Inhomogeneous cosmological model in Brans-Dicke theory

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Inhomogeneous cosmological models in Brans-Dicke scalar-tensor theory are studied and a family of exact solutions is obtained. It is conformally related to the nonsingular model recently found in general relativity. In contrast with the general relativistic solution, the present one is not invariant under time inversion.

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I. INTRODUCTION

The renewed interest in the scalar-tensor theories of gravitation is caused by two main factors. First, most of the unified theories, including superstring theory, naturally predict a dilaton theory of gravity, and it is at the present time the only theory that treats gravity in a consistent way with quantum mechanics. Second, the new scenario of extended inflation which solves the fine-tuning problem of old, new, and chaotic inflation has a scalar field that slows the expansion rate of the universe, from exponential to polynomial, allowing the completion of the phase transition from the de Sitter phase to a radiation-dominated universe; that is, scalar-tensor theories provide a possible solution to the "graceful exit" problem. There are no experiments that contradict Einstein's theory, but most of them are within the parametrized post-Newtonian (PPN) framework that has the limitation of testing objects in weak gravity. Homogeneous cosmological models in scalar-tensor theories have received a lot of attention (see, for example, Refs. [1,2] and references therein). However, inhomogeneous cosmological models have not been considered in these types of theories.

On the other hand, in general relativity some very interesting inhomogeneous cosmological models have appeared. In 1990, Senovilla [3] found a new perfect-fluid inhomogeneous cosmological solution without a big-bang singularity and without any other curvature singularity. The matter content of that solution was radiation. This solution was shown to be geodesically complete and satisfied causality conditions such as global hyperbolicity [4].

The singularity-free solution was generalized in a paper Ref. [5] to the case of G_2 diagonal cosmologies and all the different singular behavior were possible.

In a recent paper [6] a new perfect-fluid cosmological solution of Einstein's equations without a big-bang singularity or any other curvature singularities was found. Neither the energy-momentum tensor nor the Weyl tensor was singular. The equation of state corresponds to a stiff fluid $p = \rho$, with density positive and nonvanishing everywhere and satisfying a causality condition, namely, global hyperbolicity. This solution possesses a two-dimensional Abelian group of isometries acting on spacelike surfaces, but with neither of the Killing vectors being hypersurface orthogonal; that is, the metric is nondiagonal and it belongs to the class $B(i)$ of Wainwright for G_2 cosmologies [8]. This solution was obtained previously by Letelier [7]. For some interesting properties of G_2 geometries see Ref. [10].

The purpose of this paper is to study inhomogeneous cosmological models in one of the simplest scalar-tensor theories, the one due to Brans and Dicke [11].

II. FIELD EQUATIONS

The field equations of Brans-Dicke scalar-tensor theory are

$$G_{\mu\nu} = T_{\mu\nu}(\phi) = \omega\phi^{-2}(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}) + \phi^{-1}(\phi_{;\mu\nu} - g_{\mu\nu}\,\Box\,\phi), \tag{1}$$

$$\Box\phi = 0, \tag{2}$$

where we use the signature $(-,+,+,+)$, ω is the coupling parameter of this theory, and general relativity is the limit of this theory when $\omega \to \infty$; the solar system experiments imply that $|\omega| \geq 500$.

In this section we set the field equations of the scalar-tensor theory of gravitation in the metric with local spherical symmetry:

$$ds^2 = a_1(t)b_1(r)(-dt^2 + dr^2) + a_2(t)b_2(r)d\varphi^2 + a_3(t)[dz + b_3(r)d\varphi]^2, \tag{3}$$

where a_i and b_i, are functions of t and r, respectively. The field equations for this metric are given by

$$\frac{\dot{a_1}\,\dot{a_2}}{a_1\,a_2} + \frac{\dot{a}_2^2}{a_2^2} + \frac{\dot{a_1}\,\dot{a_3}}{a_1\,a_3} - \frac{\dot{a_2}\,\dot{a_3}}{a_2\,a_3} + \frac{\dot{a}_3^2}{a_3^2} + \frac{b_2'b_2'}{b_2b_2} - \frac{a_3b_3'^2}{a_2b_2} + \frac{2\,\dot{a_1}\,\dot{f}}{a_1\,f} - \frac{2\,\dot{a_2}\,\dot{f}}{a_2\,f} - \frac{2\,\dot{a_3}\,\dot{f}}{a_3\,f} - \frac{2\,w\,\dot{f}^2}{f^2} + \frac{2b_2'\,g'}{b_2\,g} + \frac{2\,b_2'\,g'}{b_2\,g} - \frac{2\,w\,g'^2}{g^2} - \frac{2\,\ddot{a_2}}{a_2} - \frac{2\,\ddot{a_3}}{a_3} - \frac{4\,\ddot{f}}{f} = 0, \tag{4}$$

$$\frac{\dot a_2\, b_2'}{a_2\, b_2}+\frac{\dot a_3\, b_2'}{a_3\, b_2}+\frac{\dot a_1\, b_2'}{a_1\, b_2}-\frac{\dot a_2\, b_2'}{a_2\, b_2}+\frac{2\, b_2'\, \dot f}{b_2\, f}+\frac{2\, \dot a_1\, g'}{a_1\, g}-\frac{4\, \dot f\, g'}{f\, g}-\frac{4\,\omega\, \dot f\, g'}{f\, g}=0, \tag{5}$$

$$\begin{aligned}&\frac{2\,\dot a_1^2}{a_1^2}+\frac{2\,a_2\,b_2\,\dot a_1^2}{a_1^2\,a_3\,b_3^2}+\frac{\dot a_2^2}{a_2^2}+\frac{a_2\,b_2\,\dot a_3^2}{a_3^3\,b_3^2}-\frac{2\,b_2'^2}{b_2^2}-\frac{2\,a_2\,b_2\,b_2'^2}{a_3\,b_2^2\,b_3^2}-\frac{b_2'^2}{b_2^2}+\frac{2\,b_2'\,b_3'}{b_2\,b_3}\\&+\frac{3\,a_3\,b_3'^2}{a_2\,b_2}-\frac{b_3'^2}{b_3^2}-\frac{2\,\dot a_2\,\dot f}{a_2\,f}-\frac{2\,a_2\,b_2\,\dot a_3\,\dot f}{a_3^2\,b_3^2\,f}-\frac{2\,\omega\,\dot f^2}{f^2}-\frac{2\,\omega\,a_2\,b_2\,\dot f^2}{a_3\,b_3^2\,f^2}+\frac{2\,b_2'\,g'}{b_2\,g}\\&-\frac{4\,b_3'\,g'}{b_3\,g}+\frac{2\,\omega\,g'^2}{g^2}+\frac{2\,\omega\,a_2\,b_2\,g'^2}{a_3\,b_3^2\,g^2}-\frac{2\,\ddot a_1}{a_1}-\frac{2\,a_2\,b_2\,\ddot a_1}{a_1\,a_3\,b_3^2}-\frac{2\,\ddot a_2}{a_2}-\frac{2\,a_2\,b_2\,\ddot a_3}{a_3^2\,b_3^2}\\&+\frac{2\,b_2''}{b_2}+\frac{2\,a_2\,b_2\,b_2''}{a_3\,b_2\,b_3^2}+\frac{2\,b_2''}{b_2}-\frac{4\,b_3''}{b_3}-\frac{4\,\ddot f}{f}-\frac{4\,a_2\,b_2\,\ddot f}{a_3\,b_3^2\,f}+\frac{4\,g''}{g}+\frac{4\,a_2\,b_2\,g''}{a_3\,b_3^2\,g}=0,\end{aligned} \tag{6}$$

$$\begin{aligned}&\frac{2\,\dot a_1^2}{a_1^2}+\frac{\dot a_2^2}{a_2^2}-\frac{2\,b_2'^2}{b_2^2}-\frac{b_2'^2}{b_2^2}+\frac{b_2'\,b_3'}{b_2\,b_3}+\frac{3\,a_3\,b_3'^2}{a_2\,b_2}-\frac{2\,\dot a_2\,\dot f}{a_2\,f}-\frac{2\,\omega\,\dot f^2}{f^2}+\frac{2\,b_2'\,g'}{b_2\,g}\\&-\frac{2\,b_3'\,g'}{b_3\,g}+\frac{2\,\omega\,g'^2}{g^2}-\frac{2\,\ddot a_1}{a_1}-\frac{2\,\ddot a_2}{a_2}+\frac{2\,b_2''}{b_2}+\frac{2\,b_2''}{b_2}-\frac{2\,b_3''}{b_3}-\frac{4\,\ddot f}{f}+\frac{4\,g''}{g}=0,\end{aligned} \tag{7}$$

$$\begin{aligned}&\frac{2\,\dot a_1^2}{a_1^2}+\frac{\dot a_2^2}{a_2^2}-\frac{2\,b_2'^2}{b_2^2}-\frac{b_2'^2}{b_2^2}+\frac{3\,a_3\,b_3'^2}{a_2\,b_2}-\frac{2\,\dot a_2\,\dot f}{a_2\,f}-\frac{2\,\omega\,\dot f^2}{f^2}\\&+\frac{2\,b_2'\,g'}{b_2\,g}+\frac{2\,\omega\,g'^2}{g^2}-\frac{2\,\ddot a_1}{a_1}-\frac{2\,\ddot a_2}{a_2}+\frac{2\,b_2''}{b_2}+\frac{2\,b_2''}{b_2}-\frac{4\,\ddot f}{f}+\frac{4\,g''}{g}=0,\end{aligned} \tag{8}$$

$$\begin{aligned}&\frac{\dot a_1\,\dot a_2}{a_1\,a_2}+\frac{\dot a_1\,\dot a_3}{a_1\,a_3}+\frac{\dot a_2\,\dot a_3}{a_2\,a_3}+\frac{b_2'\,b_2'}{b_2\,b_2}+\frac{b_2'^2}{b_2^2}-\frac{a_3\,b_3'^2}{a_2\,b_2}+\frac{2\,\dot a_1\,\dot f}{a_1\,f}+\frac{2\,\dot a_2\,\dot f}{a_2\,f}\\&+\frac{2\,\dot a_3\,\dot f}{a_3\,f}-\frac{2\,\omega\,\dot f^2}{f^2}+\frac{2\,b_2'\,g'}{b_2\,g}-\frac{2\,b_2'\,g'}{b_2\,g}-\frac{2\,\omega\,g'^2}{g^2}-\frac{2\,b_2''}{b_2}-\frac{4\,g''}{g}=0,\end{aligned} \tag{9}$$

$$\frac{\dot a_2\,\dot f}{a_2\,f}+\frac{\dot a_3\,\dot f}{a_3\,f}-\frac{b_2'\,g'}{b_2\,g}+\frac{2\,\ddot f}{f}-\frac{2\,g''}{g}=0, \tag{10}$$

where the prime means a derivative with respect to r and the overdot with respect to t and we have assumed that $\phi = f(t)g(r)$. In the next section we give a family of exact solutions to the field equations.

III. EXACT SOLUTIONS

I have found the following exact solution to the field equations:

$$ds^2 = e^{ht}\left[e^{sr^2}\cosh(2pt)\left(-dt^2+dr^2\right)+r^2\cosh(2pt)d\varphi^2 + \frac{1}{\cosh(2pt)}\left(dz+pr^2d\varphi\right)^2\right], \tag{11}$$

$$\phi = \phi_1 e^{-ht}, \tag{12}$$

where h, s, and p are constants satisfying the relation

$$h^2(3+2\omega) = 4(s-p), \tag{13}$$

and ϕ_1 is an integration constant, and the range of variation of the coordinates is

$$-\infty < t, z < \infty, \qquad 0 \le r < \infty, \qquad 0 \le \varphi \le 2\pi.$$

This spacetime has a well-defined axis of symmetry at $r = 0$ where the so-called elementary flatness [9] is satisfied and therefore the coordinate r has to be interpreted as a radial cylindrical coordinate.

We notice here that this metric is conformally related to that obtained by Mars [6] in general relativity for a stiff fluid as the material source. The conformal factor is e^{ht} and it is well behaved as well as its derivatives; therefore no curvature singularity appears in this new solution. What about the contribution of the scalar field? The components of the "energy-momentum" tensor [see Eq. (1)] of the scalar field, for this solution, are

$$T_{rr} = \frac{h^2(1+\omega) + 2hp\tanh(2pt)}{2}, \tag{14}$$

$$T_{rt} = T_{tr} = hsr, \tag{15}$$

$$T_{\varphi\varphi} = \frac{h^2 r^2(1+\omega)\left[1 + p^2 r^2 \operatorname{sech}(2pt)^2\right] + (2hpr^2)\left[1 - 2p^2 r^2 \operatorname{sech}(2pt)^2\right]\tanh(2pt)}{2e^{sr^2}}, \tag{16}$$

$$T_{\varphi z} = T_{z\varphi} = \frac{\left[h^2 pr^2(1+\omega) - 2hp^2 r^2 \tanh(2pt)\right]\operatorname{sech}(2pt)^2}{2e^{sr^2}}, \tag{17}$$

$$T_{zz} = \frac{\left[h^2(1+\omega) - 2hp\tanh(2pt)\right]\operatorname{sech}(2pt)^2}{2}, \tag{18}$$

$$T_{tt} = \frac{(3+\omega)h^2 + 2hp\tanh(2pt)}{2}. \tag{19}$$

As we can see, the contribution of the scalar field shows no singularity, that is, we have a solution without matter singularity as well. Another important property of the scalar field of Brans-Dicke is that it satisfies the weak, dominant, and strong energy conditions [12].

This metric reduces to the general relativistic case when $h = 0$, that is, when ϕ is a constant. A qualitative difference from the metric of Mars is that in the present case we do not have invariance under time reflection, and the constant h is a measure of the asymmetry and is a direct consequence of the existence of the scalar field. Since the solution given here is conformally related to that of Ref. [6] it is of type I except at $r = 0$, where it is type D in accordance with the theorems of Ref. [10].

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