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#### **S duality at the black hole threshold in gravitational collapse**

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We study gravitational collapse of the axion-dilaton field in classical low energy string theory, at the threshold for black hole formation. A new critical solution is derived that is spherically symmetric and continuously self-similar. The universal scaling and echoing behavior discovered by Choptuik in gravitational collapse appears in a somewhat different form. In particular, echoing takes the form of  $SL(2, \mathbb{R})$  rotations (cf., S duality). The collapse leaves behind an outgoing pulse of axion-dilaton radiation, with nearly but not exactly flat spacetime within it.

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The striking numerical results of Choptuik [1] on the spherically symmetric gravitational collapse of a real scalar field  $\phi$  have inspired an upsurge of interest in gravitational collapse just at the threshold for formation of black holes. Further numerical results for vacuum relativity with axial symmetry [2] and perfect fluids [3–5] suggest that the phenomena discovered by Choptuik are not just restricted to spherically symmetric real scalar fields.

The thought experiment employed by Choptuik, in the context of numerical relativity, is to “tune” across the critical threshold in the space of initial conditions that separates the non-black-hole end state from the black hole end state, and to carefully study the critical behavior of various quantities at this threshold [1]. Two distinct kinds of scaling appear. First, just above the black hole threshold, the black hole

mass is  $M_{\text{BH}}(p) \propto (p - p^*)^\gamma$ ,  $\gamma \approx 0.37$ , where  $p$  is some tuning parameter of the initial conditions.

Second, exactly at the threshold, there appears a unique critical solution acting as an attractor for all nearby initial conditions on threshold. This critical solution, which we will call a “Choptuon,” exhibits a striking, recurrent “echoing” behavior: Asymptotically at small time scales  $t$  and length scales  $r$  near the collapse, it repeats itself at ever-decreasing scales  $t' = e^{-n\Delta}t$ ,  $r' = e^{-n\Delta}r$  (for  $n \in \mathbb{Z}_+$ ). Here  $\Delta$  is either a fixed constant of the solution ( $\Delta \approx \ln 30$  for a real scalar field [1],  $\Delta \approx \ln 5$  for vacuum gravity [2]) demonstrating a discrete self-similarity; or an arbitrary constant, demonstrating a continuous self-similarity [3,6]. The Choptuon is itself a smooth solution of the field equations to the past of the asymptotic point  $(t, r) = (0, 0)$ , but clearly has some kind of spacetime singularity visible from infinity at that point; for instance, the Riemann curvature diverges at that point.

Choptuons represent in principle a means by which effects of Planck scale gravity, even quantum gravity or superstring theory, might be observable in the present universe.

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Therefore, the quantum gravitational and stringy generalizations of the Choptuon demand study [7].

In this Rapid Communication we report on threshold behavior of gravitational collapse, and Choptuik scaling, in classical (3+1)-dimensional low-energy effective string theory: i.e., in general relativity coupled to a dilaton  $\Phi$  and an axion  $a$ . The fields  $a$  and  $\Phi$  can be combined into a single complex field  $\tau \equiv a + ie^{-2\Phi}$ , and then the effective action for the model is (omitting gauge fields<sup>1</sup>)

$$S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} \frac{\partial_a \tau \partial^a \bar{\tau}}{(\text{Im}\tau)^2} \right). \quad (1)$$

The equations of motion from this action are

$$R_{ab} - \frac{1}{4(\text{Im}\tau)^2} (\partial_a \tau \partial_b \bar{\tau} + \partial_a \bar{\tau} \partial_b \tau) = 0, \quad (2a)$$

$$\nabla^a \nabla_a \tau + \frac{i \nabla^a \tau \nabla_a \bar{\tau}}{\text{Im}\tau} = 0. \quad (2b)$$

Because the model (2) arises from an underlying theory of quantum gravity, it should be an excellent starting point for investigating quantum effects.

We assume spherical symmetry. We also assume continuous self-similarity, which means we work exactly at the threshold for black hole formation, and construct a critical solution, as in [3,6]. Restricting to  $a=0$  in (2) reduces to the model studied by Choptuik, which has discrete self-similarity. Generic initial conditions for  $a$  will presumably lead at threshold either to discrete or to continuous self-similarity; further numerical work is needed to decide.

The model (2) has an extra global symmetry not present in general relativity, an  $\text{SL}(2, \mathbb{R})$  symmetry that acts on  $\Phi$  and  $a$ , but leaves the spacetime metric invariant; this is a classical version of the conjectured  $\text{SL}(2, \mathbb{Z})$  symmetry of heterotic string theory called  $S$  duality [8]. It acts on  $\tau$  as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad (3)$$

where  $(a, b, c, d) \in \mathbb{R}$  with  $ad - bc = 1$ , while leaving  $g_{\mu\nu}$  invariant.

The main results of this Rapid Communication are the following: (1) Choptuik echoing can and does occur in this model, taking the novel form of these  $\text{SL}(2, \mathbb{R})$  rotations associated with  $S$  duality; (2) a critical solution exists that is continuously self-similar, as in [3,6]; (3) the global structure of the critical spacetime is determined.

By the assumption of continuous self-similarity, there exists a homothetic Killing vector  $\xi$  that generates global scale transformations

$$\mathcal{L}_\xi g_{ab} = 2g_{ab}. \quad (4)$$

How should  $\tau$  transform under  $\xi$ ? The naive answer (from dimensional analysis) is that  $\tau$  should be invariant. However, quite generally, one must allow for spacetime symmetries to

<sup>1</sup>One might think that the Choptuon could carry gauge charge as well as mass; however, this appears not to be so.

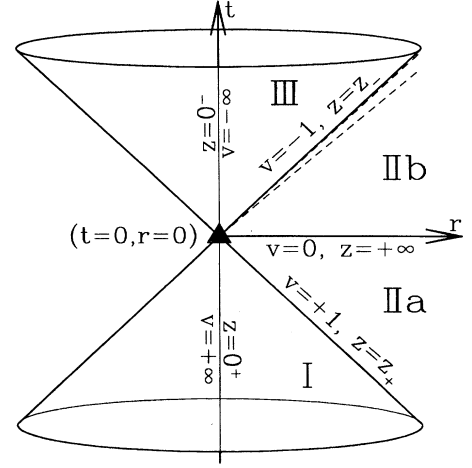


FIG. 1. Interpretation of the axion-dilaton Choptuon. There are horizons at  $v=+1$  (characteristic hypersurface) and  $v=-1$  (Cauchy horizon). There is a spacetime singularity at  $(t=0, r=0)$  which is a single point in the causal structure. Region I is a collapsing sphere of gravitationally bound field; in regions IIa and IIb this blends smoothly into an outgoing wave. The outgoing wave oscillates infinitely many times approaching the Cauchy horizon at  $v=-1$ . The dashed lines indicate the first two oscillations. Region III is nearly but not precisely flat.

mix with internal symmetries. For instance, the Maxwell equations for the potential  $A_\mu$  are conformally invariant only in the sense that a gauge transformation be allowed to accompany each conformal transformation. In our case, we must allow for  $\tau$  to be scale invariant up to an infinitesimal  $\text{SL}(2, \mathbb{R})$  transformation

$$\mathcal{L}_\xi \tau = \alpha_0 + \alpha_1 \tau + \alpha_2 \tau^2, \quad (5)$$

with  $\alpha_i \in \mathbb{R}$ , and this will be crucial as it allows echoing to occur.

In spherical symmetry the metric can be taken as

$$ds^2 = [1 + u(t, r)] [-b(t, r)^2 dt^2 + dr^2] + r^2 d\Omega^2. \quad (6)$$

The time coordinate is chosen so that gravitational collapse on the axis of spherical symmetry first occurs at  $t=0$ , and the metric is regular for  $t < 0$ . This metric remains invariant in form under transformations  $t = t(t')$  of the time coordinate, and this invariance can be used to set  $b(t, 0) = 1$  for  $t < 0$  on the axis. By regularity (no cone singularity),  $u(t, 0) = 0$  on the axis for  $t < 0$  as well. Initial conditions, assumed smooth, may be given on the hypersurface  $t = -1$ .

In these coordinates  $\xi = t\partial_t + r\partial_r$ . Continuous self-similarity (4) is then implemented by defining a scale invariant coordinate  $z \equiv -r/t$ . In terms of  $z$ ,

$$b(t, r) = b(z), \quad \text{and} \quad u(t, r) = u(z), \quad (7)$$

where the real functions  $b(z)$ ,  $u(z)$  are functions solely of the scale invariant coordinate  $z$ . From Eq. (5), the  $\tau$  field must satisfy

$$\tau(t,r) = i \frac{1 - (-t)^{i\omega} f(z)}{1 + (-t)^{i\omega} f(z)}, \quad (8)$$

where  $\omega$  is a real constant, so far arbitrary, and where the complex function  $f(z)$  depends solely on  $z$ .<sup>2</sup>

We can now use Eqs. (7) and (8) in the equations of motion (2). After some work,  $u(z)$  drops out, giving the system of ordinary differential equations (ODE's)

$$0 = b' + \frac{z(b^2 - z^2)}{b(1 - |f|^2)^2} f' \bar{f}' - \frac{i\omega(b^2 - z^2)}{b(1 - |f|^2)^2} (f\bar{f}' - \bar{f}f') - \frac{\omega^2 z |f|^2}{b(1 - |f|^2)^2}, \quad (9a)$$

$$\begin{aligned} 0 = f'' - \frac{z(b^2 + z^2)}{b^2(1 - |f|^2)^2} f' \bar{f}' + \frac{2}{(1 - |f|^2)} \left( 1 - \frac{i\omega(b^2 + z^2)}{2b^2(1 - |f|^2)} \right) \bar{f} f'^2 + \frac{i\omega(b^2 + 2z^2)}{b^2(1 - |f|^2)^2} f f' \bar{f}' \\ + \frac{2}{z} \left( 1 + \frac{i\omega z^2(1 + |f|^2)}{(b^2 - z^2)(1 - |f|^2)} + \frac{\omega^2 z^4 |f|^2}{b^2(b^2 - z^2)(1 - |f|^2)^2} \right) f' + \frac{\omega^2 z}{b^2(1 - |f|^2)^2} f^2 \bar{f}' \\ + \frac{i\omega}{(b^2 - z^2)} \left( 1 - \frac{i\omega(1 + |f|^2)}{(1 - |f|^2)} - \frac{\omega^2 z^2 |f|^2}{b^2(1 - |f|^2)^2} \right) f. \end{aligned} \quad (9b)$$

This system apparently has five singular points, as follows. First, the points  $z = \pm 0$  just represent the usual axis  $r = 0$  of spherical symmetry, and regularity is easily imposed.

The point  $z = \infty$  corresponds to the hypersurface  $t = 0$ . However, spacetime should be smooth on this hypersurface, except at the axis, since it lies in the Cauchy development of the initial conditions. Indeed, the system (9) can be rendered regular at  $z = \infty$  by the following change of variables:

$$dw = b(z) \frac{dz}{z^2}, \quad w = 0 \text{ at } z = \infty, \quad (10a)$$

$$F(w) = z^{-i\omega} f(z), \quad (10b)$$

$$v(w) = \frac{b(z)}{z}, \quad (10c)$$

$$u(w) = u(z). \quad (10d)$$

In terms of the new independent variable  $w$ , the equations of motion become

$$0 = v' + \frac{v^2 - 1}{(1 - |F|^2)^2} F' \bar{F}' + 1 - \frac{\omega^2 |F|^2}{(1 - |F|^2)^2}, \quad (11a)$$

$$\begin{aligned} 0 = F'' - \frac{2vF' + i\omega F}{(1 - |F|^2)^2} F' \bar{F}' + \frac{2\bar{F}F'^2}{1 - |F|^2} \\ + \frac{2i\omega v}{(v^2 - 1)(1 - |F|^2)} \left( 1 + |F|^2 - \frac{i\omega |F|^2}{1 - |F|^2} \right) F' \\ + \frac{i\omega}{v^2 - 1} \left( 1 + \frac{i\omega(1 + |F|^2)}{1 - |F|^2} + \frac{\omega^2 |F|^2}{(1 - |F|^2)^2} \right) F. \end{aligned} \quad (11b)$$

Finally, there are singular points wherever  $b(z) = \pm z$  [or  $v(z) = \pm 1$ ], at which the homothetic Killing vector becomes null; such characteristic hypersurfaces always occur in similarity solutions to hyperbolic equations, and in this case describe light cones in spacetime, at which new data could be injected in the  $\tau$  field. The singular point  $b(z_+) = z_+$  [or

$v(w_+) = 1$ ] describes the past light cone of the spacetime singularity at  $(t,r) = (0,0)$ ; see Fig. 1. Since this cone is within the Cauchy development of the spacetime initial conditions, the solution must be smooth across it.

The singular point  $b(z_-) = -z_-$  or  $v(w_-) = -1$  describes the future light cone of the spacetime singularity; see Fig. 1. This cone is not within the Cauchy development of the spacetime initial conditions; in particular, it is the Cauchy horizon. Therefore the solution may not be smooth there, and we should not impose any boundary conditions beyond continuity of  $\tau$ . Observers on this cone will see data coming from the singularity, and indeed it is a very interesting question to ask what they will see. Evolution across a Cauchy horizon can never be unique, but we will show below that there exists a unique extension that is spherically symmetric, self-similar, and smooth on the future axis  $t > 0$ . We expect that a generic subcritical solution near the Choptuon will be similar to the extension through  $z_-$ , and a generic supercritical solution will form a singularity near  $z_-$ .

The main advantage of the self-similarity hypothesis is that it reduces the equations of motion (2) to ordinary differential equations, (9) and (11). Unfortunately, these ODE's are still too complicated to solve explicitly. Thus we use numerical methods to determine a solution.

We are looking for a solution that is well defined everywhere, and smooth everywhere except perhaps at the Cauchy horizon  $z_-$ . Define two real functions  $f_m(z)$  and  $f_a(z)$  such that  $f(z) = f_m(z) e^{if_a(z)}$ . At any regular point  $z_r$  of (9), we need to provide the initial conditions  $b(z_r)$ ,  $f_m(z_r)$ ,  $f'_m(z_r)$ , and  $f'_a(z_r)$  to begin numerical integration [Eq. (9) is independent of  $f_a(z_r)$ ]. At any singular point, we must take care to ensure that the fields (and their first derivatives) are

<sup>2</sup>We have adopted a global  $SL(2, \mathbb{R})$  gauge here. In principle,  $\omega$  can also be purely imaginary or zero, but these do not seem to give interesting solutions.

continuous. As mentioned before, rescaling  $t$  allows us to set  $b(z=0)=1$  when  $t<0$ . Requiring regularity at  $z=0$  then fixes  $f'_m(0)$  and  $f'_a(0)$  in terms of  $f_m(0)=|f(0)|$ . A Taylor expansion of the fields  $b(z)$  and  $f(z)$  around  $z=0$  then provides initial conditions for integrating outwards from  $z=\epsilon_0$ , a small number.

We can numerically integrate out from  $z=\epsilon_0$  until the system reaches the next singular point at  $z_+$  where  $b(z_+)=z_+$ . The system must be smooth at  $z_+$ , but this is numerically difficult to impose by just integrating out from  $z=\epsilon_0$ . So instead, we require that  $b$  and  $f$  be continuous at  $z_+$ , and perform another Taylor expansion around  $z_+$ , demanding smoothness. This provides initial conditions for integrating inwards from  $z=z_+-\epsilon_+$  in terms of  $|f(z_+)|$  and  $z_+$ . We then try to match the integration out from  $z=\epsilon_0$  to the integration in from  $z=z_+-\epsilon_+$  at an intermediate point, which we choose to be  $z=1$ . Each integration gives four final conditions, and there are four parameters to adjust,  $|f(0)|$ ,  $\omega$ ,  $z_+$ , and  $|f(z_+)|$ , so we have a well-determined system of equations. We find that if

$$\omega = 1.17695272200 \pm 0.00000000270, \quad (12a)$$

$$z_+ = 2.60909347510 \pm 0.00000000216, \quad (12b)$$

$$|f(0)| = 0.892555411872 \pm 0.00000000224, \quad (12c)$$

$$|f(z_+)| = 0.364210875022 \pm 0.000000000760, \quad (12d)$$

then the equations are smooth at  $z=0$  and  $z=z_+$ .

The uncertainties quoted in (12) represent one  $\sigma$  numerical uncertainties. A good understanding of the accuracy of the parameters is crucial especially near  $z_-$  where the spacetime turns out to be nearly, but not quite, flat. To determine the errors in (12), we have performed a large number of matchings using a powerful numerical integrator, starting each time with different initial estimates for the parameters, and with different values for  $\epsilon$  and the error tolerances for the integrator. All Taylor expansions are carried out to fourth order, and should not introduce any further uncertainties.

The above equations uniquely determine the system at  $z=z_+$ . We can therefore change to the  $w, v, F$  coordinate system (10) and integrate from  $w=w_+$  until the next singularity at  $w=w_-$  where  $v(w_-)=-1$ . This takes us through  $z=\infty$ ,  $t=0$  into the region where  $t>0$ . Near the Cauchy horizon at  $w_-$ ,  $F \rightarrow \text{const}$ , but  $F'$  oscillates rapidly. To see whether it is possible to continue through  $w_-$ , or whether it is a true singularity, we must look at the behavior of  $F$  in more detail. Near  $w_-$ , Eq. (11b) becomes

$$(w-w_-)F'' - c_0F' - c_1 = 0, \quad (13)$$

which has the solution

$$F(w) = c_2 + c_3(w-w_-)^{1+c_0} - \frac{c_1}{c_0}(w-w_-), \quad (14)$$

where the  $c_i$  are complex constants. Thus,  $F(w)$  will be continuous at  $w_-$  if  $\text{Re}c_0 \geq -1$ , and  $F'(w)$  will be continuous at  $w_-$  if  $\text{Re}c_0 \geq 0$ . From Eq. (11),

$$\text{Re}c_0 = \frac{\omega^2 |F(w_-)|^2}{(1-|F(w_-)|^2)^2 - \omega^2 |F(w_-)|^2} = \frac{u(w_-)}{1-u(w_-)}. \quad (15)$$

Since  $|F(w_-)|$  is small,  $\text{Re}c_0 \geq 0$ , and both  $F(w)$  and  $F'(w)$  are continuous at  $w_-$ . Numerically, we find that

$$w_- - w_+ = 2.29540547918 \pm 0.00000000192, \quad (16a)$$

$$|F(w_-)| = 0.007603132483 \pm 0.000000000125, \quad (16b)$$

$$u(w_-) = 0.0000800854142 \pm 0.000000000212. \quad (16c)$$

Since  $u$  gives the mass aspect  $m=ru/2(1+u)$ , the Cauchy horizon nearly, but not quite, carries data for flat spacetime. Because  $\text{Re}c_0$  is only slightly larger than zero, the function  $F(w)$  is just barely  $C^1$  at  $w_-$ .  $F''$  and all higher derivatives of  $F$  are discontinuous at  $w_-$ . This is physically acceptable as the spacetime only depends on  $F$  and  $F'$ .

The remaining region to consider is  $0 > z > z_-$ . Since only  $f$  and  $f'$  are continuous at  $z=z_-$ , we cannot repeat the same procedure that we used for  $0 < z < z_+$ . Instead, we must begin integration at  $z=-\epsilon_0$  and integrate out to  $z_-$ . Since numerically we only know  $|f(z_-)|$ , we can try to achieve that final condition in the integration. As before, rescaling  $t$  allows us to set  $b(0_-)=1$  and requiring regularity at  $z=0_-$  fixes  $f'_m(0)$  and  $f'_a(0)$  in terms of  $|f(0_-)|$ . This is a standard shooting problem with one free parameter and one target. We find that

$$z_- = -1.000037262799 \pm 0.000000000113, \quad (17a)$$

$$|f(0_-)| = 0.011742998497 \pm 0.000000000195. \quad (17b)$$

This completes the last part of the spacetime.

In conclusion, full scale numerical work on this model is highly desirable, for several reasons. The solution we have given is, by assumption, continuously self-similar. It is important to see if discretely self-similar solutions also exist, and if so which is the stronger attractor. Discretely self-similar solutions may manifest themselves by a form of Choptuik [1] "echoing" in which the system is invariant after a finite scale transformation by some scale factor  $\exp(-\Delta)$ , up to an  $\text{SL}(2, \mathbb{R})$  transformation.

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