields  $Z^i$  through Eqs. (2.7) and (2.8), respectively. In this appendix we show that this action is equivalent to the (isentropic) "unbarred" perfect fluid action of Refs. [4,11,12]. In notation consistent with the present work, that action is

$$S[K^{\alpha}, \Theta, Z^i, \lambda_i; \gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4y [-\sqrt{-\gamma}\rho(|K|/\sqrt{-\gamma}) + K^{\alpha}(\Theta_{,\alpha} + \lambda_i Z^i_{,\alpha})], \quad (A2)$$

where  $|K| \equiv \sqrt{-K^{\alpha}K_{\alpha}}$  is the norm of the timelike, future-directed, spacetime four-vector density  $K^{\alpha}$ . (Here, our  $K^{\alpha}$ ,  $\lambda_i$ , and  $Z^i$  correspond, respectively, to  $J^{\alpha}$ ,  $\beta_A$ , and  $\alpha^A$  of Ref. [11] and to  $J^{\alpha}$ ,  $-W_k$ , and  $Z^k$  of Ref. [12].)

The essential step in comparing the actions (A1) and (A2) is to note that the equations of motion for some of the fields in the action (A2) can be solved algebraically for those fields as unique functions of the other fields. These "dependent" fields contain only redundant dynamical information and, when the corresponding equations of motion are solved and the solutions inserted into the action (A2), the resulting action is equivalent to (A1). We will see that this procedure produces the action (A1).

In order to identify the proper equations of motion to solve, we must first change coordinates on field space by writing  $K^{\alpha} = \kappa U^{\alpha}$ , where  $U^{\alpha}$  is a unit future pointing timelike four-vector. Thus, the action (A2) can be written as

$$S[\kappa, U^{\alpha}, \Theta, Z^i, \lambda_i; \gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4y [-\sqrt{-\gamma}\rho(\kappa/\sqrt{-\gamma}) + \kappa U^{\alpha}(\Theta_{,\alpha} + \lambda_i Z^i_{,\alpha})]. \quad (A3)$$

The variations in  $U^{\alpha}$  must preserve the normalization condition  $U^{\alpha}U_{\alpha} = -1$ .

Note, in particular, the equations of motion that follow from varying  $\lambda_i$  and  $U^{\alpha}$ :

$$\frac{\delta S}{\delta \lambda_i} = 0 \Rightarrow U^\alpha Z^i{}_{,\alpha} = 0, \quad (\text{A4})$$

$$\frac{\delta S}{\delta U^\alpha} = 0 \Rightarrow \Theta_{,\alpha} + \lambda_i Z^i{}_{,\alpha} = CU_\alpha. \quad (\text{A5})$$

Here,  $C$  is a proportionality constant that arises because of the restriction on  $\delta U^\alpha$ . These equations are easily solved for  $\lambda_i$  and  $U^\alpha$ . The solution for  $U^\alpha$  is just the expression (2.8). The solution for  $\lambda_i$  is found by contracting Eq. (A5) with  $\gamma^{\alpha\beta} Z^j{}_{,\beta}$  and using Eq. (A4). The result is

$$\lambda_i = -h_{ij} \Theta_{,\alpha} \gamma^{\alpha\beta} Z^j{}_{,\beta}, \quad (\text{A6})$$

where the fleet metric is defined in terms of  $Z^i$  by Eq. (2.7). The solution for  $U^\alpha$  and  $\lambda_i$  can be substituted into the action (A3) to yield the equivalent action

$$S[\kappa, \Theta, Z^i; \gamma_{\alpha\beta}] = \int_{\mathcal{M}} d^4 y [-\sqrt{-\gamma} \rho(\kappa/\sqrt{-\gamma}) + \kappa U^\alpha \Theta_{,\alpha}], \quad (\text{A7})$$

where  $U^\alpha$  is the functional of  $Z^i$  given by Eq. (2.8). If we now identify  $\kappa = J \underline{n} \sqrt{-\gamma}/h$  (another change of coordinates on field space), this is just the action (A1). We see that the "unbarred" fluid of Refs. [4,11,12] may be interpreted as an isentropic perfect fluid coupled to a fleet of clocks.

## APPENDIX B: FLEET METRIC AND MATTER VELOCITY AS FUNCTIONS OF THE CANONICAL VARIABLES

The velocity  $V_a X^a{}_{,i}$  and fleet metric  $h_{ij}$  that appear in the Hamiltonian constraint (4.11) are expressed in terms of the Lagrangian picture canonical variables as follows.

First, combine Eq. (3.21) for  $P_i$  with Eq. (4.5) for  $\underline{P}_a$  to obtain

$$\underline{P}_a = \Gamma(\underline{nm} + \underline{w}) V_a + \underline{t}_{ij} Z^i{}_{,a} Z^j{}_{,b} V^b / \Gamma - (\det X^b{}_{,j}) \underline{\mathcal{H}}_a^g. \quad (\text{B1})$$

By solving Eq. (3.18) for  $g^{ab}$  and inserting the result into  $Z^i{}_{,a} V^a = Z^i{}_{,a} g^{ab} V_b$ , we obtain the identity

$$Z^i{}_{,a} V^a = \Gamma^2 h^{ij} V_a X^a{}_{,j}. \quad (\text{B2})$$

Then Eq. (B1) becomes

$$\underline{P}_a X^a{}_{,i} + \underline{\mathcal{H}}_i^g = \Gamma[(\underline{nm} + \underline{w}) \delta_i^j + \underline{t}_{ik} h^{kj}] V_a X^a{}_{,j}, \quad (\text{B3})$$

where  $\underline{\mathcal{H}}_i^g$  is the pullback to  $\mathcal{S}$  of the gravitational contribution to the momentum constraint. With the identity (B2), one can confirm that

$$h_{ij} = g_{ij} + \Gamma^2 (V_a X^a{}_{,i})(V_b X^b{}_{,j}) \quad (\text{B4})$$

is indeed the fleet metric; that is,  $h_{ij}$  is the inverse of the inverse fleet metric (3.18).

Now observe that the gamma factor  $\Gamma = 1/\sqrt{1-V^2}$  can be expressed in terms of  $V_a X^a{}_{,i}$  and  $h^{ij}$ . This can be seen by again solving Eq. (3.10) for  $g^{ab}$  and inserting the result into  $V^2 = V_a g^{ab} V_b$ . This yields  $V^2/\Gamma^2 = (V_a X^a{}_{,i}) h^{ij} (V_b X^b{}_{,j})$ , which can be solved for  $\Gamma$  as a function of  $V_a X^a{}_{,i}$  and  $h^{ij}$ . We therefore see that, in principle, Eqs. (B3) and (B4) can be solved for  $h_{ij}$  and  $V_a X^a{}_{,i}$  as functions of the canonical variables. In particular,  $h_{ij}$  and  $V_a X^a{}_{,i}$  depend on  $\underline{\Pi}$  [which is contained in the argument of the clock Hamiltonian  $m$  in Eq. (B3)],  $\underline{P}_a X^a{}_{,i}$  [which appears on the left-hand side of Eq. (B3)],  $g_{ij}$  [which appears explicitly in Eq. (B4) and on the left-hand side of Eq. (B3) in the combination  $\underline{\mathcal{H}}_i^g$ ], and  $\underline{p}^{ij}$  [which appears on the left-hand side of Eq. (B3) in the combination  $\underline{\mathcal{H}}_i^g$ ].

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