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Cosmic colored black holes

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We present spherically symmetric static solutions (a particlelike solution and a black-hole solution) of the Einstein-Yang-Mills system with a cosmological constant. Although their gravitational structures are locally similar to those of the Bartnik-McKinnon particles or the colored black holes, the asymptotic behavior becomes quite different because of the existence of a cosmological horizon. We also discuss their stability by means of a catastrophe theory as well as a linear perturbation analysis and find the number of unstable modes.

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Self-gravitating structures with non-Abelian fields have been intensively studied from various points of view since Bartnik and McKinnon (BM) particle [1] and colored black-hole solutions [2] were found by the numerical method. Since such new types of solutions with non-Abelian hair show a variety of features and have many interesting properties, studying them may reveal new important aspects in general relativity such as the no hair conjecture or a stability analysis via a catastrophe theory [3,4]. Those objects may be very small even if they exist and they would be important only in the early history of the universe [5,6]. In the early universe, however, we usually expect a vacuum energy, which is equivalent to a cosmological constant. From an observational cosmological point of view, some astrophysicists have pointed out that a small cosmological constant may explain the observed number counts of galaxies [7]. We therefore have wondered why a self-gravitating non-Abelian structure with a cosmological constant has not been studied yet. Such non-Abelian structures might have been formed in the early universe and might have played an important role in cosmology. Hence it may be worthwhile studying such objects.

In this Rapid Communication we present both particlelike and black-hole non-Abelian solutions with a cosmological constant. Here we will consider only a localized object, such as the BM solution or the colored black hole, in the universe with cosmological constant. Then we assume the cosmological horizon, inside of which a localized object exists. Then the spacetime approaches the Schwarzschild–de Sitter solution or the Reissner–Nordström–de Sitter (RNdS) solution asymptotically near the cosmological horizon. Hence a new solution is a direct generalization of a RNdS solution from the U(1) gauge field to the SU(2) gauge field [8]. We study structures and thermodynamical properties of new solutions and also discuss their stability by means of a catastrophe theoretical method as well as a linear perturbation analysis.

We start with the action

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(R-2\Lambda)-\frac{1}{16\pi g^2}\mathrm{Tr}F^2\right], \quad (1)$$

where the SU(2) Yang-Mills (YM) field strength F is expressed by its potential A as $F=dA+A\wedge A$ and g is a coupling constant of the YM field.

Since we are interested in the case where the YM field is localized and the spacetime approaches asymptotically the de Sitter solution because of a cosmological constant Λ, the metric of the spherically symmetric static spacetime is written as

$$ds^2=-\left(1-\frac{2Gm(r)}{r}-\frac{\Lambda}{3}r^2\right)e^{-2\delta(r)}dt^2 +\left(1-\frac{2Gm(r)}{r}-\frac{\Lambda}{3}r^2\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \quad (2)$$

As for a SU(2) YM potential, we adopt the following form

$$A=w(r)\tau_1 d\theta+\{w(r)\tau_2+\cot\theta\tau_3\}\sin\theta d\phi, \quad (3)$$

which is the same as that of the colored black hole and is obtained from the most generic spherically symmetric one with the ansatz of being static and having no "electric" charge.

The field equations derived from (1) are

$$\tilde{m}'=\left(1-\frac{2\tilde{m}}{\tilde{r}}-\frac{1}{3}\tilde{r}^2\right)\frac{w'^2}{\alpha^2}+\frac{(1-w^2)^2}{2\alpha^2\tilde{r}^2}, \quad (4)$$

$$\delta'=-\frac{2w'^2}{\alpha^2\tilde{r}}, \quad (5)$$

$$\left[\left(1-\frac{2\tilde{m}}{\tilde{r}}-\frac{1}{3}\tilde{r}^2\right)e^{-\delta}w'\right]'-\frac{w(1-w^2)}{\tilde{r}^2}=0, \quad (6)$$

where we have used dimensionless variables normalized by Λ and G as $\tilde{r}\equiv\sqrt{\Lambda}r$, $\tilde{m}\equiv\sqrt{\Lambda}Gm$. A prime denotes a derivative with respect to $\tilde{r}$. $\alpha\equiv g/\sqrt{G\Lambda}$ is a normalized coupling constant, which is only one parameter appearing in those basic equations.

We consider two classes of solutions: One is a particlelike solution (first class) such as the BM solution and the other is a black-hole solution (second class) such as the colored black

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hole. Before showing our results, we have to discuss boundary conditions of the field functions. Because of the asymptotic structure of our spacetime, we expect a cosmological horizon. Hence we need two boundary conditions both at the origin ($r=0$) or at the black-hole horizon (r_h) and at the cosmological horizon (r_c). As for the metric functions $m(r)$ and $\delta(r)$, these must be finite in the whole region. We set $\delta=0$ at the cosmological horizon. For the YM field, a regularity at the origin requires $w=\pm 1$ and $w'=0$. We can choose $w=1$ without loss of generality. At the black-hole horizon, we use the same boundary condition as that of the colored black hole; i.e., $w'(r_h)$ is described by $w(r_h)$ from the condition for a regular horizon and $w(r_h)$ becomes a shooting parameter. At the cosmological horizon, although a value of w seems to be free in the present coordinate system, it must be analyzed more carefully. It is useful to use a new coordinate χ defined by $r=R\sin\chi$ ($\chi\in[0,\pi/2]$), where R is a radius of the cosmological horizon locating at $\chi=\pi/2$. The boundary condition at $\chi=\pi/2$ is expressed by $dw/d\chi=0$ because the spacetime approaches the RNdS spacetime asymptotically near the cosmological horizon. However, because the YM equation in terms of χ becomes singular at the cosmological horizon ($\chi=\pi/2$), we use the previous r coordinate and transform the r coordinate to the tortoise coordinate (r^*), defined by $dr/dr^*=1-2Gm/r-\Lambda r^2/3$, only near the cosmological horizon. It shifts the cosmological horizon away to infinity ($r^*\to\infty$) and makes it easy to impose the regularity at the cosmological horizon. Checking the boundary condition at the cosmological horizon in terms of the χ coordinate as well, we found new solutions numerically. We shall discuss a particlelike solution and a black hole separately.

(1) Particlelike solution. Under the above boundary conditions, we have a trivial analytic solutions: $w\equiv 1$, $m\equiv 0$, $\delta\equiv 0$, which is the de Sitter solution.

Nontrivial particlelike solutions we have found are some kind of extension of the BM solution. We show the profiles of the field functions in Fig. 1. In the region where the YM field is located, the new solution is very similar to the BM solution. We find a family of discrete solutions each of which is characterized by the node number i of the YM potential. The asymptotic behaviors, however, become quite different from those of the BM solution because of the existence of Λ. The YM field of the BM solution damps faster than $\sim r^{-2}$ and it has no global charge relating to the gauge field. This is the reason why we classified the BM solution into a globally neutral type in the previous paper [3]. In the case of the new solutions, however, the YM field does not vanish and continues to exist over the cosmological horizon. This produces an effective charge at $r=r_c$ defined by $Q_{\rm eff}=\int_{r_c}\sqrt{{\rm tr}F^2}r^2\sin\theta\, d\theta\, d\phi$; hence we expected that the spacetime approaches the RNdS spacetime asymptotically. We have also found that the effective charge gets large when the normalized coupling constant α becomes small, i.e., when g decreases or Λ increases.

A new family of solutions has a critical coupling constant $\alpha_{\rm cr}\sim 1.75$, below which no solution exists except for trivial ones. That is to say a nontrivial solution disappears when Λ gets large and/or g gets small. We can easily understand this as follows. For a general relativistic fluid with nonzero

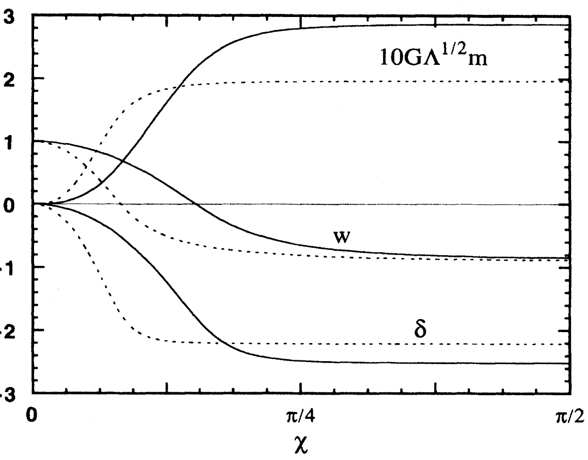

FIG. 1. The YM potential w and metric functions m, δ of particlelike solutions for $\alpha=g/\sqrt{G\Lambda}=2.5$ (solid lines) and $=4.0$ (dashed lines) in terms of χ. $\chi=\pi/2$ corresponds to the cosmological horizon. These solutions have one node in the half-sphere ($0\leq\chi\leq\pi/2$). The behaviors of the functions are similar to the BM solution near the origin. From this figure, we find that derivatives of each function with respect to χ vanish at the cosmological horizon and the spacetime has a reflection symmetry.

vacuum energy density $\rho_{\rm vac}$ (or a cosmological constant Λ), if $\rho_f/\rho_{\rm vac}\lesssim 2$, the perfect fluid cannot be localized as an isolated starlike object, where ρ_f is a fluid density [9]. For our new solution, we shall introduce the mean energy density of the YM field, $\bar\rho_{\rm YM}\equiv M/(\frac{4}{3}\pi r_0^3)$. r_0 is the effective radius at which $\rho_{\rm YM}$ of the YM field drops by half. Comparing $\bar\rho_{\rm YM}$ with $\rho_{\rm vac}=\Lambda/8\pi G$, we find that a new solution does not exist for the case of $\bar\rho_{\rm YM}/\rho_{\rm vac}\lesssim 5$. This result is consistent with the perfect fluid case and we expect that this property may be universal for any matter with a cosmological constant. The physical reason why there is the critical value $\alpha_{\rm cr}$ may be explained as follows: The size of the self-gravitating nontrivial structure is $r_0\sim\sqrt{G}/g$, while the radius of cosmological horizon is $r_c\sim(3\Lambda)^{1/2}$. Then, if $r_0>r_c$, i.e., $\alpha\lesssim 1$, no particlelike solution can exist in the de Sitter background spacetime.

(2) Black-hole solution. Now we turn to the black-hole solutions. It is easy to check that the Schwarzschild–de Sitter solution ($w\equiv\pm 1$) and the RNdS solution ($w\equiv 0$) are trivial solutions.

As for a nontrivial black-hole solution (we call it the cosmic colored black hole), we plot the mass-horizon radius relation in Fig. 2. Note that the mass of a black hole M is defined by

$$M=m(r_c)+\frac{Q_{\rm eff}^2}{2r_c}. \qquad (7)$$

The reason is as follows: A new solution has an effective charge at the cosmological horizon r_c and it approaches the RNdS spacetime asymptotically. Then the mass function m includes a contribution of a gauge field. It is plausible to subtract it in the definition of the mass of a black hole just as in the RNdS solution. Furthermore, this definition of M pro-

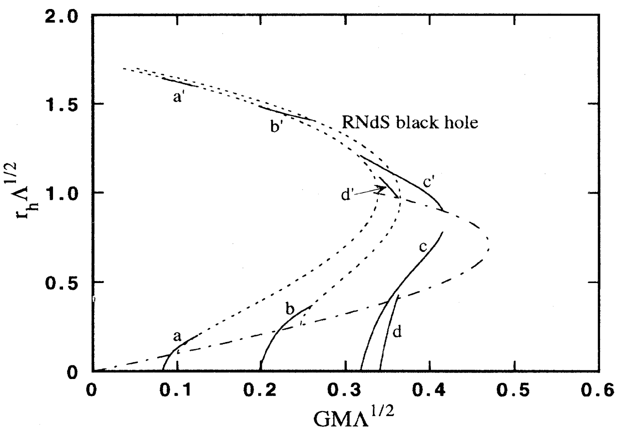

FIG. 2. The mass-horizon radius diagrams of the cosmic colored black holes for $\alpha \equiv g/\sqrt{G\Lambda}=10.0(a$[the black-hole horizon(BH)], a'[the cosmological horizon(CH)]), $=4.0[b$(BH), b'(CH)], $=2.2[c$(BH), c'(CH)], and $=1.9[d$(BH), d'(CH)]. We also plot those of the RNdS black holes for their charges $\sqrt{G\Lambda}Q$ = 0.1 and 0.25 (the dotted lines) and the extreme points of all RNdS black holes (the dot-dashed line). We find that the cosmic colored black holes for a,a' and b,b' coincide with the RNdS black holes with the same charge at the cosmological horizons, while the cosmological horizons for c' and d' finish on the extreme line.

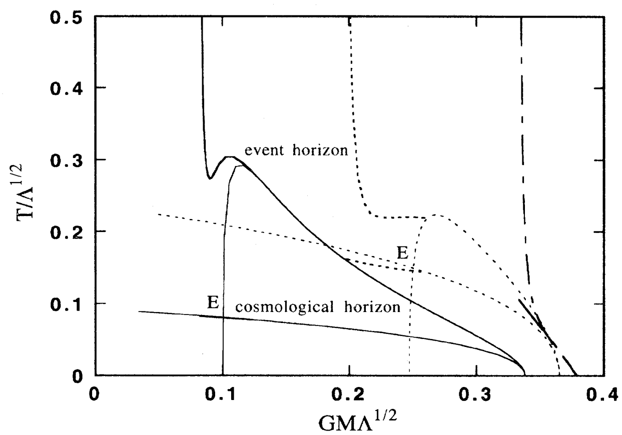

FIG. 3. The mass-temperature diagram of the cosmic colored black holes for $\alpha=10.0$ (solid lines), $=4.0$ (dotted lines), and $=2.0$ (dot-dashed lines). We also plot those for the RNdS black holes with the same charges. For the large coupling constant α (e.g., 10.0), the specific heat will change its sign two times, but there are no changes for small α (e.g., 2.0). The temperatures of both black hole and cosmological horizons of RNdS black holes intersect at the point E, but the cosmic colored black holes do not have such a point. This will change the fates of two types of black holes.

vides the conserved Abbott-Deser mass (AD mass) [10] for the RNdS black hole, although it is not certain whether this is true for the present cosmic colored black holes.

For the RNdS solution, there are in general three horizons: an inner horizon, a black-hole horizon, and a cosmological horizon (the dotted lines in Fig. 2). In Fig. 2, the solid lines denote the black hole and the cosmological horizons ($r_h<r_c$) of the cosmic colored black holes. In the limit of $r_h\rightarrow 0$, these branches end up with particlelike solutions. The behaviors of solutions depend on the coupling constant α. For a large coupling constant $\alpha>2.3$, the black-hole branch reaches that of RNdS solutions, and makes a bifurcation point. This behavior is similar to a family of monopole black holes in the Einstein-Yang-Mills-Higgs system [4], where a family of monopole black holes merges to the Reissner-Nordström black-hole branch. The monopole black hole has interesting properties depending on their self-coupling constant λ such that there are two types of nontrivial solutions for small λ, one of which is more stable than the other. One may expect that the cosmic colored black hole has similar properties, but this turns out to be forbidden by catastrophe theory. This is so because a family of monopole black holes constructs a swallow tail catastrophe, which needs at least three independent parameters, while the present system has only two parameters, i.e., α and a radius of the black-hole horizon.

For a small coupling constant $\alpha<2.3$, the cosmic colored black-hole solution is not bound up with the RNdS branch but disappears on the way. We can understand the reason for this by looking at the branch of the cosmological horizon. The end point of the branches merges to the extreme line of the RNdS black holes, where the black-hole horizon and cosmological horizon coincide with each other. When the mass of a cosmic colored black hole gets large, the radius of its cosmological horizon would become smaller than that of extreme RNdS black holes. However, although a cosmic colored black hole is different from the RNdS black hole, the asymptotic behavior should be the same. Hence it is likely for cosmic colored black holes to disappear at the extreme point of the RNdS black hole. When the coupling constant gets smaller even further ($\alpha<\alpha_{cr}\sim 1.75$), the cosmic colored black hole does not exist as the particlelike solution.

To discuss the thermodynamical properties of the cosmic colored black hole, we plot temperatures at both black hole and cosmological horizons of the cosmic colored black holes in Fig. 3. The temperature at the cosmological horizon has a similar mass dependence to that of the RNdS black-hole qualitatively, while there is a big difference for that at the black-hole horizon. In the RNdS black-hole case, when the mass of a black hole gets small, a sign of the heat capacity at the black hole horizon changes from negative to positive, and the temperatures at both horizons coincide at a point E in Fig. 3. In the limit of the extreme black hole, the temperature vanishes. On the other hand, for the cosmic colored black hole when the mass of the black hole gets small, its temperature diverges though there is some range where the sign of its heat capacity becomes positive. This behavior is similar to that of the colored black hole. This may be understood by the same mechanism explained in Ref. [11].

Those thermodynamical properties may allow us to discuss the evolution of the cosmic colored black holes. If there is initially a RNdS black hole whose mass is large enough, its mass gradually decreases via the Hawking radiation because the temperature at its black-hole horizon is higher than that at the cosmological horizon. At the point E in Fig. 3 where the two temperatures eventually become equal, however, the energy fluxes from both horizons balance and the

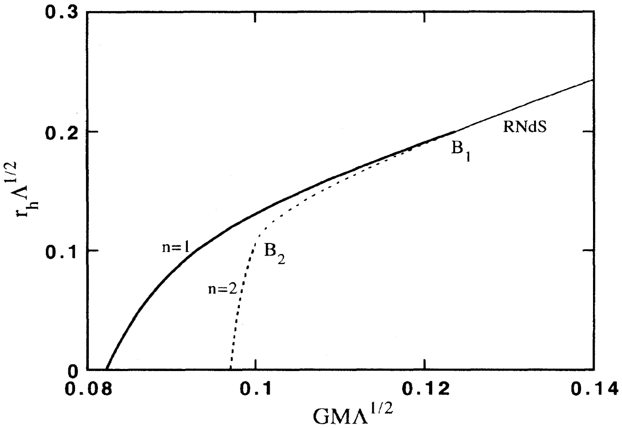

FIG. 4. The mass-horizon radius diagram of the cosmic colored black holes with node number $i=1,2$ and the RNdS black holes for $\alpha=10.0$. There are two bifurcation points B_i $(i=1,2)$, which correspond to the cusp catastrophes. The solid and the dotted lines have one and two unstable modes, respectively.

black hole does not evaporate further. On the other hand, this scenario cannot be applied to the SU(2) YM system, because of the existence of the cosmic colored black holes. The cosmic colored black hole has larger entropy than the RNdS black hole with the same mass. Hence, the RNdS black hole shifts to the cosmic colored black hole at the bifurcation point. After this, since there is no intersection of two temperature curves unlike in the RNdS case, the evaporation does not stop, but rather it will be accelerated, and finally a particlelike solution will remain.

Are the cosmic colored black holes stable or not? Here we use two methods in order to answer this question. One is the catastrophe theoretical method [12], which is useful in discussing a relative stability among several families of solutions and is widely applied in various research fields including astrophysics. The other is the usual linear perturbation method, with which we can find unstable modes explicitly and show the number of such modes.

First we show the former method. We choose the mass M and the entropy S of the black hole as a control parameter and a potential function in the catastrophe theory, respectively. The entropy S is related to a radius of the black-hole horizon as $S=\pi r_h^2$; hence, r_h is qualitatively equivalent to the entropy S. We show the $M-r_h$ relation for $\alpha=10.0$ in Fig. 4. B_i $(i=1,2)$ describe the bifurcation points consisting of branches of cosmic colored black holes with i nodes and of the RNdS black hole, and M_i $(i=1,2)$ are masses of black holes at these bifurcation points B_i. The structure at each bifurcation point in a plane spanned by the control parameter and the potential function is classified into a cusp catastrophe. Since the cosmic colored black hole has larger entropy than the RNdS black hole, the cosmic colored black hole is more stable than the RNdS black hole with the same mass by means of catastrophe theory. Provided that the RNdS black hole with $M>M_1$ has n unstable modes, it will find another unstable mode at the bifurcation point B_1 and then has $(n+1)$ unstable modes in the range of $M_2<M<M_1$. On the other hand, the cosmic colored black hole with one node has n unstable modes. Similarly, at the point B_2, the RNdS black hole will get another unstable mode while the cosmic colored black hole with two nodes has $(n+1)$ unstable modes, and so forth. The particlelike solution has the same number of unstable modes in its branch. Note that if $n=0$, i.e., the RNdS black hole with $M>M_1$ is stable, then the cosmic colored black hole with one node is also stable. Hence we only have to investigate the stability of the RNdS black hole. However it is impossible to study it by the catastrophe theory, and therefore we apply the linear perturbation method. Before showing our results it should be stressed that the RNdS black hole with a U(1) gauge field is stable.

Here we consider only radial perturbations. Writing down the perturbation equations and drawing their potential form, we can see that a cosmological constant has a tendency to stabilize the unperturbed solution. Analyzing the stability in detail numerically, however, we find that the RNdS solution with $M>M_1$ has one unstable mode. We have also confirmed that the number of unstable modes of the RNdS black hole increases one by one at the bifurcation points (B_i). Hence we conclude that the cosmic colored black hole with i nodes has i unstable modes [13]. This result does not depend on the coupling constant α.

In this paper we have investigated a particlelike solution with a cosmological constant and the cosmic colored black hole, which are the first self-gravitating non-Abelian structures with a cosmological constant. The gravitational structure is definitely changed by the cosmological constant; in particular, an effective charge appears at the cosmological horizon. Although the new solution is not stable, we may expect some important effects in the astrophysical process caused by the BM solutions and/or the colored black holes [6].

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