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Towards realistic singularity-free cosmological models

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We present an *explicit* general family of inhomogeneous cosmological models. The family contains an arbitrary function of comoving time (interpretable as the cosmological scale factor) and four arbitrary parameters. In general, it is a solution of Einstein's field equations for a fluid with anisotropic pressures, but it also includes a big subfamily of perfect-fluid metrics. The most interesting feature of this family is that it contains both *all* the diagonal separable singularity-free cosmological models recently found and *all* the Friedmann-Lemaître-Robertson-Walker standard models. This property allows one to speculate on the construction of some interesting models in which the Universe has been FLRW-like from some time on (for instance, since the nucleosynthesis time), but it also went through primordial singularity-free inhomogeneous epochs (in fact, there are quite natural possibilities in which these primordial epochs are inflationary) without ever violating energy conditions or other physical properties. Nevertheless, the physical processes leading to the isotropization and homogenization of the Universe are not fixed nor indicated by the models themselves. The interesting properties of the general model are studied in some detail.

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I. INTRODUCTION

The purpose of this paper is to present an interesting general family of explicit inhomogeneous cosmological models which contains both *all* the spatially homogeneous and isotropic cosmologies, the Friedmann-Lemaître-Robertson-Walker (FLRW) models [1–4], see also [5–7], together with *all* the G_2 diagonal and separable perfect-fluid cosmologies [8–11] considered in the general class studied in [12]. As is well known, within this last class there are all types of cosmologies with different singular behaviors [12], among which we have the solutions of [13,14], the explicit solutions presented in [12], the $p = \rho/3$ singularity-free solution of [15] (see also [16–18]), and its generalization given in Ref. [12] itself. Therefore the mentioned singularity-free models become equal to the standard models from a theoretical point of view, because they both belong to the same single explicit class and one can move from one to the other within the family "continuously," as we shall presently see.

The interest in considering spatially inhomogeneous cosmologies has been recognized since long ago [7–10,19,20], mainly due to the following fundamental reasons. The first reason is because of the obvious fact that the actual Universe is *not* exactly spatially homogeneous, not even at the largest scales. The second reason is because of the desire to avoid postulating very special initial conditions. Thus it would be very interesting to construct general inhomogeneous models which nevertheless become homogeneous and isotropic enough at late times (late might mean just 1 s after the beginning of the present expansion era) to be in accordance with the existence of the microwave background radiation or the regular distribution of helium. The third reason for interest in these cosmologies is in order to see whether or not the formation of large scale structure in the Universe, such as voids, great walls, great attractors, superclusters, etc., can be somehow scientifically explained. And the fourth reason is because inhomogeneity might be the way to avoid the initial big-bang singularity, a fundamental problem in cosmology given that no present known laws of physics can be applied to understand the breakdown of regularity at this initial singularity (remember that all physical quantities become infinite at the initial singularity and that no satisfactory quantum theory applicable to this situation has been found yet).

In this sense, the powerful singularity theorems [6] seemed to provide a final negative answer to the possibility of avoiding the initial singularity in classical general relativity. However, the general reasoning presented in [6] (p. 350) makes use of the geodesic motion of the cosmological matter, an assumption which is clearly not supported by any theoretical or observational reason. In fact, since the work of Raychaudhuri [21] it has been known that acceleration (or rotation) may prevent the existence of a universal singularity in our finite past. This was explicitly manifested by the recent appearance of the singularity-free inhomogeneous model of [15] (see also [22]). This perfect-fluid model has a well-defined cylindrical symmetry, a realistic equation of state for radiation-dominated epochs $p = \rho/3$, nice causality global properties [16], and a contracting era previous to the subsequent expansion era starting after a rebound time; it satisfies all possible energy conditions, and nevertheless is geodesically complete (therefore singularity-free) [16].

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Prior to its publication, the nonexistence of such models was thought to be established because they had not been found and the singularity theorems seemed to indicate their mathematical impossibility. This very simple model, however, was later generalized to include energy flux [17] and also to a much more general family depending on three arbitrary parameters [12] but still satisfying energy and causality conditions [12] together with geodesic completeness [23]. Other new singularity-free models have also been found quite recently [24].

Sometimes it has been considered as highly intriguing that inhomogeneous models can be free of singularities without violating energy conditions. This is due to a simplified view of the singularity theorems. As is well known, there are several types of singularity theorems. Perhaps the simplest one is that due to Raychaudhuri, stating that irrotational and geodesic expanding models must have a big-bang singularity in the finite past if the strong energy condition holds [21]. Of course, this theorem does not apply to general inhomogeneous models, because they are nongeodesic (acceleration is not zero). In fact, this has a clear explanation in physical terms, because acceleration of matter is equivalent, via conservation equations for the energy-momentum tensor, see [25], to the existence of a gradient of pressure, which is a force opposing generally gravitational attraction. With regard to the most complicated and powerful singularity theorems [6], let us remark that all of them have the same structure. Usually, they assume energy and causality conditions *plus* a boundary or initial condition. This last condition can adopt several different forms itself, although in general it amounts to the existence of a causally trapped set [6]. A causally trapped set ζ is a set such that $E^{\pm}(\zeta)$ is bound to be compact; by $E^{\pm}(\zeta)$ we mean the set of points in spacetime that can be reached from ζ by a future- (past-)directed null curve but cannot be so reached by a timelike curve. Some examples are closed trapped surfaces (the original Penrose hypothesis, see [6]), compact achronal sets without edge (such as space in closed cosmological models), points with reconverging light cones, slices with bounded-above (-below) zero expansion, etc. This boundary or initial condition is the one which is bypassed by the known singularity-free inhomogeneous models. Thus, for instance, the very first and simple singularity-free model of [15] was extensively analyzed in Ref. [16] in this sense. It was shown that the model was singularity-free *and* in accordance with all singularity theorems. These theorems do not apply to the solution because the boundary condition does not hold: there are no closed trapped surfaces, no compact achronal sets without edge, no points with reconverging light cones, etc. All these facts were explicitly proved in [16]. In summary, there were no causally trapped sets in the model of [15].

In spite of the above, the hitherto explicitly known singularity-free models are *not* realistic in the sense that they cannot explain the isotropy in the temperature of the microwave background radiation (they can certainly explain its existence, though) and some other observational features. For this reason, and given that they were not generalizations of the FLRW standard models, the relevance of these singularity-free cosmologies for the study of the actual Universe has been very limited. It is here where the new family we are about to present may have some importance, because it combines quite naturally the irrefutable good physical properties of the FLRW cosmologies with the new singularity-free models, thereby opening a *classical* way for the construction of a realistic model which never passed through the unfortunate situation of suffering the initial singularity —an indisputably desirable feature.

II. THE MODELS AND THEIR GENERAL PROPERTIES

Let us now directly present the explicit family of solutions, given in cylindrical-like coordinates $\{t, r, \phi, z\}$ by the line element

$$ds^2 = T^{2(1+n)}\Sigma^{2n(n-1)}\left(-d\tau^2 + dr^2\right) + T^{2(1+n)}\Sigma^{2n}\Sigma'^2 d\phi^2 + T^{2(1-n)}\Sigma^{2(1-n)}dz^2, \quad (1)$$

where $T(\tau)$ is an arbitrary function of only the time coordinate τ, $n \geq 0$ is a constant, primes stand for derivatives with respect to r, and $\Sigma(r)$ is a function of r which, for the only purposes of the present work,[1] will be assumed to satisfy the simple differential equation

$$\Sigma'^2 - \left(M\Sigma^2 + N - nK\Sigma^{2(1-2n)}\right) = 0, \quad (2)$$

where M, N, and K are arbitrary constants. Therefore this metric depends on one arbitrary function of time and on four arbitrary parameters. The interpretation of these parameters and of the function $T(\tau)$ is given in what follows.

Before entering into the physical properties of the models, let us just make some comments about Eq. (2) and its solutions. Of course, this is not a remaining differential equation to be solved in order to get the solution in complete closed form. As is obvious, a simple change of the coordinate r would allow one to give the metric in explicit form (for example, by choosing Σ itself as a new coordinate). However, it is better to keep the differential equation (2) because, first, Σ will not always be a good *global* coordinate, and, second and more important, because the behavior of the function Σ with the *physical* coordinate r varies depending essentially on the sign of M and the values of the other constants.

[1] In fact, the line element (1) is a solution of Einstein's equations for anisotropic fluids *in general*, that is, for any *arbitrary* function $\Sigma(r)$. However, we shall only consider here the particular case in which Σ satisfies (2) because this will better help us in our aim of combining clearly and in a simple way the general properties of FLRW models with those of the singularity-free metrics. Let us simply remark that relation (2) is a *necessary*, but *not* sufficient, condition for having a perfect-fluid energy-momentum tensor. Thus the FLRW models included in (1) will always satisfy (2).

In order to illustrate these points, let us give the solution of (2) in some particular cases. For example, if $nK = 0$, then the solution is obviously either a trigonometric function, such as $\cos(\sqrt{-M}r)$, when $M < 0$, or a linear function of r when $M = 0$, or a hyperbolic function when $M > 0$. In this last case $M > 0$, the hyperbolic function will be a hyperbolic cosine, an exponential, or a hyperbolic sine depending on whether N is less than, equal to, or greater than zero, respectively. Analogously, the general case $N = 0$ $(n \neq 0)$ can be explicitly solved and the solution of (2) is given by

$$N = 0 \;\; (n \neq 0) \;\; \Longrightarrow \;\; \Sigma(r) = \Xi^{\frac{1}{2n}}(2nr),$$

where the function $\Xi(x)$ satisfies again the trivially solvable equation

$$\left(\frac{d\Xi}{dx}\right)^2 = M\Xi^2 - nK.$$

Thus again we have the trigonometric $(M < 0)$, linear $(M = 0)$, or hyperbolic $[M > 0$, with $\sinh(\sqrt{M}x)$ or $\cosh(\sqrt{M}x)$ according to whether nK is positive or negative$]$ behaviors, adequately "weighted" in each case by the power $1/2n$ and the argument $x = 2nr$.

The properties shared by these examples are in fact general, as we are going to see. Of course, the general behavior of the function Σ can be given without having to solve (2) explicitly. The procedure is standard. We only have to notice that (2) can be seen as a typical equation with Σ'^2 as the square of a "velocity" and $V(\Sigma) = -\left(M\Sigma^2 + N - nK\Sigma^{2(1-2n)}\right) \leq 0$ as the potential. Then the behavior of Σ follows from the plot of $V(\Sigma)$ and its maxima and minima, its zeros, etc. Thus, for example, when $V(\Sigma)$ vanishes at a finite value $\Sigma_0 > 0$ from which V decreases monotonically without bound, then Σ behaves *like* a hyperbolic cosine. Similarly, all possible cases can be analyzed. In summary, and for the sake of brevity, we give the following behaviors: If $M < 0$, Σ always behaves like a trigonometric function, which can be taken as a cosine, plus perhaps a constant. The case $M = 0$ is a little bit more involved but nevertheless treatable in the above elementary form. Now, there are several cases with linear combinations of different powers of r and some other possibilities, such as trigonometric or hyperbolic functions again.

Finally, when $M > 0$, Σ always behaves like a hyperbolic function, being a sinh, a cosh, or an exponential depending on the particular values of N, K, and n. Let us remark that in this case we can always choose the value of N in such a way that Σ behaves like a hyperbolic cosine. The important thing about this particular case is that Σ *does not* vanish for any value of r. This will be of great importance for the later analysis of the singularities of the metric (1).

From the above analysis we learn two important things. First, the coordinates could be properly rescaled such that the constant M would take one of the values $\{-1, 0, +1\}$. Nevertheless, we shall not do so because the constant M carries physical dimensions (of length to the power -2) and we prefer to write $\cos(\sqrt{-M}r)$ rather than $\cos(\tilde{r})$ for a new adimensional coordinate $\tilde{r}$. And second, in most cases, the potential $V(\Sigma)$ will vanish for some values of Σ. In fact, this means that Σ' will vanish there and, from inspection of the line element (1), this implies in turn that the coefficient of $d\phi^2$ vanishes. This will not be, however, a singularity of the metric in general, as can easily be checked from the Ricci and Weyl tensors (see below). What happens here is that the above metric has well-defined cylindrical symmetry, with an axis defined by $\Sigma' = 0$, which is regular in most cases.

By choosing the natural orthonormal cobasis

$$\boldsymbol{\theta}^0 = T^{1+n}\Sigma^{n(n-1)}d\tau, \qquad \boldsymbol{\theta}^1 = T^{1+n}\Sigma^{n(n-1)}dr,$$
$$\boldsymbol{\theta}^2 = T^{1+n}\Sigma^{n}\Sigma' d\phi, \qquad \boldsymbol{\theta}^3 = T^{1-n}\Sigma^{1-n}dz, \quad (3)$$

the energy-momentum tensor of (1) takes the form $T_{\mu\nu}$=diag(ρ, p_r, p_r, p_z), so that (1) is a solution of Einstein's field equations for a fluid with anisotropic pressures (two different pressures and no energy flux) relative to the fluid velocity one-form $\boldsymbol{u} = -\boldsymbol{\theta}^0$. Obviously, the coordinates are adapted to the fluid (or comoving). The explicit form of the energy density ρ and pressures p_r and p_z of the fluid are given, when (2) is taken into account, by $(8\pi G = c = 1)$

$$\rho = \frac{(2n-1)(n-1)(n+3)nK + \Sigma^{4n}(n+1)(n-3)\left(M - \dfrac{\dot{T}^2}{T^2}\right)}{T^{2(1+n)}\Sigma^{2n(n+1)}}, \quad (4)$$

$$p_r = \frac{(2n-1)(n-1)^2 nK + \Sigma^{4n}\left((n-1)^2 M - [(n+1)(n-3)+2]\dfrac{\dot{T}^2}{T^2} - 2\dfrac{\ddot{T}}{T}\right)}{T^{2(1+n)}\Sigma^{2n(n+1)}}, \quad (5)$$

$$p_z = \frac{(2n-1)(n-1)^2 nK + \Sigma^{4n}\left((n+1)^2 M - (n+1)(n-1)\dfrac{\dot{T}^2}{T^2} - 2(n+1)\dfrac{\ddot{T}}{T}\right)}{T^{2(1+n)}\Sigma^{2n(n+1)}} \quad (6)$$

where overdots stand for derivatives with respect to τ. As is obvious, the above fluid will be *perfect* if and only if the pressures p_r and p_z are equal, so that from (5) and (6) this condition becomes

$$\text{Perfect fluid} \iff p_r = p_z \equiv p \iff n\left(\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2} - 2M\right) = 0. \tag{7}$$

Thus the matter content of the spacetime will be a perfect fluid if and only if either $n = 0$ (with arbitrary T) or the function T takes one of the explicit forms (with arbitrary n)

$$T^2(\tau) = \begin{cases} A\cosh(2\sqrt{M}\tau) + B\sinh(2\sqrt{M}\tau) & \text{if } M > 0, \\ A\tau + B & \text{if } M = 0, \\ A\cos(2\sqrt{-M}\tau) + B\sin(2\sqrt{-M}\tau) & \text{if } M < 0, \end{cases} \tag{8}$$

where A and B are arbitrary constants. We shall not assume this form of the function T, though, as we want to keep the arbitrary function T for the metric (1) and (2).

Spacetime (1) and (2) is Petrov type I in general (and at generic points), as can be trivially checked from the following Weyl scalars:

$$\Psi_1 = \Psi_3 = 0,$$

$$3\Psi_2 = \frac{n\Sigma^{2n(1-n)}}{T^{2(n+1)}}\left[n(n^2-1)\frac{K}{\Sigma^{4n}} - \frac{\ddot{T}}{2T} + \frac{2n+1}{2}\frac{\dot{T}^2}{T^2} - Mn - (n-1)\frac{3N}{2\Sigma^2}\right],$$

$$\Psi_0 - \Psi_4 = 2n(n^2-1)\frac{\Sigma^{2n(1-n)}}{T^{2(n+1)}} \times \frac{(M\Sigma^2 + N - nK\Sigma^{2(1-2n)})^{1/2}}{\Sigma}\frac{\dot{T}}{T},$$

$$\Psi_0 + \Psi_4 = \frac{n\Sigma^{2n(1-n)}}{T^{2(n+1)}}\left[(2n+1)\frac{\dot{T}^2}{T^2} - \frac{\ddot{T}}{T} + 2M(n^2-n-1) + (2n-1)(n-1)\frac{N}{\Sigma^2}\right],$$

which have been computed with respect to the natural null tetrad associated with (3). As is evident from these expressions, the metric is conformally flat for $n = 0$, and the Petrov type is D if $n = 1$ or at points where $\Sigma'(r) = 0$, that is to say, at the axis.

III. PHYSICAL ANALYSIS OF THE MODELS

First of all, let us compute the kinematic quantities of the fluid velocity vector. It is obvious that the vorticity tensor vanishes, while the expansion and the nonvanishing components of the acceleration and the shear tensor are given by, respectively,

$$\theta = (n+3)\Sigma^{n(1-n)}\frac{\dot{T}}{T^{n+2}}, \tag{9}$$

$$a_1 = n(n-1)\Sigma^{n(1-n)}\frac{\Sigma'}{T^{1+n}\Sigma}, \tag{10}$$

$$\sigma_{11} = \sigma_{22} = -\frac{\sigma_{33}}{2} = \frac{2n}{3}\Sigma^{n(1-n)}\frac{\dot{T}}{T^{n+2}} \tag{11}$$

where all the components are given relative to the cobasis (3). As we can see from (7) together with (10) and (11), the acceleration and shear vanish when $n = 0$, in which case the fluid is also perfect. As is well known (see, for instance, [5,7,25]), from this it follows that the metric for $n = 0$ is a FLRW model. Actually, *all FLRW models are included in (1).* The FLRW models are invariantly characterized within (1) and (2) by the simple condition $n = 0$ and then the function T is the FLRW scale factor. In fact, when $n = 0$ the metric (1) becomes

$$ds^2 = T^2\left(-d\tau^2 + dr^2 + \Sigma'^2 d\phi^2 + \Sigma^2 dz^2\right)$$

where now $\Sigma'^2 = M\Sigma^2 + N$. This is a familiar form (in parametric time τ) for the general FLRW model [7]. In this FLRW case, the density and pressure read

$$\rho_{\text{FLRW}} = \frac{3}{T^2}\left(\frac{\dot{T}^2}{T^2} - M\right),$$

$$p_{\text{FLRW}} = -\frac{1}{T^2}\left(2\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2} - M\right),$$

from where we immediately see that $-\text{sgn}(M)$ is the usual curvature index in FLRW models, so that the FLRW model is open, flat, or closed whenever M is positive, zero, or negative, respectively. Remember that, in the general case $n \neq 0$, the sign of M also chooses the form of the function Σ. Thus this interpretation of M as the index selecting the openness, flatness, or closedness of the model holds somehow for the general case.

But the family (1) and (2) is much richer than the FLRW model. For example, let us note that the models are locally rotationally symmetric (LRS) when $n = 1$ (acceleration is then zero). The metrics in [13,14,26] belong also to (1) and (2). However, more important for our purposes is to note that *all G_2 diagonal separable singularity-free perfect-fluid models* satisfying energy conditions [12] are also included within the family (1). They are given by one of the explicit functions T of (8) with the appropriate choice for Σ. In fact, their explicit form given in [12] can be easily recovered by setting

$$M > 0, \quad T^2 = \cosh(2\sqrt{M}\tau),$$
$$\Sigma^{2n} = \cosh(2n\sqrt{M}\tilde{r}), \quad N = nK - M,$$

together with the following elementary change of coordinate $r \to \tilde{r}$:

$$dr^2 = \frac{M\sinh^2(2n\sqrt{M}\bar{r})d\bar{r}^2}{M\cosh^2(2n\sqrt{M}\bar{r}) + N\cosh^{(2n-1)/n}(2n\sqrt{M}\bar{r}) - nK}.$$

The singularity-free models with a realistic equation of state for radiation-dominated matter $p = \rho/3$ [12] can be trivially obtained by putting then $n = 3$. The very first and simple $p = \rho/3$ singularity-free model of [15] is given by choosing further $M = 3K$. Of course, within the singularity-free family given above there are some FLRW models (when $n = 0$), but they violate the strong energy conditions [6].

As a matter of fact, there are many singularity-free metrics included in (1) and (2). Within the singularity-free metrics, we shall only consider those satisfying the strong energy conditions and with non-negative energy density (both these conditions imply also the weak energy conditions [6]). An elementary analysis shows that, under these conditions, no metric with $M < 0$ can be singularity-free. Nevertheless, in this case the singularities may be timelike (noninitial), and sometimes they only appear in the Weyl tensor, having well-behaved matter quantities. Similarly, there are no singularity-free metrics with $M = 0$ under those energy conditions. Finally, the singularity-free subclass within the family (1) and (2) and satisfying the strong energy conditions is uniquely characterized by the properties

$$M > 0,\quad K \geq 0,\quad n > 1,\quad T \neq 0,$$

$$\frac{(T^{n-1})^{\cdot\cdot}}{T^{n-1}} \leq (n-1)^2 M. \tag{12}$$

The first of these conditions implies that the singularity-free models are open in general, and together with the second it also implies that N can be chosen such that Σ behaves like a hyperbolic cosine. These singularity-free metrics have non-negative energy density everywhere if, in addition to (12), we have either

$$n = 3 \text{ or } \left\{ n > 3 \text{ and } \frac{\dot{T}^2}{T^2} \leq M \right\}. \tag{13}$$

Finally, under the above requirements (12) and (13), the dominant energy conditions [6] also hold if

$$0 < 2M \leq \frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2} \leq 2M + (n-3)\left(M - \frac{\dot{T}^2}{T^2}\right).$$

From this formula follows the remarkable fact that the general singularity-free models in (1) and (2) satisfying strong and dominant energy conditions *can have only one single rebound time* (defined by $\dot{T} = 0$). This forbids cyclic singularity-free models and gives a good invariant definition and physical sense to the "bang" happening at the rebound, where expansion changes from negative to positive values. Therefore the expanding era *starts* at the rebound and will not have an end.

Thus we have seen that, within the singularity-free metrics, those with values of n close to zero (which gives almost-FLRW models) violate energy conditions, while those with bigger n may both be singularity-free and satisfy energy conditions (even with realistic equations of state). This leads us to the interpretation of the fundamental parameter n in the general metric (1). Using (9) and (11), a straightforward calculation for the relative shear σ/θ, where $2\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu}$, gives

$$\frac{\sigma}{\theta} = \frac{2}{\sqrt{3}}\frac{n}{n+3}.$$

From this simple formula we see that the parameter n measures the anisotropy of the model. The relative shear is constant for every single model independently of the arbitrary function T, and depends only on the explicit value of n. Thus for $n = 0$ the relative shear is zero, as it should be for FLRW models, and then it is an increasing function of n. For $n \to \infty$ the relative shear approaches the maximum value $2/\sqrt{3}$. As an example, when $n = 3$ (which includes the $p = \rho/3$ singularity-free perfect-fluid models) the relative shear has an "intermediate" value of $1/\sqrt{3}$.

As we saw before, the function T is the scale factor for the FLRW models. In fact, this interpretation remains true for the general family (1) in the following sense. If we define the scale factor R of any cosmological model by the equation [25]

$$u^\mu \partial_\mu R = \frac{\theta}{3}R,$$

where $\vec{u}$ is the velocity vector of the fluid, we get for the general line element (1)

$$R(\tau) = T^{\frac{n+3}{3}}. \tag{14}$$

Mind, however, that the scale factor R is obviously defined up to an arbitrary function of the comoving spacelike coordinates, which allows one to put it as a function of the proper time if desired. Notice also that τ is not proper time for the fluid; the proper time t for the fluid of (1) can be chosen as

$$t \equiv \Sigma^{n(n-1)} \int T^{1+n} d\tau$$

which is a function not only of τ but also of r.

Let us now compute the quantity relevant for inflationary models and the variation of the expansion: the deceleration parameter q. This is defined by the general expression [25]

$$u^\mu \partial_\mu \left(\frac{1}{\theta}\right) \equiv \frac{1}{3}(1+q),$$

from where we can immediately get

$$q = \frac{1}{n+3}\left(2n+3-3\frac{T\ddot{T}}{\dot{T}^2}\right) \quad (15)$$

for the general metric (1). Let us first remark that q depends only on the time coordinate τ, a property which is due to the separability of the metric functions in comoving coordinates. On the other hand, given that the function T is completely arbitrary, we can have any desired behavior for q. In general, for very anisotropic models (for big enough n), q approaches the value 2.

In order to get a flavor of the different behaviors of q let us concentrate for a moment on the perfect-fluid models included in (1) and (2). Then we must choose the function T as in (8). The resulting $q(\tau)$ is

$$q(\tau) = \begin{cases} 2 & \text{if } M=0, \quad T^2 = A\tau + B, \\ 2 + \dfrac{6}{n+3}\tan^2(2\sqrt{-M}\tau), & \text{if } M<0, \quad T^2 = \sin(2\sqrt{-M}\tau) \\ \left.\begin{array}{l} \dfrac{2n\cosh^2(2\sqrt{M}\tau)+6}{(n+3)\cosh^2(2\sqrt{M}\tau)} \\ \dfrac{2n}{n+3} \\ \dfrac{2n\sinh^2(2\sqrt{M}\tau)-6}{(n+3)\sinh^2(2\sqrt{M}\tau)} \end{array}\right\} & \text{if } M>0, \quad \begin{cases} T^2 = \sinh(2\sqrt{M}\tau) \\ T = e^{\sqrt{M}t} \\ T^2 = \cosh(2\sqrt{M}\tau). \end{cases} \end{cases} \quad (16)$$

As we can see, the three possibilities $M=0$, $M<0$, and $M>0$ give different deceleration parameters. First, for the perfect-fluid case with $M=0$, we have that q is exactly 2 for any n. When the perfect fluid has $M<0$ (general closed models), $q(\tau) \geq 2$ as can easily be checked from (16): at the big-bang $q(0)=2$, then q increases with τ and becomes unbounded at the recollapsing time, $\tau = \frac{\pi}{4\sqrt{-M}}$, from where q decreases until it reaches again the original value 2 at the big crunch ($\tau = \frac{\pi}{2\sqrt{-M}}$). In this case, however, q does depend on the anisotropy of the model, and the bigger the value of n, the smaller the value of q at each τ. Finally, when the perfect fluid has $M>0$ (general open models), there appear the three inequivalent subcases shown in (16). When T^2 is a hyperbolic sine, the deceleration parameter satisfies $0 \leq \frac{2n}{n+3} \leq q(\tau) \leq 2$, so that q is always positive but less than the initial big-bang value $q(0)=2$ and it decreases with time, approaching asymptotically its minimum $\frac{2n}{n+3}$. This minimum depends on the anisotropy of the model in such a way that q is smaller for smaller n. When $T = e^{\sqrt{M}\tau}$, the function q is actually a constant, as we can see in (16), and its explicit value depends on n, being bigger for bigger n. Finally, when T^2 is a hyperbolic cosine, it is easily seen that q is such that $q(\tau) \leq \frac{2n}{n+3} \leq 2$. These models are singularity-free, and they have a contracting era (for $-\infty < \tau < 0$, when $\theta < 0$), an expanding epoch (for $0 < \tau < \infty$, when $\theta > 0$), and at $\tau = 0$ there is a rebound. For the expansion era, and after the nonsingular "big bang" that occurs at the rebound time $\tau = 0$ where $q = -\infty$ (it is obviously a bang, even though there is no singularity), the deceleration parameter is negative for a period given by

$$\tau_{\text{inf}} \equiv \text{inflation duration where } \sinh\left(2\sqrt{M}\tau_{\text{inf}}\right) = \sqrt{\frac{3}{n}}.$$

Actually, the fact that singularity-free models are inflationary is quite general, and all hitherto known singularity-free models [15,12,24] have inflation epochs (this is also true for FLRW models, because FLRW inflationary models have $\rho_{\text{FLRW}} + 3p_{\text{FLRW}} < 0$ so that they violate the strong energy condition and therefore can be free of the initial singularity). From the formula above we see that the duration of inflation depends on n, that is, on the anisotropy of the model. For the FLRW model ($n=0$), inflation lasts forever, which is a typical behavior. On the other hand, the non-FLRW models have a *finite* duration for inflation, which is shorter for bigger n, and they can also satisfy all energy conditions and other physical requirements [for example, the $p=\rho/3$ singularity-free model of [15] ($n=3$) is inflationary with a realistic equation of state and satisfies all energy and causality conditions [16]]. After inflation, the deceleration parameter grows with time approaching asymptotically its maximum $\frac{2n}{n+3}$, which depends on the particular value of n.

Let us now consider the question of how anisotropy and inhomogeneity affect the local relative motion of matter in the general model (1). To that end, let us calculate the generalized Hubble law given by the rate of change of relative distance [25] between neighboring particles in the cosmological fluid. The formula is [25]

$$u^\mu \partial_\mu D = v^\mu n_\mu = D\left(\frac{\theta}{3} + \sigma_{\mu\nu} n^\mu n^\nu\right)$$

where D is the relative distance and v^μ is the relative velocity vector between neighboring particles with respect to the observer defined by the fluid, and n^μ is a direction unit spacelike vector (orthogonal to the fluid velocity vector) indicating the instantaneous relative direction between those particles (and thus the relative position vector D^μ can be split as $D^\mu \equiv D n^\mu$, see [25]). The most general possible $\vec{n}$ in our general metric (1) is obviously given by

$$\vec{n} \equiv \sin\Theta\cos\Phi\vec{e}_1 + \sin\Theta\sin\Phi\vec{e}_2 + \cos\Theta\vec{e}_3 \qquad (17)$$

where Θ and Φ are angles selecting the particular direction at each point and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ form the orthonormal spacelike triad of the basis dual to (3). Using this, a straightforward calculation for the generalized Hubble law leads us to

$$\begin{aligned} u^\mu \partial_\mu D = v^\mu n_\mu &= D\left[1 - n\cos(2\Theta)\right]\Sigma^{n(1-n)}\frac{\dot{T}}{T^{n+2}} \\ &= D\left[1 - n\cos(2\Theta)\right]\frac{3}{n+3}\mathcal{H}, \qquad (18) \end{aligned}$$

where we have used (9) and have defined the *Hubble function* $\mathcal{H} \equiv \theta/3$ as usual. From this formula it is obvious that the receding velocity of typical particles in the cosmological fluid is independent of the direction $\vec{n}$ if and only if (iff) $n = 0$, that is to say, iff the model is FLRW. This is a very natural result. In the general case $n \neq 0$, the receding velocity does not depend on Φ, which is a very interesting property meaning that, at any possible event, all directions with the same Θ are equivalent from this particular local point of view. Nevertheless, for expanding epochs ($\dot{T} > 0$), neighboring particles recede in the direction $\vec{n}$ ($u^\mu \partial_\mu D = v^\mu n_\mu > 0$) if and only if

$$1 - n\cos(2\Theta) > 0$$

as follows from (18). Therefore, for $0 \leq n \leq 1$ particles recede whatever the direction $\vec{n}$. On the contrary, when $n > 1$ there always exists a set of directions in which the particles do not recede but rather come closer to each other. This set of directions is defined by (17) with

$$\cos(2\Theta) > \frac{1}{n} \quad (n > 1), \qquad 0 \leq \Phi \leq 2\pi,$$

which obviously form the solid interior of a *double cone* at each point with $\vec{e}_3$ as axis and an angle of $\frac{1}{2}\arccos\left(\frac{1}{n}\right)$. This angle is always less than $\frac{\pi}{4}$. Of course, the receding velocity (18) depends not only on the direction but also on the particular point of spacetime. In general, when $n > 1$ the magnitude of the velocity decreases with r. The time variation depends on the particular model via the explicit function T.

IV. DISCUSSION

We have thus presented a general family of spacetimes which must be considered cosmological models because they are obvious generalizations of the FLRW standard models (which are included as particular cases). The only fundamental concepts of standard cosmology, that is, the *arbitrary* scale factor and the spatial curvature index, have also been kept in the general family (1) and (2), represented by $T(\tau)$ [through formula (14)] and M, respectively, as has been carefully explained above. Nevertheless, new parameters and associated physical concepts appear in (1) and (2). These are as follows. The first is the fundamental parameter n, which defines the anisotropy and inhomogeneity of the model and measures the essential departure of any particular model with respect to its FLRW analogue (given by the same T and M but with $n = 0$). The second is the arbitrary constant K, which can be interpreted as a fixed initial or boundary energy density. Thus the maximum energy density at any instant of time τ can be chosen as large (or small) as desired, even for the singularity-free models, and independently of the scale factor. Finally, there is the constant N, which is not directly related with the matter content of the universe but only with the Weyl pure gravitational part of the curvature. N is also indirectly related to ρ, p_r, and p_z via the specific form of the function Σ. The construction of the family (1) and (2) shows that the singularity-free metrics presented in [15,12,17] (see also [23] and [24]) can be considered cosmological models as are the FLRW models, and the questions raised sometimes concerning their possible instability or zero measure in a hypothetical space of metrics may no longer be maintained unless they are also raised against FLRW cosmologies themselves.

Of course, the fundamental question arises of whether or not we can construct any *realistic* singularity-free cosmological model. In fact, one of the main purposes of this paper is to show clearly the simple fact that this question is *still* open. The family (1) and (2) cannot give a definite answer to this question. It may give some light, though, in the following sense. If we accept that the actual Universe is and has been approximately FLRW-like around us from some time up to now (for example, since nucleosynthesis time), we must look for some models which resemble FLRW models at those times. Thus what we need is, schematically,

$$T(\tau) \sim \begin{cases} T_{\text{SF}} \text{ around } \tau = 0 \text{ with any } n \\ T_{\text{FLRW}} \text{ for } \tau > \bar{\tau} \text{ with } n = 0, \end{cases} \qquad (19)$$

where $\bar{\tau}$ is some fixed time, T_{FLRW} is the scale factor of the FLRW model best describing the Universe around us now, we have set the rebound time at $\tau = 0$, and T_{SF} is any function T leading to a singularity-free scale factor there (any smooth nonvanishing function of τ with a local minimum at the rebound time $\tau = 0$). Just to fix ideas, let us note that simple examples for T_{SF} are $\left(1 + a^2\tau^2\right)^m$ or $\cosh^m(a\tau)$. Of course, there are many other different choices for singularity-free scale factors. Let us remark that $\tau = 0$ is a real *birth* time for the structured matter of the Universe, even though it is not an initial singularity. This follows because we can choose our model in such a way that any matter decomposes into its most elementary contituents (particles) in the collapsing era previous to our present expanding era. Thus the history of the Universe as usually explained will be just the same. What is really important for that history of the Universe is the existence of a highly compressed and very hot expanding phase, here represented by a period after the rebound by choosing the parameters and the scale factor appropriately. Thus the formation of light nuclei as well as the existence of the microwave background radiation might be fully explained without problems (but perhaps not their isotropy).

The simplest way to achieve properties (19) is to assume $n = 0$ everywhere, that is, a FLRW model for every time and place. As is well known, however, this would lead to the violation of energy conditions, something which is not desirable and manifestly against our purposes. In fact, if we assumed $n = 0$, we would simply have the usual standard inflationary FLRW models, which of course *do* violate the strong energy condition. Here we can remark, by the way, the very curious fact that most cosmologists are willing and happy to accept violation of energy conditions in the quite fashionable inflationary FLRW models, while they stick strongly to the view that singularity theorems imply the plausibility of the initial singularity *if* energy conditions hold. For the same price (violation of energy conditions), we could have inflationary *singularity-free* models.

But our intention is to construct a realistic singularity-free cosmological model without ever violating energy conditions. Thus we cannot assume $n = 0$ for all τ; rather we must consider a transition period with a smooth (or "adiabatic") variation of n from some $n > 1$ at small τ (say $n = 3$ so that we can have the $p = \rho/3$ realistic equation of state for highly concentrated relativistic matter), to $n = 0$ for $\tau > \bar{\tau}$. This seems the right answer, but it has some problems. Essentially the point is that, if the model is FLRW-like at some time, then the existence of closed trapped surfaces is inevitable [6], and thus the singularity theorems [6] would apply proving that the model either is singular or violates the strong energy condition somewhere. The only way out of this situation is to assume the smooth variation of n not only with τ but also with r (or other space coordinates). Thus we could have regions with $n = 0$ for $\tau > \bar{\tau}$ and at the same time some other regions with $n \neq 0$ for $\tau > \bar{\tau}$. If these regions are carefully chosen (mainly trying to avoid the situation that the FLRW regions with $n = 0$ are so big as to contain closed trapped spheres), then the model might be nonsingular and satisfy energy conditions. Notice that the horizon in FLRW models is at a distance similar to the radius of the trapped spheres around us [6], so that we could very well live in a FLRW region contained in a non-FLRW bigger Universe.

The above construction may seem somehow artificial, but it is the only thing we can do with the hitherto available singularity-free models (1) and (2). Nevertheless, there are some nice features in this construction, such as, for instance, the *natural* inflationary character of the model near the rebound time, the fact that this inflation has finite duration, and all this without ever violating energy conditions or any other physical requirement. Another virtue of these models is their testability, because their global structure has influence on the observations that can be made at the FLRW regions. For example, the redshift of distant objects depends on the shear, expansion, and acceleration of the model through an *integral along* the corresponding null geodesic, so that the inhomogeneity of faraway regions has influence on this redshift.

The physical processes leading to the variation of n with the coordinates are not clear nor are they fixed by any property at this stage. However, if we allow n to depend on τ and r, then the matter content of (1) is no longer a fluid with two pressures, but something more general. Actually, with $n(\tau, r)$ the only new nonvanishing component of the energy-momentum tensor is T_{01}. Therefore the existence of some kind of energy transport leading to the homogenization of some regions and to the greater inhomogenization of others may be at the origin of those processes. However, our own opinion is that models more general than (1) will be the right answer. In our view, the importance of this first step represented by (1) is to show the possibility of and the way towards singularity-free realistic cosmological models.

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