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Smoothness of the horizons of multi-black-hole solutions

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In a recent paper it was suggested that some multi-black-hole solutions in five or more dimensions have horizons that are not smooth. These black hole configurations are solutions to d-dimensional Einstein gravity (with no dilaton) and are extremely charged with a magnetic-type $(d-2)$-form. In this work we investigate these solutions further. It is shown that although the curvature is bounded as the horizon of one of the black holes is approached, some derivatives of the curvature are not. This shows that the metric is not C^∞, but rather is only C^k with k finite. These solutions are static so their lack of smoothness cannot be attributed to the presence of radiation.

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I. INTRODUCTION

When the Schwarzschild solution was discovered there was much confusion as to the meaning of the fact that some of the metric components were singular at the event horizon. Even after it was discovered that there exists coordinates in which the metric is smooth at the horizon, it was some time before it became clear beyond all doubt that the horizon was not singular in any physically meaningful way.

It is now well known that the Schwarzschild solution, like all known single-black-hole solutions, describes a black hole that has a smooth event horizon. There exist timelike geodesics that reach the horizon in a finite proper time and extend across it. All the curvature scalars that one can construct are well behaved at the horizon; furthermore, if one takes an orthonormal basis and parallel propagates it along a timelike geodesic then the components of the Riemann tensor in this basis will be smooth functions as one crosses the horizon [1]. They will only diverge when the singularity is approached. Similarly, if we add charge, or angular momentum, to this solution the event horizon will remain smooth. However, if sufficient charge, or angular momentum, is added the horizon will no longer exist, leaving us with a naked singularity.

An interesting question is the following: Will the horizon remain smooth if we have more than one black hole in the spacetime? In light of the results known for single-black-hole solutions it may seem likely that multi-black-hole will have smooth horizons; however, the nonlinearity of gravity makes this a difficult question to address. Having the solution for a single black hole does not mean there is an easy way to obtain multiple-black-hole solutions. Not only will multiple black hole solutions generally lack spatial symmetries, but they will not have any timelike symmetry either; in other words, they generally will be dynamic. In spite of this some multi-black-hole solutions are known.

In Newtonian theory any configuration of pointlike charged particles will remain in static equilibrium if the charges are all of the same sign and are related to their masses by $e_i^2 = Gm_i^2$. Analogous static solutions for the Einstein-Maxwell equations have been known for some time [2,3]. These correspond to configurations of extremal Reissner-Nordström black holes. Complete analytic extensions of these were given in [4]. Among the results derived in this paper were that the event horizon is smooth and that the only singularities are inside the horizon. These results support the natural extension of what is known for single-black-hole solutions, that event horizons are smooth.

Another family of multi-black-hole solutions consists of the analogues of extremal Reissner-Nordström black holes in four-dimensional asymptotically de Sitter spacetime [5]. These solutions differ from those discussed in the preceding paragraph in two ways. First, they are dynamic whereas the former, asymptotically flat, solutions are static. Second, while the asymptotically flat solutions have smooth horizons, it was shown in [6] that the asymptotically de Sitter solutions have horizons that are not smooth. However, the curvature singularities are very mild and geodesics can be extended through them. In particular, it was shown for this case that although the metric is always at least C^2, which means that the curvature is well behaved, it is not in general C^∞, so some derivatives of the curvature diverge as the horizon is approached. The fact that these solutions are dynamic means that there will be gravitational and electromagnetic radiation present. The divergences discovered were interpreted as being the result of the radiation having a distribution that is not smooth everywhere. Another result for these solutions that will be of interest here is that the differentiability of the metric increases as the order of the lowest nontrivial multipole moment of the mass distribution increases [6].

In a recent paper it was suggested that some multiple p-brane solutions in five or more dimensions have horizons that are not smooth [7]. For the special case of black

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holes, it was suggested that all the solutions in five or more dimensions are not smooth. The theory considered was d-dimensional Einstein gravity coupled to a $(d-2)$-form. The black hole solutions were extremely charged with a magnetic type $(d-2)$-form charge. In this paper the conjecture that these solutions have horizons that are not smooth will be confirmed. These multi-black-hole solutions generalize those contained in [4] by allowing the spacetime dimension to exceed four. When the spacetime dimension is set equal to four then all of the results obtained here will be consistent with those of [4].

To establish the fact that these solutions are not smooth we will consider the simplest multi-black-hole solution, that consisting of two black holes. Timelike geodesics along the axis connecting the black holes can reach the horizon in a finite proper time and can be extended through the horizon. We will find that although the components of the Riemann curvature, as measured in an orthonormal basis that is parallel propagated along one of these timelike geodesics, are bounded at the horizon, when $d \geq 5$ some derivatives of these components with respect to the proper time of geodesic will diverge at the horizon. This demonstrates that the metric is not C^∞ at the horizon, but rather it is only C^k there for some finite k. These solutions are static, so their lack of smoothness cannot be attributed to the presence of radiation, as was the case for the multi-black hole solutions in asymptotically de Sitter space.

To see the effects of adding additional black holes we will also consider the next most simple multi-black-hole configuration, that of three collinear black holes. Once again a timelike geodesic with an orthonormal basis that is parallel propagated into the central black hole along the axis connecting the black holes will be considered. It will be shown that if the outer black holes have the same mass and are the same distance from the central black hole, then in five dimensions the differentiability of the horizon, more precisely the component of the horizon surrounding the central black hole, is increased. It will be shown that in more than five dimensions the divergence is less severe in this case. This is analogous to the results of [6] that for multi-black-hole solutions in an asymptotically de Sitter space the differentiability of the horizon is increased by arranging the masses so that the lower order multipole moments vanish. Also, for these configurations, the behavior of the curvature components measured in an orthonormal basis parallel propagated into the central black hole along a geodesic that is orthogonal to the line connecting the three black holes will be briefly considered.

II. THE MULTI-BLACK-HOLE SOLUTIONS

The theory we will consider is d-dimensional Einstein gravity coupled to a $(d-2)$-form. First we summarize some previously derived results. The action we start with is [7,8]

$$\int d^dx\sqrt{-g}\left(R - \frac{2}{(d-2)!}F^2_{d-2}\right) , \tag{2.1}$$

where d is the spacetime dimension, R is the Ricci scalar, and F_{d-2} is a $(d-2)$-form. This theory has charged black hole solutions. The extremal magnetically charged versions of which are [7–9]

$$\begin{aligned} ds^2 &= -\left[1-\left(\frac{\mu}{r}\right)^{d-3}\right]^2 dt^2 \\ &\quad + \left[1-\left(\frac{\mu}{r}\right)^{d-3}\right]^{-2} dr^2 + r^2 d\Omega^2_{(d-2)} , \\ F_{(d-2)} &= Q\epsilon_{(d-2)} \end{aligned} \tag{2.2}$$

where $\epsilon_{(d-2)}$ is the volume form on the unit $(d-2)$-sphere, $\mu^{(d-3)}$ is proportional to the mass, the horizon is at $r = \mu$ and the charge Q is given by

$$Q^2 = \tfrac{1}{2}(d-2)(d-3)\mu^{2(d-3)} . \tag{2.3}$$

As one would expect for an extremal black hole, the charge is proportional to the mass.

To obtain multi-black-hole solutions it is first useful to make a coordinate transformation that puts the metric (2.2) in the isotropic form. This is done [7] by introducing a new radial coordinate, ρ, given by $r^{d-3} = \rho^{d-3} + \mu^{d-3}$. The extremal black-hole solution in these coordinates is

$$ds^2 = -H^{-2}dt^2 + H^{2/(d-3)}d\mathbf{x}\cdot d\mathbf{x} \tag{2.4}$$

$$\frac{1}{(d-2)!}\epsilon^{ij_1\cdots j_{(d-2)}}F_{j_1\cdots j_{(d-2)}} = \left(\frac{d-2}{2(d-3)}\right)^{1/2}\partial_i H ,$$

where $\epsilon^{i_1\cdots i_{(d-1)}}$ is the constant alternating tensor density of the Euclidean $(d-1)$-space, the spatial coordinates, $\{x^i\}$, are related to ρ by $\rho = (\sum_{i=1}^{d-1} x^i x^i)^{1/2}$ and

$$H = 1 + \left(\frac{\mu}{\rho}\right)^{d-3} . \tag{2.5}$$

We will consider solutions where all of the black holes have charge with the same sign. This means that the gravitational and Coulomb forces on each hole will be of equal magnitude and in opposite directions; hence, we can have static solutions. It is possible to obtain multi-black-hole solutions with H being any solution of Laplace's equation in Euclidean $(d-1)$-space with k point sources located at $\mathbf{x} = \mathbf{x}_a$, that is with

$$H = 1 + \left(\frac{\mu}{\rho}\right)^{d-3} + \sum_{a=2}^{k}\left(\frac{M_a}{|\mathbf{x}-\mathbf{x}_a|}\right)^{d-3} , \tag{2.6}$$

where the black hole with a mass parameter μ was chosen to be at the origin. It should be noted that the "point" sources are actually the horizons of the individual black holes and there are no material sources there. In spite of the fact that they appear as points here, they have a nonzero area [more precisely, nonzero $(d-2)$-volume].

In this paper the multi-black-hole solutions that will be considered are the two- and three-black-hole systems. For the two-black-hole solution we will choose one of the coordinate axes to be connecting the black holes, this will be called the w axis. In the case of three black holes we will choose all of them to be on the w axis. This loss of generality in the three-black-hole case is compensated for by an increased symmetry that allows us to more easily find geodesics.

Now we establish our conventions and state some equations that will be used later. To start, consider the three-black-hole case. The two-black-hole case can be recovered from this by taking the mass of the third black hole equal to zero. The black hole into which the geodesic, along which we will calculate curvature components and their derivatives, travels will be taken to be at $\rho = 0$ and to have mass parameter μ. The second black hole will be at $\mathbf{x} = a\mathbf{w}$ ($\mathbf{w}$ being the coordinate basis vector along the w axis) and have mass parameter M. The third black hole will be at $\mathbf{x} = -a_2\mathbf{w}$ and have mass parameter M_2. We will primarily be concerned with geodesics that are along the w axis, without loss of generality these geodesics will be taken to be along the positive w axis (to get the results for one along the negative w axis we only need to exchange a and a_2). In our configuration we have

$$H = 1 + \left(\frac{\mu}{\rho}\right)^{d-3} + M^{d-3}\left[(a-w)^2 + \sum_{i\neq w} x^i x^i\right]^{-(d-3)/2}$$

$$+ M_2^{d-3}\left[(a_2+w)^2 + \sum_{i\neq w} x^i x^i\right]^{-(d-3)/2} . \qquad (2.7)$$

In addition, we will use the common convention that Latin indices $\{i, j, k, \ldots\}$ take only spatial values.

III. CALCULATIONS OF CURVATURE

A. Preliminaries

In the previous section the metric of a general two-black-hole solution, and that of a three-black-hole solution with a special symmetry, were given. These solutions clearly have coordinate singularities at the horizon. We now want to determine if this is due to a bad choice of coordinates, or if the metric is actually singular there. The most natural thing one might try doing to address this question is to consider the behavior of curvature scalars near the horizon. However, this would give us an incomplete picture because it is possible to have a divergent Reimann tensor and still have curvature scalars that are well behaved. In light of this we choose to consider how the components of the Riemann tensor behave as we approach the horizon in a "good" coordinate system in order to examine the possible singular nature of the solution.

To construct this "good" coordinate system we will start with an orthonormal basis and parallel propagate it along a timelike geodesic that goes into the horizon. In our case this is equivalent to considering the components in a static orthonormal basis, here formed by the vectors

$$\left\{\hat{e}_t = H\frac{\partial}{\partial t},\ \hat{e}_i = H^{-1/(d-3)}\frac{\partial}{\partial x^i}\right\} , \qquad (3.1)$$

and boosting it with the velocity parameter that a free-falling observer would have with respect to this static basis. Of course the exact value of the velocity parameter will depend on the initial conditions of the observer, but as the horizon is approached it will diverge in a manner that is independent of the initial conditions. To see the equivalence note that the geodesic is along the axis connecting the black holes, so the symmetry of the solutions (and the parallel transport equations) leads to the transverse basis vectors, $\{\hat{e}_i | i \neq w\}$, being unchanged by the parallel transport, just as in the case of boosting along the axis. The timelike basis vector in our initial static frame, $\hat{e}_t^0$, when parallel transported is just the covariant velocity vector $\mathbf{u}$ of the free-falling observer. It can be obtained by boosting the timelike basis vector $\hat{e}_t$ of the locally static orthonormal frame by the appropriate Lorentz transformation. Finally, consider that the longitudinal spatial basis vector $\hat{e}_w^0$. In the initial static frame $\hat{e}_t^0 \cdot \hat{e}_w^0 = 0$. Vector products are preserved under parallel transport, so the basis vector obtained by parallel transporting $\hat{e}_w^0$, say $\hat{e}'_w$, must be orthogonal to $\mathbf{u}$ and the transverse basis vectors. This shows that $\hat{e}'_w$ can be obtained by boosting the vector $\hat{e}_w$ of the locally static basis with the same Lorentz transformation that we used to get $\mathbf{u}$ from $\hat{e}_t$. Therefore, boosting our locally static orthonormal basis is equivalent to parallel transporting an orthonormal basis along a timelike geodesic. Basically, any divergences we find in this basis will be divergences that a free-falling observer could measure.

To start we calculate the components of the totally covariant form of the Riemann tensor in the coordinate basis, $R_{\alpha\beta\sigma\gamma}$, using the usual method. From these we obtain the components in a static orthonormal frame, $R_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}$, by multiplying by a power of H that depends on how many of the indices of the tensor are t. If two of the indices are t then we must multiply by $H^{2(d-4/d-3)}$ and if all of the indices are spatial we must multiply by $H^{-4/(d-3)}$. If only one index is t then the component is zero.

Carrying out this procedure we find that the nonzero components of the Riemann tensor in a static orthonormal frame are

$$R_{\hat{t}\hat{i}\hat{t}\hat{i}} = H^{-2(d-2)/(d-3)}\left\{-H\partial_i^2 H + 2\frac{d-2}{d-3}(\partial_i H)^2 - (d-3)^{-1}\sum_{\text{all } k}(\partial_k H)^2\right\},$$
$$R_{\hat{t}\hat{i}\hat{t}\hat{j}} = H^{-2(d-2)/(d-3)}\left\{-H\partial_i\partial_j H + 2\frac{d-2}{d-3}\partial_i H\partial_j H\right\}, \quad i \neq j,$$
$$R_{\hat{i}\hat{j}\hat{i}\hat{j}} = H^{-2(d-2)/(d-3)}(d-3)^{-1}\left\{-H\partial_i^2 H - H\partial_j^2 H + (\partial_i H)^2 + (\partial_j H)^2 - (d-3)^{-1}\sum_{k\neq i,j}(\partial_k H)^2\right\},$$
$$R_{\hat{i}\hat{j}\hat{i}\hat{k}} = H^{-2(d-2)/(d-3)}(d-3)^{-1}\left\{-H\partial_j\partial_k H + \frac{d-2}{d-3}\partial_j H\partial_k H\right\}, \quad j \neq k \tag{3.2}$$

with H given by (2.6) and d is equal to the spacetime dimension.

B. Solutions with two black holes

Equation (3.2) displays the curvature components of the metric given in (2.4) for any H. Now, consider the special case of two black holes, this corresponds to taking H as in Eq. (2.7) with $M_2 = 0$. The components of the Riemann tensor will now be calculated along the axis connecting the black holes. Taking the derivatives of H as prescribed in (3.2) and then taking $x^i = 0$ for $i \neq w$ gives

$$\begin{aligned} R_{\hat{t}\hat{w}\hat{t}\hat{w}} &= A^{-2(\beta+1)/\beta}\beta\{\beta\mu^{2\beta} - (\beta+1)\mu^\beta w^\beta - (\beta+1)(M\mu)^\beta f^\beta - (4\beta+2)(M\mu)^\beta f^{\beta+1} \\ &\quad -(\beta+1)(M\mu)^\beta f^{\beta+2} - (\beta+1)M^\beta f^{\beta+2}w^\beta + \beta M^{2\beta}f^{2\beta+2}\}, \\ R_{\hat{t}\hat{x}\hat{t}\hat{x}} &= A^{-2(\beta+1)/\beta}\beta\{\mu^\beta w^\beta + (M\mu)^\beta f^\beta + 2(M\mu)^\beta f^{\beta+1} + (M\mu)^\beta f^{\beta+2} + M^\beta f^{\beta+2}w^\beta\} \\ &= -R_{\hat{w}\hat{x}\hat{w}\hat{x}}, \\ R_{\hat{x}\hat{y}\hat{x}\hat{y}} &= A^{-2(\beta+1)/\beta}\{\mu^{2\beta} + 2\mu^\beta w^\beta + 2(M\mu)^\beta f^\beta + 2(M\mu)^\beta f^{\beta+1} + 2(M\mu)^\beta f^{\beta+2} + M^{2\beta}f^{2\beta+2} + 2M^\beta f^{\beta+2}w^\beta\}, \end{aligned} \tag{3.3}$$

where, x (and y) is any of the transverse spatial coordinates, $\beta = d-3$, f is defined by

$$f = \frac{w}{a-w}, \tag{3.4}$$

and A is defined by

$$A = \mu^\beta + w^\beta + M^\beta f^\beta. \tag{3.5}$$

There are two important things to notice about (3.3). One is that the components of the Riemann tensor are invariant under boosts in the w direction [10]. This means that the components we just calculated for a static orthonormal basis are also the components in a *free-falling* orthonormal basis. Another is that the components are finite at the horizon, that is, at $w = 0$, clearly the metric is at least C^2. We now proceed to calculate derivatives of these components with respect to the proper time of a free-falling observer.

In order to do this we first need to find the geodesics. Because of the symmetry of the solution we can take $\mathbf{u} = \dot{t}\partial/\partial t + \dot{w}\partial/\partial w$, where the overdot denotes taking the derivative with respect to proper time, which will simply be referred to as the derivative from now on. The t-geodesic equation gives $\dot{t} = kH^2$, with k begin some constant that depends on the initial conditions. The fact that $\mathbf{u}\cdot\mathbf{u} = -1$ can then be used to find $\dot{w}$. It is given by

$$\dot{w} = -H^{-1/\beta}\sqrt{k^2H^2 - 1}. \tag{3.6}$$

The negative sign is there because we are considering geodesics going in from positive w. The symbol $|_h$ will be used to denote the value of a quantity as the horizon is approached. Notice that as we approach the horizon $\dot{w}$ is unbounded if $d \geq 5$:

$$\dot{w}|_h = -k\mu^{d-4}w^{4-d}[1 + O(w^{d-3})]. \tag{3.7}$$

This suggests the possibility that some derivatives of the Riemann tensor may diverge at the horizon. It should be noted that this is finite if $d = 4$ [in which case the $(d-2)$-form is just the Maxwell tensor] and that all of the results that will be presented here are consistent with those of [4,7], where it was demonstrated that the horizons of multi-extremal black-hole solutions in four dimensions have smooth horizons. The fact that $\dot{w}$ diverges as we approach the horizon may make one worry that our results will imply that single-black-hole solutions in more than four dimensions have nonsmooth horizons; this would be in direct conflict with known results [7]. However, one can show that all derivatives of the Riemann tensor for single-black-hole solutions are finite at the horizon by the following argument. To differentiate the components of the Riemann tensor we take $\dot{w}\partial w$ of (3.2). Using (3.7) we see that $\dot{w}\partial_w$ acting on a term proportional to w^n gives one proportional to $w^{n-\beta}$. This shows that, if n is an integer multiple of β, then we will never get negative powers of w by taking derivatives of w^n. The reason is that taking derivatives reduces the power of w by an integer multiple of β; hence, some order derivative of w^n will give a constant. Taking more derivatives of this will give zero. An examination of (3.3) and (3.5) shows that when $M = 0$, i.e., the single-black-hole solution, w only appears as w^β. From the preceding argument one can see that all derivatives of the Riemann tensor of the single-black-hole solutions will be well behaved.

The smallest power of w appearing in (3.3) is w^β, and as expected from the above argument the first derivatives of the components (3.3) are finite. One can also see that when $M \neq 0$ some $w^{\beta+1}$ powers appear in (3.3) near the horizon, this suggests that second derivatives of (3.3) will diverge at the horizon. We now go on and calculate the second derivatives of (3.3).

We now state some formulas that will be necessary for calculating the second derivatives of (3.3). We can get $\ddot{w}$ from the geodesic equation for u^w. Expanding this function near the horizon gives

$$\ddot{w}|_h = -k^2\mu^{2d-8}w^{7-2d}(d-4)[1+O(w^{d-3})] \ . \qquad (3.8)$$

We now calculate the second derivative of a general power of f using (3.7) and (3.8). These are the terms that will give the divergences of derivatives of the Riemann tensor because elsewhere w only appears raised to the β power. For reasons that will soon be clear the first two terms in the expansion will be kept

$$(f^n)^{\cdot\cdot}|_h = k^2\mu^{4/n-2}a^{-n}w^{n-2\beta}n\{(n-\beta) + \frac{w}{a}(n+1)(n+1-\beta) + O([w/a]^2)\} \ . \qquad (3.9)$$

There are several properties of this that we will use. The most important feature of this is that it will diverge at the horizon if $\beta > 1$ (i.e., $d > 4$) and $n < 2\beta$. Also, the larger n is the less divergent (3.9) is, with one exception. The leading term for the second derivative of f^β and $f^{\beta+1}$ are both of order $w^{1-\beta} = w^{4-d}$ near the horizon. By inspection of (3.3), (3.5), and (3.9), we see that these will give the leading-order terms in the second derivatives of (3.3). For $R_{\hat{x}\hat{y}\hat{x}\hat{y}}$ these cancel to leading order, so the $f^{\beta+2}$ term must also be considered. It also suggests the possibility that these second derivatives will diverge at the horizon if $d \geq 5$. One can also see from (3.9) that, as one may have anticipated, the coefficient of any divergent terms will decrease as the separation between the black holes increases, because of the negative power of a. To confirm that there is a divergence we must add the leading-order contributions and see if their coefficients add to some nonzero value, which is indeed what they do.

Taking the second derivatives of (3.2) and only keeping the leading-order term as w approaches zero gives (an alternate notation one may want to use for these is $\nabla_{\hat{t}}\nabla_{\hat{t}}R_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}$ or $\nabla_{\mathbf{u}}\nabla_{\mathbf{u}}R_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}$ where all coordinates refer to those in the free falling frame)

$$\ddot{R}_{\hat{t}\hat{w}\hat{t}\hat{w}}|_h = -k^2\frac{1}{a}\left(\frac{M}{a}\right)^\beta \mu^{\beta-4}w^{4-d}(d-3)(d-2) \times(d-1)(3d-8) \ ,$$

$$\ddot{R}_{\hat{t}\hat{x}\hat{t}\hat{x}}|_h = k^2\frac{1}{a}\left(\frac{M}{a}\right)^\beta \mu^{\beta-4}w^{4-d}(d-3)(d-2)(d-1) = -\ddot{R}_{\hat{w}\hat{x}\hat{w}\hat{x}}|_h \ ,$$

$$\ddot{R}_{\hat{x}\hat{y}\hat{x}\hat{y}}|_h = -k^2\frac{1}{a^2}\left(\frac{M}{a}\right)^\beta \mu^{\beta-4}w^{5-d}2d(d-1) \ , \qquad (3.10)$$

if $d \geq 5$ the first two clearly diverge at the horizon. This demonstrates that the metric is not C^∞ at the horizon. This is the main result of this paper, that multiple-black-hole solutions in five or more dimensions need not have smooth horizons.

C. Solutions with three black holes

We not briefly consider three-black-hole solutions. To do this we evaluate the components of the Riemann tensor (3.2) using H as given in (2.7), with $M_2 \neq 0$. Because the behavior of the geodesics is dominated by the black hole at $\rho = 0$ we again use (3.7) and (3.8) to get the second derivatives of the Riemann tensor. Doing this to obtain $\ddot{R}_{\hat{t}\hat{w}\hat{t}\hat{w}}$ we find that the leading-order term is the same as that of (3.10), but with

$$\left(\frac{M}{a}\right)^\beta \frac{1}{a} \to \left(\frac{M}{a}\right)^\beta \frac{1}{a} - \left(\frac{M_2}{a_2}\right)^\beta \frac{1}{a_2} \ . \qquad (3.11)$$

By the following simple argument one can see that this will be the case for all components of (3.10).

First notice that to find the leading-order behavior of the second derivatives we only need to consider terms in $R_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}|_h$ that are proportional to $w^{\beta+1}$. In the two-black-hole solution these come from the f^β and the $f^{\beta+1}$ terms in (3.3).

Consider the $f^{\beta+1}$ terms in (3.3), they arise from the $(\partial_w H)^2$ terms in (3.2). The analogous terms in the three-black-hole solution are cross terms we get by squaring;

$$\partial_w H = -\beta\mu^\beta w^{-\beta-1} + \beta M^\beta(a-w)^{-\beta-1} - \beta M_2^\beta(a_2+w)^{-\beta-1} \qquad (3.12)$$

and then multiplying by the $w^{2-2\beta}$ factor we get by factoring a w^β out of the H prefactor in (3.2) [as we did in the two-black-hole solution when going from (3.2) to (3.3)]. The result is that, keeping only the terms that will give the leading-order divergences when derivatives are taken, we now have

$$M^\beta\left(\frac{w}{a-w}\right)^{\beta+1} - M_2^\beta\left(\frac{w}{a_2+w}\right)^{\beta+1} \qquad (3.13)$$

in the equations for the Riemann tensor components where we had $M^\beta f^{\beta+1}$ terms in (3.3). As we approach the horizon this becomes

$$\left[\left(\frac{M}{a}\right)^{\beta}\frac{1}{a}-\left(\frac{M_2}{a_2}\right)^{\beta}\frac{1}{a_2}\right]w^{\beta+1}+O(w^{\beta+2})\ . \quad (3.14)$$

This shows that some of the $w^{\beta+1}$ terms in $R_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}$ for the three-black-hole solution can be obtained from those of the two-black-hole solution by making the substitution (3.11). We will now show that the remaining $w^{\beta+1}$ terms in the three-black-hole solution can also be obtained in this way.

The other sources of $w^{\beta+1}$ terms in the two-black-hole solution are the f^{β} terms in (3.3) and (3.5). These $w^{\beta+1}$ terms come from the first correction to the leading term in the expansion of f^{β}. In the two-black-hole solution the f^{β} terms come from the H prefactor and the $\partial_i^2 H$ terms in (3.2), when the w^{β} term is factored out of the H prefactor. In the three-black-hole solution the terms analogous to f^{β} will have the same sign for the contributions from the second and third blackholes. In other words, the three-black-hole solution will have (again ignoring terms that may lead to lower-order divergences)

$$M^{\beta}\left(\frac{w}{a-w}\right)^{\beta}+M_2^{\beta}\left(\frac{w}{a_2+w}\right)^{\beta}\ , \quad (3.15)$$

where $M^{\beta}f^{\beta}$ apepars in (3.3) and (3.5). The leading-order term in (3.15) is proportional to w^{β} and all order derivatives of this are finite. The first correction to this gives $w^{\beta+1}$ terms, that are in fact the same as those given by (3.14). This, along with (3.14) itself, shows that all of the $w^{\beta+1}$ terms in $R_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}|_h$ for the three-black-hole solution can be obtained from those in the two-black-hole solution by making the replacement (3.11).

When we take the second derivatives of the Riemann tensor components the $w^{\beta+1}$ terms will give the leading-order contribution. Therefore, $\ddot{R}_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}|_h$ for the three-black-hole solution is given by (3.10), but with the replacement (3.11) for all components. If we take $M_2 = M$ and $a_2 = a$, then we find that what was the leading-order divergence for each component vanishes.

In five dimensions these are the only terms that diverge; therefore, the second derivatives of the Riemann tensor components are well behaved, at the horizon of the central black hole. In fact, as long as we approach the central black hole along the axis connecting it with the outer black holes, all order derivatives of the Riemann tensor components will be bounded. The reason is that along the w axis these components are functions of w^2 and in five dimensions taking derivatives lowers the power of w by integer multiples of two. The same argument that was used to demonstrate that single-black-hole solutions are smooth can now be used to show this result. Nevertheless, it is possible that higher-order derivatives of $R_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}$ will still diverge as we approach the horizon from other directions. One should also note that this is just one component of the event horizon of the spacetime. If instead we consider the component of the horizon surrounding one of the outer two black holes, then the second derivative of the Riemann tensor will still diverge there. The reason is that to calculate this we can take $M_2 = M$ and $a_2 = -2a$ in (2.7) and then repeat the previous analysis [also taking $\mu \to M$ in (3.7)]. This will give divergences like those in (3.10), with the coefficient slightly changed.

In more than five dimensions there are lower-order divergences in addition to the leading-order ones in (3.10). One would expect that the cancellation obtained by taking $M_2 = M$ and $a_2 = a$ would only occur in the leading-order terms and not in all the corrections to them. To see that this is the case and that there are still divergences of $\ddot{R}_{\hat{\alpha}\hat{\beta}\hat{\sigma}\hat{\gamma}}$ we choose to use the fact that if we take $M_2 = M$ and $a_2 = a$, then the symmetry of our three-black-hole configuration allows us to easily find timelike geodesics along a transverse axis. We can then repeat what was done for timelike geodesic along the w axis. Doing this for the $R_{\hat{t}\hat{x}\hat{t}\hat{x}}$ component yields

$$\ddot{R}_{\hat{t}\hat{x}\hat{t}\hat{x}}|_h = k^2\left(\frac{M}{a}\right)^{\beta}\frac{1}{a^2}\mu^{\beta-4}x^{5-d}\beta(\beta+2) \times(5\beta^2+21\beta+12)\ , \quad (3.16)$$

where x is the transverse spatial direction that the geodesic travels along. In five dimensions the second derivative is finite at the horizon, as was the case for geodesics along the w axis. In more than five dimensions this diverges at the horizon. This confirms that the second derivatives of the Riemann tensor still diverge if $d > 5$, albeit less severely than the two-black-hole case. As claimed, it is indeed only the leading-order divergences that vanish and the other divergences are still present for $d > 5$. Presumably these divergences could be removed by adding a sufficient number of additional black holes.

IV. DISCUSSION

The primary purpose of this paper was to examine the smoothness of event horizons when there is more than one black hole. This was done for static configurations of extremely charged black holes. Two classes of such solutions were considered: general two-black-hole solutions and solutions with three collinear black holes. The components of the Riemann tensor were evaluated in an orthonormal basis that was parallelly propagated along a timelike geodesic through one of the horizons. While these components are well behaved, in more than four dimensions some of their derivatives diverge on the horizon. This shows that these multi-black-hole solutions have nonsmooth horizons, thus confirming the conjecture of [7].

The results obtained here are similar in many ways to those of [6]. Both have rather mild singularities that allow geodesics to be extended through the black-hole horizons, and both demonstrated that by adding more black holes with the proper masses and coordinates the differentiability of the solutions can be improved (the demonstration in [6] was much more general). There are, however, substantial differences. In [6] the cosmological constant is nonzero and the solutions are four dimensional. In the present work the cosmological constant is zero, the divergences only occur in more than four dimensions and are more mild than those of [6]. Perhaps the biggest difference is that the solutions considered in [6] are dynamical and the singularities were attributed to the presence of electromagnetic and gravitational radiation. In this paper the solutions are static and their lack of smoothness can have no such cause.

Other single-black-hole solutions that can be made into static multi-black-hole solutions by using methods such as that of section two were derived in [8]. These solutions have a dilaton in addition to the $(d-2)$-form. In particular, their five-dimensional solution is a solution to the low-energy string equations with F_{d-2} being the familiar antisymmetric tensor field from string theory. While these solutions have Riemann tensor components similar to those presented here, it takes an infinite proper time to reach the horizons for these solutions, so they will not have singularities like those seen here.

One might question the significance of the second-, or higher-, order derivatives of the Riemann tensor diverging. One might consider it analogous to the situation in quantum mechanics where it is one thing to say the operator $\hat{x}^3\hat{p}+\hat{p}\hat{x}^3$ is Hermitian, and therefore an observable. However, it is quite another thing to say how one would actually measure it. Nevertheless, the fact that the horizon of a single black hole is smooth and adding another black hole anywhere, no matter how small its mass, spoils this smoothness, is quite surprising. It is also possible that if a string fell into the horizon of one of these multi-black-hole solutions then some coupling of it to the derivatives of the curvature would cause it to behave in a way that has interesting physical consequences.

It may seem strange that multi-black-hole solutions in four dimensions have smooth horizons, while those in higher dimensions have horizons with finite differentiability. There is no obvious reason why increasing the spacetime dimension from four to five (or more) would cause this change. Understanding this is likely to be the key to obtaining a physical explanation of why the higher-dimensional solutions have nonsmooth horizons.

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