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ARTICLES

Stochastic dynamics of large-scale inflation in de Sitter space

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In this paper we derive exact quantum Langevin equations for stochastic dynamics of large-scale inflation in de Sitter space. These quantum Langevin equations are the equivalent of the Wigner equation and are described by a system of stochastic differential equations. We present a formula for the calculation of the expectation value of a quantum operator whose Weyl symbol is a function of the large-scale inflation scalar field and its time derivative. The quantum expectation value is calculated as a mathematical expectation value over a stochastic process in an extended phase space, where the additional coordinate plays the role of a stochastic phase. The unique solution is obtained for the Cauchy problem for the Wigner equation for large-scale inflation. The stationary solution for the Wigner equation is found for an arbitrary potential. It is shown that the large-scale inflation scalar field in de Sitter space behaves as a quantum one-dimensional dissipative system, which supports the earlier results of Graziani and of Nakao, Nambu, and Sasaki. But the analogy with a one-dimensional model of the quantum linearly damped anharmonic oscillator is not complete: the difference arises from the new time-dependent commutation relation for the large-scale field and its time derivative. It is found that, for the large-scale inflation scalar field, the large time asymptotics is equal to the "classical limit." For the large time limit the quantum Langevin equations are just the classical stochastic Langevin equations (only the stationary state is defined by the quantum field theory).

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I. INTRODUCTION

While the quasiclassical picture of the inflationary universe scenario, based on a Fokker-Planck evolution equation for the probability distribution of the inflation field, is almost complete (see the basic papers [1–8]), the essentially quantum-mechanical features of inflation are now a subject of investigation.

We would like to note recent investigations into this problem carried out by Graziani [9–12], Nakao, Nambu, and Sasaki [13], Hu, Paz, and Zhang [14], Nambu [15], and Habib [16]. (The aim is not to present the full list of papers on this subject but to mention those which are closest to our considerations.)

In Ref. [13] the dynamics of an inflationary scalar field in a de Sitter background is investigated on the basis of the extended version of the stochastic approach proposed by Starobinsky [5]. In this approach, the scalar field operator is split into the long wavelength mode and the short wavelength mode. This split allows the reduction of the operator equation for the scalar field to Langevin equations of order $\sqrt{\hbar}$.

In the series of papers [9–12] of Graziani, dealing with quantum probability distributions and the dynamics of the early Universe, the approach is based on the Wigner function and its evolution equation, which is the Wigner equation. The author concentrates attention on the large-scale inflation, where the spatial variable of the inflation scalar field is removed by averaging over a causal horizon volume. It is established that the quantum description of large-scale inflation in de Sitter space is equal to the quantum mechanics of a one-dimensional dissipative system. In Ref. [10] it is shown how quantum Langevin equations can be derived to any order of $\sqrt{\hbar}$ when they correspond to the Wigner equation expanded in powers of $\hbar$ and truncated at some power of $\hbar$. However the Wigner equation for large-scale inflation presented in Refs. [9–12] is not accurate and we will return to this point in our paper.

In paper [14] dealing with quantum Brownian motion Hu, Paz, and Zhang have derived an exact master equation for a quantum open system, which is an extension of earlier results obtained by Dekker [17], and Caldeira and Leggett [18]. As was established by Vilenkin [1], Linde [3], and Starobinsky [5], the basic stochastic equation for the large-scale quantum evolution of inflation is similar to Brownian motion, that is the quantum Langevin equations of order $\sqrt{\hbar}$. So the investigation of quantum Brownian motion is closely connected to further investigations of statistical and quantum effects at the early Universe.

In Ref. [15] the master equation for the long wavelength mode for the scalar field is derived and the Wigner representation is given for the space-homogeneous case of this mode. Such a Wigner representation should be equivalent to the Wigner equation for the large-scale scalar field, which is derived in our paper. However, the Wigner

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equation presented by Nambu [15] differs from ours by a term with a second derivative over the space coordinate and by a term with mixed derivatives over space and momentum coordinates. The origin of the difference is the use of different phase space variables. We have found an apparent contradiction between the Wigner representation in Ref. [15] and the assumption under which it was derived. This will be discussed at the end of Sec. IV.

In paper [16] a quantum-mechanical phase-space picture is constructed for the coarse-grained free scalar field in an inflationary universe. The distinction of Habib's paper is that the evolution picture is considered in a conformal time and for the conformal field, related to the original scalar field by a time-dependent canonical transformation. The evolution equation for the conformal field does not contain any damping term. Through this approach one can see an elegant analogy between the original problem and a quantum-mechanical system with a time-dependent mass. An odd thing about the Wigner representation in Ref. [16] is that the Wigner distribution function, governed by Eq. (55) in [16], is not a deterministic function. It is not clear how the stochastic function can be understood as a distribution function. To avoid this confusion the author works with an averaged Wigner function.

In this paper we restrict our consideration of the inflation scalar field to investigation of the dynamics of the large-scale or coarse-grained ($\geq$ causal horizon) scalar field in a de Sitter background. Following the main line of Graziani [9–12], we describe the evolution of the large-scale inflation by the Wigner equation. Then the aim of this work is to derive exact quantum Langevin equations (to all orders of $\sqrt{\hbar}$), which describe the stochastic dynamics of the large-scale inflation.

Thus, our approach is opposite in some sense to the approach of Hu, Paz, and Zhang [14]. These authors move toward the possible observation of macroscopic effects from the search for an adequate description for statistical and quantum effects, while our way is to start from the macrolevel to obtain an equivalent stochastic description.

II. BASIC FORMULATIONS

The Lagrangian density of the inflation scalar field $\Phi(x,t)$ in a de Sitter background is

$$\mathcal{L} = -\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi + \mathcal{V}(\Phi)\right], \tag{1}$$

where the background metric is assumed to have the form

$$ds^2 = -dt^2 + a^2(t)dx^2, \tag{2}$$

g denotes a metric, t is a time, x is a three-dimensional spatial coordinate, $\mathcal{V}(\Phi)$ is a potential, and $a(t)$ is a scale factor (some positive function).

Then the equation for the classical inflation scalar field $\Phi(x,t)$ is

$$\left[\frac{\partial^2}{\partial t^2} + 3\frac{\dot{a}(t)}{a(t)}\frac{\partial}{\partial t} - \left(\frac{1}{a(t)}\right)^2\Delta_x\right]\Phi(x,t) + \mathcal{V}'[\Phi(x,t)] = 0, \tag{3}$$

where Δ_x is the spatial Laplace operator, the dot over a symbol means a time derivative $\dot{y}(t) = dy/dt$, and $\mathcal{V}'[\Phi] = d\mathcal{V}[\Phi]/d\Phi$.

The expansion of the Universe is assumed,

$$\frac{\dot{a}(t)}{a(t)} = H(t), \tag{4}$$

where $H(t)$ is some non-negative integrable function of time. In other words, the scale factor is

$$a(t) = a(0)\exp\left\{\int_0^t H(\tau)d\tau\right\}, \tag{5}$$

where $a(0)$ is the value of the scale factor at $t=0$, the beginning of inflation. In this model the beginning of inflation is taken to coincide with the origin of the Universe.

Strictly speaking, in our consideration we deal with an expanding Friedmann-Robertson-Walker (FRW) universe, "approximately" close to a de Sitter universe in the sense: $H(t) \approx$ const.

From Eq. (1) one obtains the canonical momentum conjugate to the field $\Phi(x,t)$,

$$\Pi(x,t) = \frac{\partial\mathcal{L}}{\partial(\partial\Phi/\partial t)} = a(t)^3\frac{\partial\Phi(x,t)}{\partial t}, \tag{6}$$

and the Hamiltonian density

$$\begin{aligned} H &= \Pi\frac{\partial\Phi}{\partial t} - \mathcal{L} \\ &= \tfrac{1}{2}a(t)^{-3}\Pi^2 + \tfrac{1}{2}a(t)[\nabla_x\Phi]^2 + a(t)^3\mathcal{V}(\Phi). \end{aligned} \tag{7}$$

When interested in large-scale ($\geq$ causal horizon) physics, a coarse-graining procedure is utilized and it leads to a coarse-grained or averaged scalar field $\Phi_X(t)$:

$$\Phi_X(t) = \frac{1}{V}\int_{\Omega_X}\Phi(x,t)dx, \tag{8}$$

where the index X is a label referring to the center of a region Ω, over which $\Phi(x,t)$ is averaged, and V is its volume. The volume of spatial averaging is taken not smaller than a causal horizon volume:

$$V \geq \tfrac{4}{3}\pi\ell^3(t) \tag{9}$$

with the causal horizon (or the "coordinate horizon" in terms of Ellis and Rothman [19]) given by

$$\ell(t) = \int_0^t a(\tau)^{-1}d\tau.$$

For de Sitter space,

$$a(t) = \exp\{Ht\}, \qquad H = \text{const}; \tag{10}$$

then $\ell(t) = H^{-1}(1 - e^{-Ht})$ and the volume of averaging is chosen to be $V = (4/3)\pi H^{-3}$.

For "approximately" de Sitter space we will think about a volume of averaging as constant for all time. This always can be assumed if $\ell(t) \leq$ const. The techniques we use allow considering time-dependent volume $V(t)$ along the same lines but we shall not consider this case for the sake of brevity. (See remarks in Sec. VIII.)

Each large-scale region Ω (labeled by X) can be considered as a separate quantum-mechanical system because each lies outside of its neighbors' light cone: there is no exchange of information between large-scale regions. The profit of the coarse-graining procedure is that it reduces the quantum field problem to a quantum-mechanical problem. At the same time there are still some peculiarities, following from the field theory, which do not make the analogy with quantum mechanics complete. We will point out these peculiarities later in this section.

After performing the coarse graining (8), the spatial varying term in the Euler-Lagrange equation (3) can be neglected because of a smaller factor $a^{-2}(t)V^{-2/3}$ (see [20]) and the equation for the large-scale inflation field $\Phi_X(t)$ becomes

$$\ddot{\Phi}_X(t) + 3H(t)\dot{\Phi}_X(t) + U'[\Phi_X(t)] = 0, \tag{11}$$

where $U(\Phi)$ is the coarse-grained potential.

The averaging of the momentum (6) gives

$$\Pi_X(t) = a(t)^3\dot{\Phi}_X(t). \tag{12}$$

The Lagrangian (density) for the coarse-grained field $\Phi_X(t)$ is

$$\mathcal{L}[\Phi_X] = a(t)^3[\tfrac{1}{2}\dot{\Phi}_X^2 - U(\Phi_X)]. \tag{13}$$

This gives the Euler-Lagrange equation (11) and keeps the averaged momentum (12) canonically conjugate to $\Phi_X(t)$:

$$\Pi_X(t) = \frac{\partial\mathcal{L}[\Phi_X]}{\partial\dot{\Phi}_X}. \tag{14}$$

Now the Hamiltonian (density) for the coarse-grained field $\Phi_X(t)$ is

$$\mathcal{H}(\Phi_X, \Pi_X; t) = \tfrac{1}{2}a(t)^{-3}\Pi_X^2 + a(t)^3U(\Phi_X), \tag{15}$$

it can be considered as the classical Hamiltonian for the large-scale inflation field, in the sense that the equations of motion produced by this Hamiltonian

$$\frac{\partial\Phi_X}{\partial t} = \frac{\partial\mathcal{H}}{\partial\Pi_X}, \frac{\partial\Pi_X}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\Phi_X} \tag{16}$$

are equivalent to the field equation (11).

To make the step from the classical equation (11) to the quantum equation it is necessary to quantize Eq. (11) taking into account Eqs. (12) and (15). This leads at the end to the quantum mechanics of a one-dimensional dissipative system in the description on Φ_X, $\dot{\Phi}_X$ variables.

One can apply the canonical procedure, which is based on quantal noise operators and conserves the fundamental commutator for canonical position and momentum operators in the course of time, or the influence-functional method of Feynman and Vernon [21]. Both ways give the master equation for the "reduced" density operator $\hat{\rho}$ (see [17,18] for techniques), describing the time evolution for the large-scale inflation:

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{V}{i\hbar}[\hat{\mathcal{H}}, \hat{\rho}] - \frac{V}{\hbar}a(t)^6D(t)[\hat{\Phi}_X, [\hat{\Phi}_X, \hat{\rho}]]. \tag{17}$$

Here, $\hat{\mathcal{H}}$ is an operator form for the classical Hamiltonian (15), [,] stands for a commutator, and the diffusion coefficient $D(t)$ is, in general, some non-negative function. In particular, the diffusion coefficient can be assumed to be $D(t) = 3H(t)\times$ const, where const is determined by physical parameters of the system at equilibrium.

Someone who would follow, formally, the quantization procedure in quantum mechanics would notice immediately two points by which the master equation (17) differs from the formally obtained one: first, the new constant $\hbar/V$ instead of Planck's constant $\hbar$, and second, the factor $a(t)^6$ in the diffusion term, which reflects the explicit time dependence of the Hamiltonian (15). It is time to discuss how these peculiarities come from the quantum field theory.

The origin of the scaling

$$\hbar \rightarrow \hbar/V \tag{18}$$

in Eq. (17), where V is a volume of averaging (9), is that the fundamental commutator for canonical position and momentum operators in the relativistic quantum field theory

$$[\hat{\Phi}(x,t), \hat{\Pi}(y,t)] = i\hbar\,\delta(x-y)$$

is transformed by coarse-graining procedure (8) to

$$[\hat{\Phi}_X(t), \hat{\Pi}_X(t)] = i\hbar/V, \tag{19}$$

$$[\hat{\Phi}_X(t), \hat{\Pi}_Y(t)] = 0, \qquad X \neq Y.$$

Another way to derive the master equation (17) would be to use the momentum and Hamiltonian

$$\Pi_\Omega(t) = V\Pi_X(t), \tag{20}$$

$$\mathcal{H}_\Omega = V\mathcal{H}(\Phi_X, \Pi_X; t).$$

For momentum Π_Ω, the fundamental commutator is

$$[\hat{\Phi}_X(t), \hat{\Pi}_\Omega(t)] = i\hbar. \tag{21}$$

The description in terms of Eqs. (20) and (21) is natural for the large-scale region Ω. Returning back to the spatially dependent field $\Phi(x,t)$, the Hamiltonian for Ω is found to be

$$\mathcal{H}_\Omega = \int_\Omega H(x,t)dx, \tag{22}$$

or, in view of the Hamiltonian (density) (15),

$$\mathcal{H}_\Omega = V\mathcal{H}(\Phi_X, \Pi_X; t) \tag{23}$$

[in (23) the spatially varying term of the original field $(\nabla_x\Phi)^2$ was neglected].

Now let us discuss the diffusion term in the master equation (17). It should be mentioned that the general form of the diffusion term in the master equation is

$$-\frac{V^2}{\hbar^2}C(t)[\hat{\Phi}, [\hat{\Phi}, \hat{\rho}]],$$

where $C(t)$ is some time-dependent coefficient. This coefficient is defined with respect to a suitably chosen vacuum state in the field theory. For de Sitter space it is the so-called Bunch-Davies vacuum [22,23]. In terms of Eq. (17) this means that it is determined by the physical parameters of the system at equilibrium.

If in accordance with the field theory (of inflation) we assume that the energy density of the equilibrium state is an invariant in de Sitter space, then

$$\langle\hat{\mathcal{H}}_\Omega(t)\rangle/a(t)^3V = \langle\hat{\mathcal{H}}(\hat{\Phi}_X, \hat{\Pi}_X; t)\rangle/a(t)^3 = \text{const}, \tag{24}$$

where, to obtain the energy density, the Hamiltonian $\mathcal{H}_\Omega$ is divided by the proper volume $a(t)^3V$. The expectation values, $\langle\,\rangle$, of the operators are taken on the stationary solution of the master equation. For "approximately" de Sitter space, $\dot{H}(t)$ is neglected so the condition (24) can be assumed.

To fulfill Eq. (24) we have obtained the factor $a(t)^6$ in the diffusion term of (17). Taking into account that the constant in Eq. (24) is proportional to $(\hbar/V)$, in Eq. (17) we can show explicitly the dependence

$$C(t) = \frac{\hbar}{V}a(t)^6D(t),$$

which is valid for large-scale inflation. We will use master equation (17) as a starting point for investigation of the stochastic dynamics of the large-scale inflation.

From this point onwards in the paper the large-scale scalar field is referred to as $\Phi(t)$, omitting the index X.

III. THE WIGNER EQUATION FOR LARGE-SCALE INFLATION

The Wigner function $W(q,p;t)$ [24,25] is a function on the classical phase space and describes the distribution of position and momentum. The Wigner function is not a probability distribution since it can assume negative values; $W(q,p;t)$ is a real function.

As a density matrix the Wigner function contains all of the information corresponding to the quantum state. The expectation value for an arbitrary operator $\hat{A}(\hat{\Phi},\hat{\dot{\Phi}})$ can be calculated by the formula

$$\langle\hat{A}\rangle = \int dq\,dp\,A(q,p)W(q,p;t), \tag{25}$$

where $A(q,p)$ is a Weyl symbol for the operator $\hat{A}(\hat{\Phi},\hat{\dot{\Phi}})$.

Let us derive the time-evolution equation for the Wigner function for large-scale inflation, that is equivalent to the master equation (17).

For the Wigner operator $\hat{W}$ the equation has the same form as for the density operator $\hat{\rho}$ (17) [because coefficients $a(t)$ and $D(t)$ depend only on t],

$$\frac{\partial\hat{W}}{\partial t} = \frac{V}{i\hbar}[\hat{\mathcal{H}},\hat{W}] - \frac{V}{\hbar}\,a(t)^6D(t)[\hat{q},[\hat{q},\hat{W}]], \tag{26}$$

where Weyl symbols for $\hat{W}$ and $\hat{\mathcal{H}}$ are

$$\hat{W} \leftrightarrow W(q,\Pi;t),$$
$$\hat{\mathcal{H}} \leftrightarrow \mathcal{H}(q,\Pi) = \tfrac{1}{2}a(t)^{-3}\Pi^2 + a(t)^3V(q). \tag{27}$$

Here, we assume the following correspondence between phase-space variables (q,Π) or (q,p) and variables of the large-scale inflation scalar field $(\Phi,\dot{\Phi})$:

$$\Phi = q, \qquad \dot{\Phi} = p, \qquad a(t)^3\dot{\Phi} = \Pi. \tag{28}$$

To obtain an equation for the Wigner function $W(q,\Pi;t)$ from its operator form (26) we will use the formula for the composition of operators in the Weyl formalism. Because of this formula [26] and the commutator relation (19), a Weyl symbol $A(q,\Pi)$ for the composition of two operators $\hat{A} = \hat{B}\hat{C}$ is defined via the correspondence

$$A(q,\Pi) = B\left(\overset{\mathrm{II}}{q} + \frac{i\hbar}{2V}\overset{\mathrm{I}}{\frac{\partial}{\partial\Pi}}, \overset{\mathrm{II}}{\Pi} - \frac{i\hbar}{2V}\overset{\mathrm{I}}{\frac{\partial}{\partial q}}\right)C(q,\Pi)$$
$$= C\left(\overset{\mathrm{II}}{q} - \frac{i\hbar}{2V}\overset{\mathrm{I}}{\frac{\partial}{\partial\Pi}}, \overset{\mathrm{II}}{\Pi} + \frac{i\hbar}{2V}\overset{\mathrm{I}}{\frac{\partial}{\partial q}}\right)B(q,\Pi)\,, \tag{29}$$

where $B(q,\Pi)$ and $C(q,\Pi)$ are Weyl symbols for the operators $\hat{B}$ and $\hat{C}$, and the numbers I,II over operators show the order in which the operators act.

Using Eqs. (27) and (29), one can obtain the following correspondence between operators and their Weyl symbols:

$$[\hat{\mathcal{H}},\hat{W}] \leftrightarrow -\frac{i\hbar}{V}\,a(t)^{-3}\Pi\frac{\partial}{\partial q}W(q,\Pi;t)$$
$$+a(t)^3\left[U\left(q + \frac{i\hbar}{2V}\frac{\partial}{\partial\Pi}\right) - U\left(q - \frac{i\hbar}{2V}\frac{\partial}{\partial\Pi}\right)\right]$$
$$\times W(q,\Pi;t), \tag{30}$$

$$[\hat{q},[\hat{q},\hat{W}]] \leftrightarrow -\frac{\hbar^2}{V^2}\frac{\partial^2}{\partial\Pi^2}W(q,\Pi;t). \tag{31}$$

Combining relations (30) and (31) in accordance with Eq. (26), one has the time-evolution equation for the Wigner function $W(q,\Pi;t)$:

$$\frac{\partial W(q,\Pi;t)}{\partial t} = -a(t)^{-3}\Pi\frac{\partial W}{\partial q} + \frac{\hbar}{V}a(t)^6D(t)\frac{\partial^2W}{\partial\Pi^2}$$
$$+\frac{V}{i\hbar}a(t)^3\left[U\left(q + \frac{i\hbar}{2V}\frac{\partial}{\partial\Pi}\right)\right.$$
$$\left.-U\left(q - \frac{i\hbar}{2V}\frac{\partial}{\partial\Pi}\right)\right]W(q,\Pi;t). \tag{32}$$

Equation (32) is called the Wigner equation. [Note that the commonly used way to derive the Wigner equation is to apply the coordinate representation for the Wigner function,

$$W(q,\Pi;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\Pi x}\left\langle q - \frac{\hbar}{2V}x \,\middle|\, \hat{\rho} \,\middle|\, q + \frac{\hbar}{2V}x \right\rangle dx \,, \tag{33}$$

to the master equation (17). Actually it is a longer way because it deals with multiple integrations and integrations by parts while the Weyl formalism gives an answer in a plain algebraic way.]

Because we are interested in the distribution of Φ and $\dot{\Phi}$ let us make the transition from $W(q,\Pi;t)$ to $W(q,p;t)$ by relations (28) and the equality

$$W(q,p;t) = a(t)^3 W(q,\Pi;t) \,, \tag{34}$$

which follows from Eq. (25). For the Wigner function $W(q,p;t)$, one has the time-evolution equation

$$\begin{aligned}\frac{\partial W(q,p;t)}{\partial t} &= -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2}\\ &\quad + \frac{V}{i\hbar}a(t)^3\left[U\left(q + \frac{i\hbar}{2V}a(t)^{-3}\frac{\partial}{\partial p}\right) - U\left(q - \frac{i\hbar}{2V}a(t)^{-3}\frac{\partial}{\partial p}\right)\right]W(q,p;t),\end{aligned} \tag{35}$$

where $H(t)$ is defined by Eq. (4).

The Wigner equation (35) can be rewritten in the equivalent form

$$\frac{\partial W(q,p;t)}{\partial t} = -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2} + \frac{V}{i\hbar}a(t)^3\int_{-\infty}^{\infty} du\, W(q,p-u;t)\mathcal{I}(q,u;t), \tag{36}$$

where

$$\mathcal{I}(q,u;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-iuy}\left[U\left(q + \frac{\hbar}{2V}a(t)^{-3}y\right) - U\left(q - \frac{\hbar}{2V}a(t)^{-3}y\right)\right]. \tag{37}$$

Equation (35), or Eq. (36), is the complete Wigner equation (to all orders of $\hbar$) for the large-scale inflation in de Sitter space. It describes the time evolution for the distribution of Φ and $\dot{\Phi}$ in a sense of equality (25). The expansion of the Universe, which is described by the scale factor $a(t)$ and equality (4), gives the origin for a dissipation term, with the dissipation coefficient $3H(t)$, in the Wigner equation. In this sense, large-scale inflation can be considered as a quantum one-dimensional dissipative system, which supports Graziani's statement [9]. At the same time, the expansion of the Universe influences the potential terms of Eq. (35) [or (36)], which was missed in Refs. [9–12].

In Refs. [17,14], the Wigner equation for a quantum dissipative system was derived for a harmonic oscillator's potential $U(q) = \omega^2 q^2/2$. For an arbitrary potential $U(q)$, a truncated Wigner equation, or a Kramers-Moyal equation, is commonly used in the literature instead of the complete Wigner equation (see for example [18]). In our case (35), it would be

$$\begin{aligned}\frac{\partial W(q,p;t)}{\partial t} &= -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2}\\ &\quad + \frac{\partial U(q)}{\partial q}\frac{\partial W}{\partial p} + O(\hbar^2),\end{aligned} \tag{38}$$

where $O(\hbar^2)$ is a value of order $\hbar^2$.

The potential term in Eq. (35) can be expanded in powers of $\hbar$ by Taylor's series, which leads to the equation

$$\begin{aligned}\frac{\partial W(q,p;t)}{\partial t} &= -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2}\\ &\quad + \sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2V}a(t)^{-3}\right)^{k-1}\\ &\quad \times\frac{\partial^k U(q)}{\partial q^k}\frac{\partial^k W}{\partial p^k}.\end{aligned} \tag{39}$$

In the series of papers [9–12], the expansion in powers of $\hbar$ of the Wigner equation was used to show how to improve the accuracy of Eq. (38).

The potential term in Wigner equation (35) [or in the equivalent representations (36), (37), and (39)] on a macrolevel contains information about quantum noise on the microlevel, when

$$\frac{\partial^k U(q)}{\partial q^k} \not\equiv 0 \qquad \text{for } k \geq 3. \tag{40}$$

This quantum noise is non-Gaussian noise. Condition (40) is always satisfied when a potential deviates from a harmonic oscillator's potential. The diffusion term, containing $\partial^2 W/\partial p^2$, in the Wigner equation also represents quantum noise (Gaussian noise) on a macrolevel. Some authors consider a possibility that the Wigner equation may contain second order derivatives $\partial^2 W/\partial q^2$, $\partial^2 W/\partial q\partial p$, which are also of quantum origin [14–17]. (These terms would correspond to additional terms $[\hat{\Pi},[\hat{\Pi},\hat{\rho}]]$, $[\hat{\Phi},[\hat{\Pi},\hat{\rho}]]$, $[\hat{\Pi},[\hat{\Phi},\hat{\rho}]]$ in the master equation (17).) If one works with the phase-space variables (28), these terms have no physical sense because they lead to wrong Langevin equations: $dq_t/dt \neq p_t$ or $d\Phi/dt \neq \dot{\Phi}$.

In this paper we will work with the complete Wigner equations (36) and (37) for large-scale inflation to derive equations for the corresponding stochastic dynamics. Let us supply the evolution equation (36) with the initial condition

$$W(q,p;0) = W_0(q,p), \tag{41}$$

where the initial Wigner function is chosen to satisfy the properties

$$\text{(a)} \qquad W_0(q,p) \text{ is a real function}, \tag{42}$$

$$\text{(b)} \qquad \int W_0(q,p)dq\,dp = 1, \tag{43}$$

$$\text{(c)} \qquad \int [W_0(q,p)]^2 dq\,dp \leq \frac{V}{2\pi\hbar}a(0)^3, \tag{44}$$

which follow from general properties for the Wigner function (see [25]).

IV. QUANTUM LANGEVIN EQUATIONS

In this section we deduce quantum Langevin equations which are equal to the complete Wigner equations (36) and (37), but we start with some assumption about the potential $U(q)$:

$$U(q) = \frac{\omega^2}{2}q^2 + \int \exp\{-iqp\}\,\mu(dp), \tag{45}$$

where $\mu(p)$ is a bounded measure such that

$$\int \mu(dp) \leq \text{const},$$

$$\int p^2\mu(dp) \leq \text{const}. \tag{46}$$

The first term in the right-hand side of Eq. (45) is just the harmonic oscillator's potential while the second term can be considered as a deviation from it. The potential is assumed to be real.

To include more model potentials used in the theory of inflation, ω^2 is allowed to be either positive or negative, or to be zero. A parabolic potential connected with "chaotic inflation" is included in Eq. (45) with $\mu(dp) \equiv 0$. A double-well potential, representable in the form (45) is, for example,

$$U(q) = \omega^2 q^2/2 + K\cos(kq) I_{[-(3\pi/2k),(3\pi/2k)]}(q), \tag{47}$$

where ω,K,k are some parameters (real values) and $I_A(q)$ is the identificator of a set A:

$$I_A(q) = \begin{cases} 1 & \text{if} q \in A, \\ 0 & \text{if} q \notin A. \end{cases}$$

Form (45) rules out model potentials which are polynomials higher than second order in q. Remember here that model potentials of polynomial type have appeared in the theory of inflation through expansion of $U(q)$ in powers of q near $q = 0$ (see, for example, [5]). Thus, almost all physical potentials for the inflation field $\Phi(t)$ can be represented as Eq. (45).

For potential (45) the Wigner equation becomes

$$\begin{aligned} \frac{\partial W(q,p;t)}{\partial t} = & -p\frac{\partial W}{\partial q} + \omega^2 q\frac{\partial W}{\partial p} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2} - \frac{iV}{\hbar}a(t)^3\int d\mu(p')\exp\{-iqp'\} \\ & \times\left[W\left(q,p+\frac{\hbar}{2V}a(t)^{-3}p';t\right) - W\left(q,p-\frac{\hbar}{2V}a(t)^{-3}p';t\right)\right], \end{aligned} \tag{48}$$

$q,p,p' \in \mathbf{R}^1$, $t \in [0,T]$, supplied by the initial condition

$$W(q,p;0) = W_0(q,p), \tag{49}$$

such that properties (42)–(44) are satisfied.

In order to solve this equation and to deduce stochastic equations, which describe a path in a phase space $(q,p) \equiv (\Phi,\dot{\Phi})$, we should reduce it to the form of a forward Kolmogorov equation [27] (p. 102). Such an approach was first proposed by Maslov and Chebotarev [28] and developed much further by Comber, *et al.* [29].

To reduce Eq. (48) to a forward Kolmogorov equation we need to transform the last term in the right-hand side of it to a standard form. It can be done by introduction of a new function

$$f(q,p,\theta;t) = W(q,p;t)\exp\left\{-\frac{2V}{\hbar}A(t) + i\theta\right\}, \quad \theta \in \mathbf{R}^1/\text{mod}\,2\pi, \tag{50}$$

where θ is a new variable, and function $A(t)$ will be defined later.

Inserting Eq. (50) into Eq. (48), one has

$$\begin{aligned} \frac{\partial f(q,p,\theta;t)}{\partial t} = & -p\frac{\partial f}{\partial q} + \omega^2 q\frac{\partial f}{\partial p} + 3H(t)\frac{\partial}{\partial p}(pf) + \frac{\hbar}{V}D(t)\frac{\partial^2 f}{\partial p^2} + \frac{V}{\hbar}a(t)^3\int_{-\infty}^{\infty}du\int\mu(dp') \\ & \times\left[f\left(q,p+\frac{\hbar}{2V}a(t)^{-3}p'u,\theta+\frac{\pi}{2}u-qp';t\right) - f(q,p,\theta;t)\right][\delta(u+1)+\delta(u-1)] \\ & + \frac{2V}{\hbar}a(t)^3 f(q,p,\theta;t)\int\mu(dp') - \frac{2V}{\hbar}\frac{dA(t)}{dt}f(q,p,\theta;t), \end{aligned} \tag{51}$$

where $\delta(u)$ is a Dirac delta function.

Let us put

$$A(t) = \int_0^t a(\tau)^3 d\tau \int \mu(dp). \tag{52}$$

Note that, for the scale factor (10),

$$A(t) = \int \mu(dp)(e^{3Ht} - 1)/3H. \tag{53}$$

Let $m_\hbar(dpdu;t)$ be the measure

$$m_\hbar(dpdu;t) = \frac{V}{\hbar} a(t)^3 \mu(dp)[\delta(u+1) + \delta(u-1)]du. \tag{54}$$

The measure $m_\hbar(dp\,du;t)$ is a bounded measure on $\mathbf{R}^1 \otimes (-1) \otimes (+1)$ multiplied by time-dependent function $a(t)^3$. For an arbitrary function $\psi(p,u)$, an integral over this measure is

$$\int \psi(p,u) m_\hbar(dp\,du;t) = \frac{V}{\hbar} a(t)^3 \int [\psi(p,+1) + \psi(p,-1)]\mu(dp). \tag{55}$$

Now, with Eqs. (52) and (54), Eq. (51) becomes

$$\begin{aligned} \frac{\partial f(q,p,\theta;t)}{\partial t} &= -p\frac{\partial f}{\partial q} + \omega^2 q \frac{\partial f}{\partial p} + 3H(t)\frac{\partial}{\partial p}(pf) + \frac{\hbar}{V}D(t)\frac{\partial^2 f}{\partial p^2} \\ &+ \int \left[f\left(q, p + \frac{\hbar}{2V}a(t)^{-3}p'u, \theta - \frac{\pi}{2}u - qp';t\right) - f(q,p,\theta;t)\right] m_\hbar(dp'du;t), \end{aligned} \tag{56}$$

which is the forward Kolmogorov equation.

If Eq. (56) is formally supplied by the initial condition

$$f(q,p,\theta;t) = \delta(q-q_0)\,\delta(p-p_0)\,\delta(\theta-\theta_0), \tag{57}$$

then, according to the theory of stochastic differential equations (SDE's), there exists a three-dimensional stochastic process

$$\xi_t \equiv (\Phi_t, \dot{\Phi}_t, \Theta_t), \tag{58}$$

for which the function $f(q,p,\theta;t)$ is a probability distribution. This means that for an arbitrary function $h(q,p,\theta)$, which is continuous and periodic with the period 2π on variable θ and for fixed θ belongs to the class $\mathbf{C}_b^{1,2}(\mathbf{R}^1 \otimes \mathbf{R}^1)$,

$$\int h(q,p,\theta) f(q,p,\theta;t)dq\,dp\,d\theta = \mathbf{E}h(\xi_t), \tag{59}$$

where the symbol $\mathbf{E}$ denotes mathematical expectation and

$$\xi_{t=0} = (q_0, p_0, \theta_0) \tag{60}$$

[see equality (57)].

Using the generalized Ito formula for stochastic differentials [see [27], p. 270, formula (13)], one deduces an SDE for the stochastic process ξ_t (58):

$$d\Phi_t = \dot{\Phi}_t dt, \tag{61}$$

$$d\dot{\Phi}_t = -[3H(t)\dot{\Phi}_t + \omega^2\Phi_t]dt + \left[\frac{2\hbar}{V}D(t)\right]^{1/2} dw_t - \frac{\hbar}{2V}a(t)^{-3}\int pu\nu_\hbar(dp\,du;dt), \tag{62}$$

$$d\Theta_t = \int \left[\frac{\pi}{2}u + \Phi_t p\right] \nu_\hbar(dp\,du;dt). \tag{63}$$

Here, w_t is a one-dimensional Wiener process (or Brownian motion) and $\nu_\hbar(dp\,du;dt)$ is a Poisson measure on $\mathbf{R}' \otimes (-1) \otimes (+1) \otimes [0,T]$ nonhomogeneous with respect to translation on $[0,T]$ such that

$$\mathbf{E}[\nu_\hbar(dp\,du;dt)] = m_\hbar(dp\,du;t)dt. \tag{64}$$

Θ_t is an additional stochastic variable, which can be interpreted as a stochastic phase [see Eq. (50)]. It appears only due to deviation of the potential (45) from the harmonic oscillator's potential.

In order for the stochastic process (61)–(63) to have a unique solution on a time interval $[0,T]$, right continuous with probability 1, it is enough to have conditions (46), and to assume that functions $a(t)$, $H(t)$, $a(t)^{-3}$, $D(t)$, and their first derivatives are continuous functions.

Let us denote the stochastic process (61)–(63) with initial condition (57) by

$$\begin{aligned} \xi_t(q_0,p_0,\theta_0) = (&\Phi_t(q_0,p_0,\theta_0), \dot{\Phi}_t(q_0,p_0,\theta_0), \\ &\times\Theta_t(q_0,p_0,\theta_0)). \end{aligned} \tag{65}$$

If the initial condition for Eq. (56),

$$f(q,p,\theta;0) = f_0(q,p,\theta)\ , \tag{66}$$

belongs to the class of generalized functions, then instead of (59), one has

$$\begin{aligned} &\int h\ (q,p,\theta) f(q,p,\theta;t)dq\,dp\,d\theta \\ &= \int \mathbf{E}h(\Phi_t(q,p,\theta), \dot{\Phi}_t(q,p,\theta), \Theta_t(q,p,\theta)) \\ &\quad\times f_0(q,p,\theta)dq\,dp\,d\theta. \end{aligned} \tag{67}$$

Now one can readily find the correspondence between an integral over the Wigner function $W(q,p;t)$, governed by Eq. (48), and the stochastic process (61)–(63). Let us assume that

$$f_0(q,p,\theta) = W_0(q,p)$$

and let the function $h(q,p,\theta)$ in Eq. (67) be of the form

$$h(q,p,\theta) = h(q,p)\exp\{-i\theta\}.$$

Then from Eq. (67) and (50), one finds

$$\int h\ (q,p)W(q,p;t)dq\,dp = \exp\left\{\frac{2V}{\hbar}A(t)\right\}\int \mathbf{E}\{h(\Phi_t(q,p,0),\dot{\Phi}_t(q,p,0)) \times \exp[-i\Theta_t(q,p,0)]\}W_0(q,p)dq\,dp, \tag{68}$$

where $\int d\theta = 2\pi$ was used and the function $A(t)$ is defined by Eq. (52).

Formula (68) gives one the expectation value of a quantum operator $\hat{A}(\hat{\Phi},\hat{\dot{\Phi}})$ with its Weyl symbol $h(q,p)$ [compare Eq. (68) with Eq. (25)].

Equations (61)–(63) are exact quantum Langevin equations for the large-scale inflation in de Sitter space associated with the master equation (17).

In Ref. [10] an attempt was made to derive the quantum Langevin equations for potentials with a polynomial growth higher than second order in q. As was mentioned, such potentials are excluded in our consideration, which is restricted by Eq. (45). In Ref. [10] the deduction of the quantum Langevin equations is based on a general expansion in powers of $\hbar$ of the Wigner equation, as in our representation (39). From Ref. [10] it follows that for polynomial potentials, of order higher than second, the quantum Langevin equations can be derived exactly only to order $\sqrt{\hbar/V}$ (corresponding to the truncated Wigner equation of order $\hbar/V$). Already, the first correction of order $\hbar/V$ to the quantum Langevin equations cannot be calculated precisely (there is no explicit representation for the noise terms in the quantum Langevin equations). The origin of this problem is related to the term $\partial^3W/\partial p^3$ in the truncated Wigner equation of order $(\hbar/V)^2$.

Let us rewrite the quantum Langevin equations (61)–(63) in an integral form to show explicitly dependence on initial data $(q,p,0)$:

$$\Phi_t(q,p,0) = q + \int_0^t \dot{\Phi}_\tau(q,p,0)d\tau, \tag{69}$$

$$\dot{\Phi}_t(q,p,0) = p - \int_0^t [3H(\tau)\dot{\Phi}_\tau(q,p,0) + \omega^2\Phi_\tau(q,p,0)]d\tau + \left[\frac{2\hbar}{V}\right]^{1/2}\int_0^t D(\tau)^{1/2}dw_\tau - \frac{\hbar}{2V}\int_0^t a(\tau)^{-3}p'u\nu_\hbar(dp'du;d\tau), \tag{70}$$

$$\Theta_t(q,p,0) = \left\{\int_0^t\int\left[\frac{\pi}{2}u + \Phi_\tau(q,p,0)p'\right]\nu_\hbar(dp'du;d\tau)\right\}\Big/\mathrm{mod}\,2\pi. \tag{71}$$

Equations (69)–(71) describe a stochastic path $\Phi_t,\dot{\Phi}_t$ in phase space $(\Phi,\dot{\Phi})$ starting at a point (q,p) when $t=0$. The stochastic phase Θ_t plays role in the final formula (68) for the expectation value of a quantum operator and can be interpreted as a contribution of the stochastic path $\Phi_t(q,p),\dot{\Phi}(q,p)$.

Equation (68) together with Eqs. (61)–(63) can be used for numerical simulations to calculate the expectation value for quantum operator $\hat{A}(\hat{\Phi},\hat{\dot{\Phi}})$, which is much simpler than solving the Wigner equation because SDE's (61)–(63) are of first order.

Now it is clear how to derive the quantum Langevin equations that are equivalent to the Wigner equation. In particular, we can discuss the contradiction inherent to the Wigner representation given by Nambu [15], mentioned earlier in the introduction.

The Wigner representation in Ref. [15] [Eq. (26) or (27)] is given for the space-homogeneous long wavelength mode of the scalar field ϕ_t. The result is obtained as a limit when some small parameter ϵ goes to zero. The stating point was the equations of motion. In the first equation of motion the small parameter can be extracted through

$$d\phi_t = v_tdt + \epsilon\gamma(dt),$$

where γ is some function [Eqs. (3) and (4) in [15]]. Thus, the Wigner equation in Ref. [15], derived for $\epsilon \to \infty$, should correspond to the equation of motion

$$d\phi_t = v_tdt. \tag{72}$$

On the other hand, the Wigner function $W(\phi,v;t)$ is governed by an equation containing second derivatives $\partial^2W/\partial\phi^2$ and $\partial^2W/\partial\phi\partial v$ [Eq. (27) in [15]]. Because of these derivatives, the Langevin equations which are equivalent to Nambu's Wigner representation, give the following correspondence between the phase-space variables ϕ and v:

$$d\phi_t = v_tdt + \sigma_{1j}dw_t^j, j=1,2, \tag{73}$$

where (w_t^1,w_2^2) is a two-dimensional Wiener process and $\{\sigma_{ij}\}$ is a 2×2 matrix with constant coefficients. $\{\sigma_{ij}\}$ does not depend on ϵ.

Equation (73) by itself would mean that Nambu's Wigner representation and ours are given for different phase spaces, ignoring the contradiction in the frame of Ref. [15]. Equality (73) contradicts Eq. (72).

V. SOLUTION OF THE WIGNER EQUATION

The aim of this section is to show how the Wigner function itself can be expressed by the expectation value with respect to a stochastic process in the extended phase space (q,p,θ). In Ref. [29] such an expression is found for the case

$$\frac{\partial \hat{W}}{\partial t} = \frac{1}{i\hbar}[\hat{H},\hat{W}], \quad H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 q^2 + U(q,p), \tag{74}$$

while in the case under consideration (26) and (15), the diffusion term and the expansion of the Universe play an essential role.

To find a solution for the Wigner equation (48) one should transform it to a backward Kolmogorov equation [27] (p. 300). This can be done by introduction of the function

$$f(q,p,\theta;t) = W(q,p;t)\exp\left\{-3\int_0^t H(\tau)d\tau - \frac{2V}{\hbar}A(t) + i\theta\right\}, \tag{75}$$

where $\theta \in \mathbf{R}^1/\text{mod}\,2\pi$, and $A(t)$ is the same function (52) as in Sec. IV.

The time evolution of f is governed now by the equation

$$\frac{\partial f(q,p,\theta;t)}{\partial t} = -p\frac{\partial f}{\partial q} + [\omega^2 q + 3H(t)p]\frac{\partial f}{\partial p} + \frac{\hbar}{V}D(t)\frac{\partial^2 f}{\partial p^2} + \int\left[f\left(q,p+\frac{\hbar}{2V}a(t)^{-3}p'u,\theta-\frac{\pi}{2}u - qp';t\right) - f(q,p,\theta;t)\right]m_\hbar(dp'du;t), \tag{76}$$

which would be a backward Kolmogorov equation for the backward time t', by setting $t = T - t'$, where $0 \le t \le T$. The measure $m_\hbar(dp\,du;t)$ is defined by Eq. (54).

If Eq. (76) is supplied by an initial condition

$$f(q,p,\theta;0) = f_0(q,p,\theta), \tag{77}$$

where f_0 is a continuous function, then the solution for Eq. (76) can be represented as

$$f(q,p,\theta;t) = \mathbf{E}f_0(Q_T(q,p,\theta;T-t),P_T(q,p,\theta;T-t),\Theta_T(q,p,\theta;T-t)), \quad 0 \le t \le T, \tag{78}$$

where the stochastic process can be found by applying the generalized Ito formula for stochastic differentials [27] (p. 270),

$$Q_s(q,p,\theta;T-t) = q - \int_{T-t}^{s} P_\tau(q,p,\theta;T-t)d\tau, \tag{79}$$

$$P_s(q,p,\theta;T-t) = p + \int_{T-t}^{s}[3H(\tau)P_\tau(q,p,\theta;T-t) + \omega^2 Q_\tau(q,p,\theta;T-t)]d\tau + \left[\frac{2\hbar}{V}\right]^{1/2}\int_{T-t}^{s} D(\tau)^{1/2}dw_\tau + \frac{\hbar}{2V}\int_{T-t}^{s}\int a(\tau)^{-3}p'u\nu_\hbar(dp'du;d\tau), \tag{80}$$

$$\Theta_s(q,p,\theta;T-t) = \left\{\theta - \int_{T-t}^{s}\int\left[\frac{\pi}{2}u + Q_\tau(q,p,\theta;T-t)p'\right]\nu_\hbar(dp'du;d\tau)\right\}\Big/\text{mod}\,2\pi, \tag{81}$$

where $0 \le T-t \le s \le T$ and $\nu_\hbar(dp'du;d\tau)$ is a Poisson measure on $\mathbf{R}^1 \otimes (-1) \otimes (+1) \otimes [0,T]$ with the intensity defined by Eq. (64).

Returning to the Wigner function, one has

$$W(q,p;t) = \exp\left\{3\int_0^t H(\tau)d\tau + \frac{2V}{\hbar}A(t)\right\}\mathbf{E}\{W_0(Q_T(q,p,0;T-t),P_T(q,p,0;T-t))\exp[i\Theta_T(q,p,0;T-t)]\}, \tag{82}$$

where the stochastic process is defined by Eqs. (79)–(81) with $\theta = 0$.

Conditions for existence and uniqueness of the solution for SDE's (79)–(81) are the same as for SDE's (69)–(71).

Let us assume additionally that

$$\int p^k \mu(dp) \le \text{const}, \qquad k = 3,4, \tag{83}$$

and that the initial Wigner function $W_0(q,p)$ is twice continuously differentiable in p and once in x, and that its first and second order partials are bounded. Then the Wigner function (82) is twice continuously differentiable in p and once in x, differentiable in t, and is the unique solution for the Cauchy problem (76), (77).

Let us consider how properties (42)–(44) for the initial Wigner function are preserved in the course of time.

(a) The reality of the Wigner function is preserved. To prove this proposition, it is necessary to use the decomposition of the Wigner function into a difference of two positive functions [see Ref. [29] formula (4.4)]. The group of transformation (82) preserves this decomposition.

(b) The normalization for the Wigner function

$$\int W(q,p;t)dq\,dp = 1, \qquad t \geq 0,$$

is satisfied because of the equality

$$\frac{d}{dt}\int W(q,p;t)dq\,dp = 0,$$

which follows directly from the Wigner equation.

(c) The presence of the dissipation and diffusion terms in the Wigner equation destroys the inequality (44). It now becomes

$$\int W^2(q,p;t)dq\,dp \leq \int W_0^2(q,p)dq\,dp\exp\left\{3\int_0^t H(\tau)d\tau\right\} \leq \frac{V}{2\pi\hbar}a(t)^3. \tag{84}$$

To derive Eq. (84), it is necessary to take a time derivative of the expression on the left-hand side of Eq. (84) and to use the Wigner equation. After this, one has

$$\int W^2(q,p;t)dq\,dp = \left[\frac{a(t)}{a(0)}\right]^3\left\{\int W_0^2(q,p)dq\,dp - \frac{2\hbar}{V}\int_0^t\left[\frac{a(0)}{a(\tau)}\right]^3 D(\tau)\left[\int\left(\frac{\partial W(q,p;\tau)}{\partial p}\right)^2 dq\,dp\right]d\tau\right\}, \tag{85}$$

and Eq. (84) follows from Eq. (85).

If there is no diffusion, $D(t)\equiv 0$, for such a system, starting at $t=0$ from a pure state

$$\int [W_0(q,p)]^2 dq\,dp = \frac{V}{2\pi\hbar}a(0)^3 \quad \left(< \frac{V}{2\pi\hbar}a(0)^3 \text{ for a mixed state}\right),$$

it is possible to follow the pure state in the course of time because of equality

$$\int W^2(q,p;t)dq\,dp = \frac{V}{2\pi\hbar}a(t)^3 \quad \left(< \frac{V}{2\pi\hbar}a(t)^3 \text{ for a mixed state}\right). \tag{86}$$

However, diffusion $[D(t)\not\equiv 0]$ smears the picture and one cannot disinguish pure and mixed states by inequality (84).

For the Wigner function $W(q,\Pi;t)$ [see Eq. (34)] relations (85), (84) become

$$\int W^2(q,\Pi;t)dq\,d\Pi = \int W_0^2(q,\Pi)dq\,d\Pi - \frac{2\hbar}{V}\int_0^t a^6(\tau)D(\tau)\left[\int\left(\frac{\partial W(q,\Pi;\tau)}{\partial \Pi}\right)^2 dq\,d\Pi\right]d\tau, \tag{87}$$

$$\int W^2(q,\Pi;t)dq\,d\Pi \leq \frac{V}{2\pi\hbar}. \tag{88}$$

VI. LARGE-TIME ASYMPTOTICS

If $H(t)\approx$ const, the scale factor $a(t)$ increases exponentially with time and for large time the Wigner equation becomes

$$\frac{\partial W(q,p;t)}{\partial t} = -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\partial U(q)}{\partial q} \times\frac{\partial W}{\partial p} + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2} \tag{89}$$

to order $a(t)^{-3}$, what follows from Eq. (39).

The expectation value at large time T for an operator $\hat{A}$ with Weyl symbol $h(q,p)$ is

$$\begin{aligned}\langle\hat{A}\rangle_T &= \int h(q,p)W(q,p;T)dq\,dp \\ &= \int \mathbf{E}\,h(\Phi_T(q,p,0),\dot{\Phi}_T(q,p,0))\,W_0(q,p)dq\,dp,\end{aligned} \tag{90}$$

where the stochastic process $(\Phi_t,\dot{\Phi}_t)$ is governed now by

$$\begin{aligned} d\Phi_t &= \dot{\Phi}_t dt, \\ d\dot{\Phi}_t &= -[3H(t)\dot{\Phi}_t + \dot{U}(\Phi_t)]dt + \left[\frac{2\hbar}{V}D(t)\right]^{1/2} dw_t\,, \end{aligned} \tag{91}$$

with the initial condition $(\Phi_0,\dot{\Phi}_0)=(q,p)$.

Note that the stochastic differentials (89) are equivalent to the quantum Langevin equations (61)–(63) for the scalar field $\Phi(t)$ in the large-time limit [not for the beginning of the inflation if the potential $U(q)$ satisffies the inequality (40)].

VII. STATIONARY STATES FOR LARGE-SCALE INFLATION

Stationary or equilibrium states for large-scale inflation are described by the stationary Wigner function $W(q,p) = \lim_{t\to\infty} W(q,p;t)$ governed by the Wigner equation

$$0 = -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\partial U(q)}{\partial q}\frac{\partial W}{\partial p} + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2} \tag{92}$$

for an arbitrary potential $U(q)$. This equation has time-dependent coefficients coming from the time dependence of the scale factor.

For the stationary Wigner function $W(q,p)$, one can easily find the following expectation values for the operators:

$$\langle\hat{\dot{\Phi}}\rangle = 0, \tag{93}$$

$$\langle\hat{U}(\hat{\Phi})\rangle = 0, \tag{94}$$

$$\langle\hat{\dot{\Phi}}^2\rangle = \langle\hat{\Phi}\hat{U}(\hat{\Phi})\rangle = \frac{\hbar D(t)}{V\cdot 3H(t)}. \tag{95}$$

From the last equality, one can see that the diffusion coefficient $D(t)$ should be

$$D(t) = 3H(t)\,\sigma \tag{96}$$

(at least for large time), where the constant σ is defined by the choice of stationary state

$$\langle\hat{\dot{\Phi}}^2\rangle = \frac{\hbar}{V}\sigma. \tag{97}$$

The stationary Wigner function is found to be

$$W(q,p) = N\exp\left\{-\left(\frac{p^2}{2} + U(q)\right)V/(\hbar\sigma)\right\}, \tag{98}$$

where N is the normalization constant.

For a particular case, with the scale factor given by Eq. (10) and potential $U(q) = \omega^2 q^2/2$, the Bunch-Davies vacuum is given by

$$\langle\hat{\Phi}^2\rangle = \frac{3H^4\hbar}{8\pi^2\omega^2} = \frac{\hbar H}{V2\pi\omega^2}\ , \tag{99}$$

which yields $\sigma = H/2\pi$ and $D = 3H^2/2\pi$.

For this case, the stationary solution is

$$W(q,p) = \frac{V\omega}{\hbar H}\exp\left\{-(p^2+\omega^2q^2)\frac{V\pi}{\hbar H}\right\},\quad V = \frac{4\pi}{3H^3}, \tag{100}$$

where one must assume that

$$H \geq \omega/\pi \tag{101}$$

for the inequality (44) to be satisfied by the Wigner function (100).

VIII. CONCLUSIONS

(1) The appearance of the dissipation term $3H(t)(\partial/\partial p)(pW(q,p;t))$ in the Wigner equation, after transition from phase space (Φ,Π) to $(\Phi,\dot{\Phi})$, supports the earlier result of Graziani [9] and Nakao, Nambu, and Sasaki [13], that the large-scale inflation scalar field behaves as a quantum one-dimensional dissipative system. Nevertheless, this analogy is not complete: it is destroyed by a new commutation relation

$$[\hat{\Phi},\hat{\dot{\Phi}}] = \frac{i\hbar}{V}a(t)^{-3}, \tag{102}$$

where $a(t)$ is the scale factor in de Sitter metric and it reflects the expansion of the Universe [see Eq. (4)].

Comparing the Wigner equation (48) for the large-scale inflation [or more general case (36) and (37)] with the Wigner equation for a quantum one-dimensional linearly damped unharmonic oscillator (see Refs. [14,17,18]), one can see that the expansion of the Universe amplifies the role of the potential term, which is a deviation from the harmonic oscillator potential. Explicitly, this amplification appears in formula (68) for the expectation value of a quantum operator and in formula (82) for the Wigner function, where function $A(t)$ (52) has the factor $\int_0^t a(\tau)^3 d\tau$ (without expansion this factor would be 1). At the same time, as $t\to\infty$, the commutation relation (103) leads to degeneration of the jump process, in the quantum Langevin equations, into a continuous process (see Sec. VI).

(2) As a consequence of our investigation we have the following statement: for the large-scale inflation scalar field, the asymptotic $t\to\infty$ is equal to the classical limit.

In the limit $\hbar\to 0$, the Wigner equations (36) and (37) [or Eq. (48)] turns out to be

$$\frac{\partial W(q,p;t)}{\partial t} = -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\partial U(q)}{\partial q}\frac{\partial W}{\partial p}, \tag{103}$$

and the corresponding Langevin equations are

$$d\Phi_t = \dot{\Phi}_t dt,$$
$$d\dot{\Phi}_t = -[3H(t)\dot{\Phi}_t + U'(\Phi_t)]dt. \tag{104}$$

Equations (104) are just the classical deterministic equations of motion, equivalent to the field equation (11).

However, Eq. (103) is not the "classical limit" for the Wigner equations (36) and (37) (only the truncated Wigner equation of order $\hbar_0$). In the classical limit, the diffusion term in the Wigner equations (36) and (37) is not proportional to $\hbar/V$ because, instead of Eq. (97) for a stationary state, one has

$$\langle\hat{\dot{\Phi}}^2\rangle = \sigma_{\rm cl}. \tag{105}$$

The "classical limit" of the Wigner equation for the large-scale inflation scalar field is

$$\frac{\partial W(q,p;t)}{\partial t} = -p\frac{\partial W}{\partial q} + 3H(t)\frac{\partial}{\partial p}(pW) + \frac{\partial U(q)}{\partial q}\frac{\partial W}{\partial p} + D_{\rm cl}(t)\frac{\partial^2 W}{\partial p^2}\ , \qquad (106)$$

with $D_{\rm cl}(t) = 3H(t)\sigma_{\rm cl}$. The classical stochastic Langevin equations are

$$d\Phi_t = \dot{\Phi}_t dt,$$
$$d\dot{\Phi}_t = -[3H(t)\dot{\Phi}_t + U'(\Phi_t)]dt + [2D_{\rm cl}(t)]^{1/2} dw_t. \qquad (107)$$

One can see that in the "classical limit", the Wigner equation and Langevin equations [(106) and (107)] are the same as those of the large-time asymptotics [Eqs. (89) and (91)]. For complete coincidence, the coefficient of the second derivative in the Wigner equations (89) and (106) should be presented in the form $3H(t)\langle\dot{\Phi}^2\rangle_{\rm st}$, where the expectation value $\langle\dot{\Phi}^2\rangle_{\rm st}$ is taken on the corresponding stationary state.

(3) In this paper, each large-scale region (for the coarse-graining procedure) is considered as an independent quantum-mechanical system. If, nevertheless, it is necessary to take into account interaction with the environment, then, for linear interaction, the master equation for the "reduced" density operator $\hat{\rho}$ is

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{V}{i\hbar}[\hat{\mathcal{H}},\hat{\rho}] + \frac{\lambda(t)V}{4i\hbar}[\{\hat{\Pi},\hat{\Phi}\},\hat{\rho}] + \frac{\lambda(t)V}{2i\hbar}([\hat{\Phi},\hat{\rho}\,\hat{\Pi}] - [\hat{\Pi},\hat{\rho}\,\hat{\Phi}]) - \frac{V}{\hbar}a(t)^6 D(t)[\hat{\Phi},[\hat{\Phi},\hat{\rho}]], \qquad (108)$$

where $\lambda(t)$ is the dissipation coefficient (originated by interaction with the environment), $\hat{\mathcal{H}}$ is an operator form for the Hamiltonian (15) of the system without dissipation, and $\{\,,\,\}$ stands for an anticommutator.

In phase space $(\Phi,\dot{\Phi})$, the Wigner equation corresponding to Eq. (108) is

$$\frac{\partial W(q,p;t)}{\partial t} = -p\frac{\partial W}{\partial q} + [3H(t)+\lambda(t)]\frac{\partial}{\partial p}(pW) + \frac{\hbar}{V}D(t)\frac{\partial^2 W}{\partial p^2} + \frac{V}{i\hbar}a(t)^3\int_{-\infty}^{\infty} du\, W \times(q,p-u;t)\mathcal{I}(q,u;t), \qquad (109)$$

where $\mathcal{I}(q,u;t)$ is defined by Eq. (37).

Equation (109) is an equation of the same type as the Wigner equation (36). Thus, Eq. (109) can be treated as the Wigner equation considered in this paper.

(4) Since in our consideration on phase space $(\Phi,\dot{\Phi})$, the scaling $\hbar a(t)^{-3}/V$ depends already on t, it is not a problem to take into account a time dependence of the coarse-graining volume $V(t)$. The scaling becomes $\hbar a(t)^{-3}/V(t)$, where $V(0) = \text{const} \neq 0$ is assumed.

Therefore the result can be extended on an expanding FRW space-time. The principal point in this case is the new time dependence of the canonical momentum conjugate to $\Phi_X(t)$,

$$\Pi_\Omega(t) = V(t)a(t)^3\dot{\Phi}_X(t)\ , \qquad (110)$$

and of the Hamiltonian,

$$\mathcal{H}_\Omega = \tfrac{1}{2}V(t)^{-1}a(t)^{-3}\Pi_\Omega^2 + V(t)a(t)^3 U(\Phi_X) \qquad (111)$$

[compare with Eqs. (20) and (23)]. Also the condition (24), taken for "approximately" de Sitter space, should be replaced by a condition corresponding to each concrete FRW space-time.

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