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Effective energy for (2+1)-dimensional QED with semilocalized static magnetic fields: A solvable model

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We evaluate the exact (2+1)-dimensional QED effective energy for charged spin-0 and spin-$\frac{1}{2}$ fields in the presence of a family of static magnetic field profiles localized in a strip of width λ. The exact result yields an infinite set of relations between the terms in the derivative expansion of the effective energy for a general magnetic field. Upon addition of the standard Maxwell magnetostatic energy, the minimum energy configuration at fixed flux corresponds to a uniform magnetic field.

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The effective action and the effective energy are important tools for the study of quantum electrodynamics [1,2]. Using the proper-time technique, Schwinger showed [1] that the QED effective action can be computed exactly for either a *constant* or a *plane wave* electromagnetic field. This result was later adapted to (2+1)-dimensional QED (QED_{2+1}) with constant fields by Redlich [3]. For more general electromagnetic fields one generally performs some sort of perturbative expansion, such as the derivative expansion [4] which yields the large distance behavior of the theory. This approach was recently applied to QED_{2+1} [5], showing, (for example) for the special case of zero electric field and static magnetic field, that while the zero derivative term increases the effective energy, the next order correction term with two derivatives tends to decrease the effective energy. The question of vacuum stability is inaccessible in a derivative expansion, so more powerful tools are required to study it. In this paper we make a first step towards the nonperturbative understanding of such a system by considering a new exactly solvable model which has a spatially varying magnetic field. We expect our model to be relevant for recent investigations of symmetry breaking [6,7] and finite temperature effects [8,9] in QED_{2+1}.

We show that the QED_{2+1} effective energy for charged spin-0 and spin-$\frac{1}{2}$ particles of arbitrary mass can be computed *exactly* in the presence of time-independent but spatially varying magnetic fields of the form

$$B(x,y)=\frac{B}{[\cosh(x/\lambda)]^2}. \tag{1}$$

Here, the magnetic field is localized in a strip of infinite extent in the y direction and of width λ in the x direction. In the limit $\lambda\to\infty$ the magnetic field in (1) tends to a uniform one of strength B. The constant strength B sets a length scale $1/\sqrt{eB}$ known as the magnetic length, and the derivative expansion regime corresponds to $\lambda \gg 1/\sqrt{eB}$. In practice, the system will be considered in a box in the y direction of size L. The total flux Φ of the magnetic field is then finite: $\Phi=eB\lambda L/\pi$. We show that the effective energy has a simple *exact* integral representation involving elementary functions for all values of mass m, width λ, electric charge e, and strength B.

The motivation for the choice of these particular profiles for the magnetic field resides in the fact that they may be fairly representative of the type of inhomogeneity that we expect in the system, while still being exactly solvable. They have been chosen here in the form of stringlike flux tubes with finite width and infinite length. While we cannot, of course, analyze exactly the dynamics for arbitrary magnetic field configurations, the exact solution will permit us to study the system nonperturbatively within this family of profiles and gain rigorous constraints on the derivative expansion for more general fields.

The starting point for the evaluation of the effective energy in the presence of the magnetic field (1) is the functional determinant expression of the effective action, which we quote in Minkowski space-time:

$$i\int d^3x\ \mathscr{L}_\pm=\mp\ln\ \mathrm{Det}\{D_\mu D^\mu+m^2+e\Sigma_\pm^{\mu\nu}F_{\mu\nu}-i\epsilon\}. \tag{2}$$

Here, $\mathscr{L}_\pm$ are the effective Lagrangians for bosons (+) and fermions (−) of electric charge e in the presence of a gauge potential A_μ with $D_\mu=\partial_\mu+ieA_\mu$. The Lagrangian $\mathscr{L}_-$ with $\Sigma_-^{\mu\nu}=(i/4)[\gamma^\mu,\gamma^\nu]$ produces the effective action for a four-component spinor consisting of two-component spinors of

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masses m and $-m$, respectively, whereas $\mathscr{L}_+$ with $\Sigma_+^{\mu\nu}=0$ produces the effective action for spin-0 complex scalars with mass m. Both systems are invariant under parity and time reversal. The determinants are understood to be regulated by Pauli-Villars masses in the ultraviolet, which will not be exhibited explicitly. Also, we calculate the effective action relative to that for zero electromagnetic fields and therefore drop all contributions independent of the fields. The effective energy for static magnetic fields is then given by $\mathscr{E}_\pm=-\int dxdy\ \mathscr{L}_\pm$.

For the family of magnetic fields in (1), we may use the translation invariance of the problem in time and in the y direction to work in an eigenbasis of frequency ω and y momentum k. Then the operator $D_\mu D^\mu+m^2+e\Sigma_\pm^{\mu\nu}F_{\mu\nu}$ coincides with the Schrödinger operator of a solvable one-dimensional quantum-mechanical system. (Recall that the solvability of the constant field case is based on its relation to the solvable one-dimensional harmonic oscillator system [10].) To see this, we choose the gauge potential $A_x=0$ and $A_y=\lambda B\tanh(x/\lambda)$, which reproduces the magnetic field in (1):

$$D_\mu D^\mu+m^2+e\Sigma_\pm^{\mu\nu}F_{\mu\nu}=-\frac{d^2}{dx^2}+V_k(x)-\omega^2, \qquad (3a)$$

$$V_k(x)=-\frac{1}{\lambda^2}\left(\gamma_\pm^2-\frac{1}{4}\right)\left[1-\left(\tanh\frac{x}{\lambda}\right)^2\right]+\frac{1}{2}\alpha_k^2\left(1+\tanh\frac{x}{\lambda}\right)+\frac{1}{2}\alpha_{-k}^2\left(1-\tanh\frac{x}{\lambda}\right). \qquad (3b)$$

Here, the following assignments for the parameters of this potential have been made, with σ^3 denoting the spin projection eigenvalue $+1$ for spin up fermions and -1 for spin down fermions:

$$\alpha_k=\sqrt{(k-eB\lambda)^2+m^2}, \qquad (4a)$$

$$\gamma_\pm=\begin{cases}\frac{1}{2}\sqrt{1+4(eB\lambda^2)^2} & (+)\ \text{bosons},\\ \frac{1}{2}+eB\lambda^2\sigma^3 & (-)\ \text{fermions}.\end{cases} \qquad (4b)$$

The one-dimensional Schrödinger operator (3a) is of the supersymmetric quantum mechanical form [11], but with a k-dependent superpotential $W_k(x)=\lambda B\tanh(x/\lambda)-k$. The k-dependent potential $V_k(x)$ is of the modified Pöschl-Teller form, with asymptotic energy barrier heights $\alpha_{\pm k}^2$ as $x\rightarrow\pm\infty$; the corresponding incoming and outgoing momenta are then just $\alpha_{\pm k}$, while ω^2 is merely an overall shift in energy. The origin of the difference between the two asymptotic energy barriers may be simply understood in terms of classical electrodynamics. As a particle of charge e enters the magnetic strip from $-\infty$, its kinetic energy is conserved while its momentum in the y direction behaves as $p_y(x)=p_y(-\infty)+eB\lambda[\tanh(x/\lambda)+1)]$. If the momentum in the x direction is large enough, the particle traverses the strip, but for small x momentum, it will instead be reflected off the strip and return to $-\infty$.

The evaluation of the determinants in (2) thus reduces to a spectral problem for the Schrödinger operator in (3a). Note that this Schrödinger operator has both a discrete and a continuous spectrum, in contrast with the constant B field case for which the spectrum is purely discrete. It is convenient at this point to analytically continue frequencies to imaginary values, $\omega\rightarrow i\omega$, as usual; the effective energy is then given by

$$\mathscr{E}_\pm=\pm\frac{L}{4\pi^2}\int_{-\infty}^{+\infty}dk\int_{-\infty}^{+\infty}d\omega\ \ln\mathrm{Det}\left(-\frac{d^2}{dx^2}+V_k(x)+\omega^2\right). \qquad (5)$$

[Notice that we actually compute the difference between $\mathscr{E}_\pm$ and the corresponding free field ($B=0$) case; thus we may drop terms independent of B.] Integrating by parts in ω and omitting B-independent terms, the effective energy may be recast as

$$\mathscr{E}_\pm=\mp\frac{2L}{4\pi^2}\int_{-\infty}^{+\infty}dk\int_{-\infty}^{+\infty}d\omega\ \omega^2\ \mathrm{Tr}\ G_{-\omega^2,k}. \qquad (6)$$

The resolvent Green function $G_{E,k}$ is the inverse of the Schrödinger operator, for general complex parameter E:

$$\left(-\frac{d^2}{dx^2}+V_k(x)-E\right)G_{E,k}(x,x')=\delta(x-x'). \qquad (7)$$

It is standard to obtain $G_{E,k}(x,x')$ by matching the independent solutions of the homogeneous equation, which are proportional to hypergeometric functions in terms of the new variable $\xi=[1+\tanh(x/\lambda)]/2$:

$$u_1(x)=\xi^\alpha(1-\xi)^\beta F(z+\tfrac{1}{2}-\gamma_\pm,z+\tfrac{1}{2}+\gamma_\pm;1+2\alpha;\xi), \qquad (8a)$$

$$u_2(x)=\xi^\alpha(1-\xi)^\beta F(z+\tfrac{1}{2}-\gamma_\pm,z+\tfrac{1}{2}+\gamma_\pm;1+2\beta;1-\xi). \qquad (8b)$$

Here, α and β are defined as the roots with positive real parts of the equations

$$\alpha=\frac{\lambda}{2}\sqrt{\alpha_{-k}^2-E},\quad \beta=\frac{\lambda}{2}\sqrt{\alpha_k^2-E},\quad z=\alpha+\beta. \qquad (9)$$

With these conventions, u_1 is regular at $\xi=0$ (i.e., as $x\rightarrow-\infty$), whereas u_2 is regular at $\xi=1$ (i.e., as $x\rightarrow+\infty$). The Wronskian of these solutions is a constant, given by

$$W=u_1'(x)u_2(x)-u_1(x)u_2'(x)=\frac{2}{\lambda}\frac{\Gamma(1+2\alpha)\Gamma(1+2\beta)}{\Gamma(z+\frac{1}{2}-\gamma_\pm)\Gamma(z+\frac{1}{2}+\gamma_\pm)} \qquad (10)$$

and the Green function $G_{E,k}$ is therefore given by

$$G_{E,k}(x,x')=\theta(x'-x)\frac{u_1(x)u_2(x')}{W}+\theta(x-x')\frac{u_1(x')u_2(x)}{W}. \qquad (11)$$

All information concerning the spectrum of the Schrödinger operator is contained in the trace of the Green function $G_{E,k}$. We first compute

$$\text{Tr } G_{E,k}=\int_{-\infty}^{+\infty}dx\ G_{E,k}(x,x)=\frac{1}{W}\int_{-\infty}^{+\infty}dx\ u_1(x)u_2(x). \qquad (12)$$

Some care is needed in regularizing this integral at $x=\pm\infty$; for example, one can multiply the arguments ξ and $1-\xi$ in the hypergeometric functions in u_1 and u_2, respectively, by a factor $1-\epsilon$ for $\epsilon>0$ and infinitesimal. With this regularization, the integration may be performed exactly in terms of the Euler psi function $\psi(x)=\Gamma'(x)/\Gamma(x)$, and we find

$$\text{Tr } G_{E,k}=-\frac{\lambda^2}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)[\psi(z+\tfrac{1}{2}-\gamma_\pm)+\psi(z+\tfrac{1}{2}+\gamma_\pm)]+f_\epsilon(\alpha)+f_\epsilon(\beta), \qquad (13)$$

where $f_\epsilon(\alpha)$ is a function of α, but not of β, whose precise form is regulator dependent, but which does not contribute to the effective energy once we integrate over k. Here and in the following, it is understood that for the case of four-spinors $(-)$ both spin states are to be summed over. The spectrum contains a finite number of bound states, which arise from the (simple) poles of the ψ functions in (13) at $z+\frac{1}{2}-\gamma_\pm=-n$ for $0\leqslant n<\gamma_\pm-1/2-\sqrt{|\alpha_k^2-\alpha_{-k}^2|}$:

$$E_n=\tfrac{1}{2}(\alpha_k^2+\alpha_{-k}^2)-\lambda^{-2}(n+\tfrac{1}{2}-\gamma_\pm)^2-\tfrac{1}{16}\lambda^2(\alpha_k^2-\alpha_{-k}^2)^2(n+\tfrac{1}{2}-\gamma_\pm)^{-2}. \qquad (14)$$

The same discrete spectrum may be obtained by solving the homogeneous Schrödinger equation (7) for real E and requiring normalizability of the eigenfunctions [12]. The spectrum also contains a cut starting at α_k^2 and another cut starting at α_{-k}^2, corresponding to the two barrier thresholds. For $B=0$, the discrete spectrum is absent, whereas for the constant magnetic field case (i.e., $\lambda\to\infty$), it reduces to $E_n=2eB(n+1/2)$ for bosons and $2eBn$ for fermions, as expected.

We now complete the calculation of the effective energy, using the result of (13) in the expression (6) for the effective energy. First, since α only depends upon $k+eB\lambda$, we may shift k by $-eB\lambda$ in the contribution of $f_\epsilon(\alpha)$ in (13). Thus, the regulator-dependent f_ϵ terms in (13) yield only B-independent contributions to the effective energy in (6) and may be omitted. Next, we use the identity

$$-\frac{\lambda^2}{8}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=\frac{\partial z}{\partial E}, \qquad (15)$$

which suggests the change of variable from ω to $z=\alpha+\beta$ and yields

$$\mathscr{E}_\pm=\pm\frac{L}{2\pi^2\lambda}\int_{-\infty}^{+\infty}dk\int_{|\alpha_k+\alpha_{-k}|}^{\infty}dz\ \frac{1}{z}\times\sqrt{[z^2-(\alpha_k+\alpha_{-k})^2][z^2-(\alpha_k-\alpha_{-k})^2]}\times[\psi(z+\tfrac{1}{2}-\gamma_\pm)+\psi(z+\tfrac{1}{2}+\gamma_\pm)]. \qquad (16)$$

The integration over k can be performed and we end up with the remarkably simple expression

$$\mathscr{E}_\pm=\pm\frac{L}{4\pi\lambda^2}\int_{z_0}^{\infty}dz\ \frac{z(z^2-z_0^2)}{\sqrt{z^2-z_0^2+(\lambda m)^2}}\times[\psi(z+\tfrac{1}{2}-\gamma_\pm)+\psi(z+\tfrac{1}{2}+\gamma_\pm)], \qquad (17)$$

where $z_0\equiv\lambda\sqrt{(eB\lambda)^2+m^2}$. This expression can be rewritten in terms of elementary integrals by making use of the following representation of the ψ function [13]:

$$\psi(x)=\ln x-\frac{1}{2x}-2\int_0^\infty\frac{tdt}{(t^2+x^2)(e^{2\pi t}-1)}. \qquad (18)$$

The first two terms on the right-hand side in (18) contribute B-independent terms to $\mathscr{E}_\pm$. (It is necessary to sum over both spin states to see this in the fermion case.) For the third term, the z integration can be carried out exactly and we obtain the following finite integral representation for the effective energy:

$$\mathscr{E}_\pm(L,m\lambda,eB\lambda^2)=\frac{L}{4\pi\lambda^2}\int_0^\infty dt\ \frac{1}{e^{2\pi t}\pm1}\left((b_\pm-it)\times\frac{(\lambda^2m^2+v_\pm^2)}{v_\pm}\ln\frac{\lambda m-iv_\pm}{\lambda m+iv_\pm}+\text{c.c.}\right), \qquad (19)$$

where c.c. denotes the complex conjugate, and

$$b_\pm=\begin{cases}\sqrt{(eB\lambda^2)^2+1/4} & (+)\ \text{bosons},\\ eB\lambda^2 & (-)\ \text{fermions},\end{cases} \qquad (20a)$$

$$v_\pm^2=\begin{cases}t^2+2i\ t\ b_+-1/4 & (+)\ \text{bosons},\\ t^2+2i\ t\ b_- & (-)\ \text{fermions}.\end{cases} \qquad (20b)$$

Expression (19) gives the exact effective action for a background field (1) and is the main result of this paper. For definiteness, we now concentrate on the fermion case, but analogous discussions can be made for bosons.

In the limit of vanishing mass, $m=0$, the only relevant dimensionless parameter is $eB\lambda^2$, and so one finds an asymptotic expansion

$$\mathscr{E}_- = \frac{L\lambda(eB)^{3/2}}{8\pi} \sum_{j=0}^{\infty} \frac{1}{(4\pi eB\lambda^2)^j} \frac{\Gamma(j-3/2)\Gamma(j+5/2)}{\Gamma(j+1)\Gamma(j/2+1/4)\Gamma(3/4-j/2)} \zeta(j+3/2) \tag{21}$$

$$= \frac{L\lambda(eB)^{3/2}}{8\sqrt{2}\pi} \left[\zeta(3/2) - \frac{15}{16\pi} \zeta(5/2) \frac{1}{eB\lambda^2} + \cdots \right], \tag{22}$$

where $\zeta(z)$ is the Riemann zeta function [13]. The first term in this expansion agrees with the uniform B field case [3,14], while the next term agrees with the first-order derivative expansion computation in [5].

For nonzero mass m, there is another dimensionless parameter eB/m^2 which is the ratio of the cyclotron energy to the rest mass energy. A double expansion of (19) yields

$$\mathscr{E}_- = \frac{Lm^3\lambda}{8\pi} \sum_{j=0}^{\infty} \frac{1}{j!} (2eB\lambda^2)^{(-j)} \sum_{k=1}^{\infty} \frac{(2k+j-1)!}{(2k)!} \frac{\mathscr{B}_{2k+2j}}{(2k+j-1/2)(2k+j-3/2)} \left(\frac{2eB}{m^2} \right)^{2k+j} \tag{23}$$

with $\mathscr{B}_n$ the nth Bernoulli number [13]. Each power in λ^{-2} corresponds to a fixed order in a derivative expansion of the effective action. The zeroth-order term agrees with the eB/m^2 expansion of the exact constant B field answer [3,14], while the first-order term agrees with the eB/m^2 expansion of the leading derivative expansion contribution found in [5].

In fact, the specific configurations (1) give some insight in the more general case of a background magnetic field $B(x)$ depending on one coordinate only. In a derivative expansion of the effective Lagrangian the terms with a total of $2j$ derivatives, $\mathscr{L}^{[2j]}$, have, up to integration by parts, a unique structure:

$$\mathscr{L}^{[0]} = m^3 F_0^{[0]}\left(\frac{eB(x)}{m^2} \right), \tag{24a}$$

$$\mathscr{L}^{[2]} = m F_2^{[2]}\left(\frac{eB(x)}{m^2} \right) \left[\frac{eB'(x)}{m^2} \right]^2, \tag{24b}$$

$$\mathscr{L}^{[2j]} = m^{3-2j} \sum_{l=1}^{2j-2} F_l^{[2j]}\left(\frac{eB(x)}{m^2} \right) \frac{eB^{(2j-l)}(x)}{m^2} \left[\frac{eB'(x)}{m^2} \right]^l, \quad 2j=4,6,8,\ldots, \tag{24c}$$

where $B^{(l)}(x)$ denotes the lth derivative of $B(x)$. Parity invariance forces $F_l^{[2j]}(x)$ to be even (odd) for l odd (even). A comparison with (23) entirely determines the zero- and two-derivative terms (in agreement with [5]) and gives at each order $(2j>2)$ a relation among the $(2j-2)$ functions $F_l^{[2j]}(x)$:

$$F_0^{[0]}(x) = -\sum_{k=1}^{\infty} \frac{1}{8\pi^{3/2}} \frac{\Gamma(2k-3/2)}{\Gamma(2k+1)} \mathscr{B}_{2k} (2x)^{2k}, \tag{25a}$$

$$F_2^{[2]}(x) = -\sum_{k=1}^{\infty} \frac{1}{4\pi^{3/2}} \frac{\Gamma(2k-1/2)}{\Gamma(2k)} \mathscr{B}_{2k+2} (2x)^{2k-2}, \tag{25b}$$

$$\sum_{l=1}^{j-1} \sum_{s=1}^{j-l+1} (-2)^{2j-s} \frac{\Gamma(j+3/2-s)}{\Gamma(3/2)} \left(\frac{d}{dx} \right)^{s-1} x^{2l+s-2} \{ [s\, a_{j-l+1}^s + 2(j-l+2-s)\, a_{j-l+1}^{s-1}] F_{2l-1}^{[2j]}(x) + x\, a_{j-l+1}^s F_{2l}^{[2j]}(x) \}$$

$$= \sum_{k=1}^{\infty} \frac{1}{8\pi^{3/2}} \frac{\Gamma(2k+j)}{\Gamma(2k+1)} \frac{\Gamma(2k+j-3/2)}{\Gamma(2k)} \mathscr{B}_{2k+2j} (2x)^{2k-1}, \tag{25c}$$

where a_p^s are the coefficients in the polynomial of degree $2p$,

$$\left[(1-t^2)\frac{d}{dt} \right]^{2p-1} t \equiv \sum_{s=1}^{p} a_p^s\, (-2)^{2p-s-1}\, t^{2p-2s}\, (1-t^2)^s. \tag{26}$$

As an example, the two functions appearing in the four-derivative terms $(2j=4)$ obey

$$\left(4\frac{d}{dx} + \frac{5}{x} \right) F_1^{[4]}(x) + x \frac{d}{dx} F_2^{[4]}(x) = \sum_{k=1}^{\infty} \frac{1}{8\pi^{3/2}} (2k+1) \frac{\Gamma(2k+1/2)}{\Gamma(2k)} \mathscr{B}_{2k+4} (2x)^{2k-3}. \tag{27}$$

Thus, Eqs. (25) provide an infinite number of relations among the coefficients of the effective action for any background magnetic field with translation invariance in one direction.

Finally, we may apply the results on the effective energy obtained here and in [5] to the study of the full QED_{2+1} theory around classical electromagnetic configurations. In [5], the effective energy is obtained for large wavelength fluctuations in the fields. While the leading order term contributes positively to the energy, the next, two-derivative term contributes negatively. It was proposed in [5] that this behavior may drive the system towards a lowest energy state with inhomogeneous magnetic field. Indeed, assuming flux conservation, it is natural to consider static fluctuations $\delta B(\vec{x})$ that leave the total flux unchanged. Under this constraint, the energy fluctuation is given by

$$\begin{aligned}\mathscr{E}[B+\delta B(\vec{x})]-\mathscr{E}(B) &= \frac{1}{2}\int d^2x\ \{1+\alpha\ (e^2/\sqrt{eB}) \\ &\quad \times[1-\beta\ (eB)^{-1}|\vec{\partial}\delta B/\delta B|^2]\}(\delta B)^2, \quad (28)\end{aligned}$$

where the coefficients α and β are positive functions of $m^2/(eB)$, except for bosons with sufficiently small mass as explained in [5]. The coefficient β is typically of order 1, and within the derivative expansion regime $(eB)^{-1}|\vec{\partial}\delta B/\delta B|^2 \ll 1$, indicating that the constant background magnetic field is stable under variations that conserve flux.

In the case of the special family of profiles for the magnetic field in (1) we can go beyond the leading order derivative expansion, using the exact effective energy (19), together with the Maxwell term, to give the total energy as

$$\begin{aligned}\mathscr{E}_{\pm}^{\mathrm{tot}} &= L\lambda B^2+\mathscr{E}_{\pm}(L,m\lambda,eB\lambda^2) \\ &= \frac{\pi^2}{e^2}\frac{\Phi^2}{\lambda L}+\mathscr{E}_{\pm}(L,m\lambda,\pi\Phi\lambda/L) \quad (29)\end{aligned}$$

with flux $\Phi=eB\lambda L/\pi$. We have shown that the total energy (29) at fixed flux Φ is a positive monotonically decreasing function of the parameter λ. As a result, a system with magnetic profile (1) is driven towards a system with uniform magnetic field. Thus, within this family, we have not found further support for our suggestion in [5] that magnetic field inhomogeneities lower the total energy at fixed nonzero flux.

This raises a number of open questions. (a) Are any other families of magnetic field profiles integrable, within which nonuniform magnetic fields minimize the energy at fixed flux? The recursion relations (25) suggest that magnetic field profiles with x dependence only may be such candidates. (b) More generally, can inhomogeneous magnetic fields be found that minimize the energy for fixed flux?

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