Black hole formation by sine-Gordon solitons in two-dimensional dilaton gravity

Hak-Soo Shin* and Kwang-Sup Soh†
Department of Physics Education, Seoul National University, Seoul 151-742, Korea
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The CGHS model of two-dimensional dilaton gravity coupled to a sine-Gordon matter field is considered. The theory is exactly solvable classically, and the solutions of kink- and two-kink-type solitons are studied in connection with black hole formation.

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I. INTRODUCTION

The two-dimensional dilaton gravity coupled with scalar matter fields proposed by Callan, Giddings, Harvey, and Strominger (CGHS) [1] has been extensively studied with the aim of gleaning useful information about black hole formation and evaporation. In spite of the initial high hopes, the theory turned out to be intractable even in semiclassical approximations [2–4], let alone in a full quantum analysis.

In the CGHS model [1] the scalar matter was free fields, and a black hole was formed by a shock wave of the free scalar fields. Black hole formation by an interacting scalar field has not been considered. On the other hand, in two-dimensional spacetime there has been much interest in integrable models of nonlinear partial differential equations, and especially in the soliton solutions [5]. In particular, the sine-Gordon theory of the interacting scalar fields provides a good example of solitons and their scatterings [6].

In this paper we consider a sine-Gordon-type matter field coupled with dilaton gravity, and investigate black hole formation by solitons. In Sec. II we introduce the model by giving the action and gauge fixing, and in Sec. III we study the black hole geometry formed by a kink-type soliton. Scattering of two solitons is considered in Sec. IV, and brief discussions are given in the last section.

II. ACTION AND GAUGE FIXING

We begin with the action in two spacetime dimensions:

$$S=\frac{1}{2\pi}\int_M d^2x\sqrt{-g}\left[e^{-2\phi}[R+4(\nabla\phi)^2+4\lambda^2]-\frac{1}{2}(\nabla f)^2+4\mu^2(\cos f-1)e^{-2\phi}\right], \tag{1}$$

where g, ϕ, and f are metric, dilaton, and matter fields, respectively, and λ^2 is a cosmological constant. This action is the CGHS action except the last sine-Gordon term which we added to study formation of black holes by solitons.

The classical theory described by (1) is most easily analyzed in the conformal gauge

$$ds^2=-e^{2\rho}dx^+dx^-, \tag{2}$$

where $x^\pm=t\pm x$. The action then reduces to

$$S=\frac{1}{\pi}\int d^2x\left[e^{-2\phi}(+2\partial_+\partial_-\rho-4\partial_+\phi\partial_-\phi+\lambda^2e^{2\rho})+\frac{1}{2}\partial_+f\partial_-f+\mu^2(\cos f-1)e^{2\rho-2\phi}\right], \tag{3}$$

and the metric equations of motion are

$$T_{++}=e^{-2\phi}(4\partial_+\rho\partial_+\phi-2\partial_+^2\phi)+\frac{1}{2}(\partial_+f)^2=0, \tag{4}$$

$$T_{--}=e^{-2\phi}(4\partial_-\rho\partial_-\phi-2\partial_-^2\phi)+\frac{1}{2}(\partial_-f)^2=0, \tag{5}$$

$$T_{+-}=e^{-2\phi}(2\partial_+\partial_-\phi-4\partial_+\phi\partial_-\phi-\lambda^2e^{2\rho})-\mu^2(\cos f-1)e^{2\rho-2\phi}=0. \tag{6}$$

The dilaton and matter equations are

$$(-4\partial_+\partial_-\phi+4\partial_+\phi\partial_-\phi+2\partial_+\partial_-\rho)e^{-2\phi}+[\lambda^2+\mu^2(\cos f-1)]e^{2\rho-2\phi}=0, \tag{7}$$

$$\partial_+\partial_-f+\mu^2\sin f e^{2(\rho-\phi)}=0. \tag{8}$$

Adding Eqs. (6) and (7) we get

$$\partial_+\partial_-(\rho-\phi)=0, \tag{9}$$

which has the general solution

$$\rho-\phi=w_+(x^+)+w_-(x^-). \tag{10}$$

The arbitrary functions w_+ and w_- can be eliminated by fixing the subconformal gauge freedom. From now on we will take the simplest gauge fixing such that $\omega_+=\omega_-=0$.

*Electronic address: hsshin@ibm3090.snu.ac.kr
†Electronic address: kssoh@phyb.snu.ac.kr

Since $\rho = \phi$ the field equations reduce to the following simple ones:

$$\partial_+^2(e^{-2\phi}) + \frac{1}{2}(\partial_+ f)^2 = 0, \quad (11)$$

$$\partial_-^2(e^{-2\phi}) + \frac{1}{2}(\partial_- f)^2 = 0, \quad (12)$$

$$\partial_+\partial_-(e^{-2\phi}) + \lambda^2 + \mu^2(\cos f - 1) = 0, \quad (13)$$

$$\partial_+\partial_- f + \mu^2 \sin f = 0, \quad (14)$$

whose solutions we will consider in the rest of this paper.

III. KINK SOLUTION AND BLACK HOLE FORMATION

The sine-Gordon equation (14) is well known to have a soliton solution [7]:

$$f(z) = 4\mathrm{arctan}\exp\left[2\frac{z - z_0}{\sqrt{1-v^2}}\right], \quad (15)$$

where

$$z = \mu(x + vt), \quad (16)$$

and the center of the soliton is at z_0. This is a traveling wave of kink type with velocity v. Another soliton solution of antikink type is given by

$$f_{\mathrm{antikink}}(z) = 4\mathrm{arctan}\exp\left[-2\frac{z - z_0}{\sqrt{1-v^2}}\right]. \quad (17)$$

In order to study black hole formation by a kink we solve Eqs. (11)–(13) with f given by (15). First, we note that

$$f(z) = 4\mathrm{arctan}\exp\left[\gamma_+ x^+ + \gamma_- x^- - \Delta_0\right], \quad (18)$$

where $\Delta_0 = \frac{2z_0}{\sqrt{1-v^2}}$,

$$\gamma_+ = \mu\sqrt{\frac{1+v}{1-v}}, \quad \gamma_- = -\mu\sqrt{\frac{1-v}{1+v}}, \quad (19)$$

and

$$\cos f = \frac{1 - 6\exp(2\Delta - 2\Delta_0) + \exp(4\Delta - 4\Delta_0)}{[1 + \exp(2\Delta - 2\Delta_0)]^2}, \quad (20)$$

where

$$\Delta \equiv \gamma_+ x^+ + \gamma_- x^- = \frac{2z}{\sqrt{1-v^2}}. \quad (21)$$

It is straightforward to show that Eq. (13),

$$\partial_+\partial_-(e^{-2\phi}) = -[\lambda^2 + \mu^2(\cos f - 1)], \quad (22)$$

can be integrated as

$$e^{-2\phi} = a(x^+) + b(x^-) - \lambda^2 x^+ x^- - 2\ln[1 + \exp(2\Delta - 2\Delta_0)], \quad (23)$$

where $a(x^+)$ and $b(x^-)$ are arbitrary functions which are to be determined by the constraint equations (11) and (12). Inserting the solutions (18) and (23) into (11) we obtain

$$\frac{d^2 a(x^+)}{dx^{+2}} = 0, \quad (24)$$

and similarly

$$\frac{d^2 b(x^-)}{dx^{-2}} = 0, \quad (25)$$

which yields

$$e^{-2\phi} = C + ax^+ + bx^- - \lambda^2 x^+ x^- - 2\ln[1 + \exp(2\Delta - 2\Delta_0)], \quad (26)$$

where a, b, and C are constants. By shifting the origin appropriately we can take $a = b = 0$. Hence the full solution of kink type is

$$f = 4\mathrm{arctan}\exp(\gamma_+ x^+ + \gamma_- x^- - \Delta_0), \quad (27)$$

$$e^{-2\rho} = e^{-2\phi} = C - \lambda^2 x^+ x^- - 2\ln[1 + \exp 2(\gamma_+ x^+ + \gamma_- x^- - \Delta_0)], \quad (28)$$

where $\gamma_+ = \mu\sqrt{\frac{1+v}{1-v}}$, $\gamma_+\gamma_- = -\mu^2$, and C and Δ_0 are constants. An antikink solution is similarly obtainable and the result is simply given by replacing $(\gamma_+ x^+ + \gamma_- x^- - \Delta_0)$ by $-(\gamma_+ x^+ + \gamma_- x^- - \Delta_0)$ in Eqs. (27) and (28).

The geometry can be most easily analyzed by dividing the spacetime into three regions: $\Delta - \Delta_0 \ll -1$, $\Delta - \Delta_0 \simeq 0$, and $\Delta - \Delta_0 \gg 1$. In the first region ($\Delta - \Delta_0 \ll -1$) we ignore the exponential term in (28), and we have

$$e^{-2\rho} \simeq -\lambda^2 x^+ x^-, \quad (29)$$

where we take the constant $C = 0$. It is simply the linear dilaton vacuum. In the third region ($\Delta - \Delta_0 \gg 1$), we have

$$\begin{aligned} e^{-2\rho} &\simeq -\lambda^2 x^+ x^- - 4(\gamma_+ x^+ + \gamma_- x^- - \Delta_0) \\ &= 4\left(\Delta_0 - \frac{4\mu^2}{\lambda^2}\right) - \lambda^2\left(x^+ + \frac{4\gamma_-}{\lambda^2}\right)\left(x^- + \frac{4\gamma_+}{\lambda^2}\right), \end{aligned} \quad (30)$$

which is the geometry of a black hole of mass $4\lambda(\Delta_0 - \frac{4\mu^2}{\lambda^2})$ after shifting x^+ by $\frac{4\gamma_-}{\lambda^2}$, and x^- by $\frac{4\gamma_+}{\lambda^2}$. The two solutions are joined along the soliton wave. At the center of the soliton ($\Delta - \Delta_0 \simeq 0$) we have

$$e^{-2\rho} \simeq 2\left(\Delta_0 - \ln 2 - \frac{2\mu^2}{\lambda^2}\right) - \lambda^2\left(x^+ + \frac{2\gamma_-}{\lambda^2}\right)\left(x^- + \frac{2\gamma_+}{\lambda^2}\right) \quad (31)$$

which is again the geometry of a black hole of mass $2\lambda(\Delta_0 - \ln 2 - \frac{2\mu^2}{\lambda^2})$.

This analysis shows that a black hole is formed following the input of the soliton wave. The position of its apparent horizon, $\partial_+\phi = 0$, is given by

$$\lambda^2 x^- + \frac{4\gamma_+}{1+\exp[-2(\Delta-\Delta_0)]} = 0, \tag{32}$$

which, at the region $\Delta - \Delta_0 \gg 1$, is simply $x^- = -4\frac{\gamma_+}{\lambda^2}$, which coincides with the event horizon of the black hole, and, at the center of the soliton wave, $x^- = -2\frac{\gamma_+}{\lambda^2}$.

In a relativistic soliton ($v \to 1$) we obtain the shock wave geometry of CGHS [1] as

$$e^{-2\rho} \to \begin{cases} -\lambda^2 x^+ x^-, & x^+ - x_0^+ < 0, \\ -\lambda^2 x^+ x^- - 4\mu\sqrt{\frac{1+v}{1-v}}(x^+ - x_0^+), & x^+ - x_0^+ > 0, \end{cases} \tag{33}$$

which coincides with the solution of CGHS when the magnitude of the shock wave a is $4\mu\sqrt{\frac{1+v}{1-v}}$.

IV. TWO-KINK SOLUTION

Kinks emerge unscathed from collision, suffering only a phase shift. A two-kink solution that demonstrates this property was derived by Perring and Skyrme [8], and takes the form

$$f(x,t) = 4\arctan\left[\frac{v\sinh(\frac{\mu}{\sqrt{1-v^2}}x)}{\cosh(\frac{\mu}{\sqrt{1-v^2}}vt)}\right]. \tag{34}$$

The limit $t \longrightarrow -\infty$ yields

$$\lim_{t\to-\infty} f(x,t) = 4\arctan\left[\exp\left(\frac{\mu(x+vt-\delta)}{\sqrt{1-v^2}}\right) - \exp\left(-\frac{\mu(x-vt+\delta)}{\sqrt{1-v^2}}\right)\right], \tag{35}$$

which represents a kink and an antikink that are separated very far and approaching each other with the same speed v. Here δ is a phase shift given as

$$\delta = \sqrt{1-v^2}\ln\frac{1}{v}. \tag{36}$$

The kink and antikink collide, emerge again, and run away from each other as $t \longrightarrow \infty$:

$$\lim_{t\to\infty} f(x,t) = 4\arctan\left[-\exp\left(-\frac{\mu(x+vt+\delta)}{\sqrt{1-v^2}}\right) + \exp\left(\frac{\mu(x-vt-\delta)}{\sqrt{1-v^2}}\right)\right]. \tag{37}$$

For the two-kink solution the metric function is obtained as

$$e^{-2\rho} = C + a(x^+) + b(x^-) - \lambda^2 x^+ x^- - 2\ln[\cosh^2\beta t + v^2\sinh^2\gamma x], \tag{38}$$

where

$$\beta = \frac{\mu v}{\sqrt{1-v^2}}, \qquad \gamma = \frac{\mu}{\sqrt{1-v^2}}. \tag{39}$$

By inserting this solution (38) into (11) and (12) we find that

$$e^{-2\rho} = C + ax^+ + bx^- - \lambda^2 x^+ x^- - 2\ln[\cosh^2\beta t + v^2\sinh^2\gamma x]. \tag{40}$$

Shifting the origin of the coordinates appropriately we have the full solution associated with the two-kink soliton as

$$f(x,t) = 4\arctan\left[\frac{v\sinh(\frac{\mu(x-x_0)}{\sqrt{1-v^2}})}{\cosh(\frac{[\mu v(t-t_0)]}{\sqrt{1-v^2}})}\right], \tag{41}$$

$$e^{-2\phi} = e^{-2\rho} = C - \lambda^2 x^+ x^- - 2\ln\left[\cosh^2\left(\frac{\mu v(t-t_0)}{\sqrt{1-v^2}}\right) + v^2\sinh^2\left(\frac{\mu(x-x_0)}{\sqrt{1-v^2}}\right)\right], \tag{42}$$

where we restored x_0 and t_0 which we had omitted for convenience. The constant C is to be determined from the condition that the region of spacetime which is not affected by the incoming soliton is a linear dilaton vacuum. For this we consider the region

$$\gamma(x-x_0) + \beta(t-t_0) \ll -1, \\ -\gamma(x-x_0) + \beta(t-t_0) \ll -1, \ \beta(t-t_0) \ll -1, \tag{43}$$

in which region the metric function becomes

$$e^{-2\rho} \to C - \lambda^2 x^+ x^- - 2\ln\left[\frac{e^{-\beta[(x^+-x_0^+)+(x^--x_0^-)]}}{4}\right] = C + 4\ln 2 - 2\beta(x_0^+ + x_0^-) + 4\frac{\beta^2}{\lambda^2} - \lambda^2\left(x^+ - \frac{2\beta}{\lambda^2}\right)\left(x^- - \frac{2\beta}{\lambda^2}\right). \tag{44}$$

In order to have a linear dilaton vacuum we take C as

$$C = 4\beta t_0 - 4\left(\ln 2 + \frac{\beta^2}{\lambda^2}\right), \tag{45}$$

such that, in this region,

$$e^{-2\rho} = -\lambda^2 \left(x^+ - \frac{2\beta}{\lambda^2}\right)\left(x^- - \frac{2\beta}{\lambda^2}\right). \tag{46}$$

With this C we have the solution of two approaching solitons into a region of the linear dilaton vacuum.

The singularity of the curvature is easily located by considering the following three regions separately: (R_1) $\gamma(x-x_0)-\beta(t-t_0) \ll \gamma(x-x_0)+\beta(t-t_0) \ll -1$, $\beta(t-t_0) \gg 1$, (R_2) $\gamma(x-x_0)-\beta(t-t_0) \gg 1$, $\gamma(x-x_0)-\beta(t-t_0) \ll -1$, $\beta(t-t_0) \gg 1$, and (R_3) $\gamma(x-x_0)+\beta(t-t_0) \gg \gamma(x-x_0)-\beta(t-t_0) \gg 1$, $\beta(t-t_0) \gg 1$. In these three regions we have

$$e^{-2\rho} = \begin{cases} \left(4(\beta t_0 - \gamma x_0) - 2\ln v^2 - 4\frac{\beta^2+\gamma^2}{\lambda^2}\right) - \lambda^2(x^+ + \frac{2\gamma}{\lambda^2})(x^- - \frac{2\gamma}{\lambda^2}) & (R_1), \\ 8\beta t_0 - \lambda^2(x^+ + \frac{2\gamma}{\lambda^2})(x^- + \frac{2\gamma}{\lambda^2}) & (R_2), \\ \left(4(\beta t_0 + \gamma x_0) - 2\ln v^2 - 4\frac{\beta^2+\gamma^2}{\lambda^2}\right) - \lambda^2(x^+ - \frac{2\gamma}{\lambda^2})(x^- + \frac{2\gamma}{\lambda^2}) & (R_3), \end{cases} \tag{47}$$

which represent the black hole metrics in each sector. The singularity of each region are obtained from the equation $e^{-2\rho}=0$, and we see that the effective masses of the black hole in each region is different from one another. There are two event horizons $x^- = -\frac{2\gamma}{\lambda^2}$ and $x^+ = -\frac{2\gamma}{\lambda^2}$.

V. DISCUSSION

There are other soliton solutions besides the simple ones we have considered in this paper [6]. In connection with integrable nonlinear systems, research on solitons is one of the most interesting and active areas. It will be quite fruitful to combine the knowledge on solitons with theories of two-dimensional gravity, and in particular with black hole physics.

The next step is to consider Hawking radiation, a conformal anomaly, and semiclassical analysis, etc. We plan to deal with these topics in a separate article using semiclassical models that have explicit analytic solutions such as those proposed by Bilal and Callan [9], de Alwis [10], and Russo, Susskind, and Thorlacius [3,11].

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[1] C. G. Callan, Jr., S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **45**, R1005 (1992).

[2] S. W. Hawking, Phys. Rev. Lett. **69**, 406 (1992); B. Birnir, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **46**, 638 (1992); T. Banks, A. Dabholkar, M.R. Douglas, and M. O' Loughlin, *ibid.* **45**, 3607 (1992); J.G. Russo, L. Susskind, and L. Thorlacius, Phys. Lett. B **292**, 13 (1992); D. A. Lowe, Phys. Rev. D **47**, 2446 (1993); T. Piran and A. Strominger, *ibid.* **48**, 4729 (1993).

[3] J.G. Russo, L. Susskind, and L. Thorlacius, Phys. Rev. D **46**, 3444 (1992); **47**, 533 (1993).

[4] A. Mikovic, Phys. Lett. B **291**, 19 (1992); J. Navarro-Salas, M. Navarro, and V. Aldaya, Nucl. Phys. B **403**, 291 (1993).

[5] R. Jackiw, Rev. Mod. Phys. **49**, 681 (1977); R. Rajarman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982); E. Abdalla, M. Cristina Abdalla, and K. Rothe, *Nonperturbative Methods in Two-Dimensional Quantum Field Theory* (World Scientific, Singapore, 1991).

[6] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **30**, 1262 (1973); R. Dodd, J. Eilbeck, J. Gibbon, and H. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, New York, 1982).

[7] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics* (John Wiley & Sons, New York, 1989).

[8] K. K. Perring and T. H. R. Skyrme, Nucl. Phys. **31**, 550 (1962).

[9] A. Bilal and C. G. Callan, Nucl. Phys. **B394**, 73 (1993).

[10] S. P. de Alwis, Phys. Lett. B **289**, 278 (1992); **300**, 330 (1993); Phys. Rev. D **46**, 5429 (1992).

[11] L. Thorlacius, "Black hole evolution," Institute for Theoretical Physics, Report No. NSF-ITP-94-109, hep-th 9411020, 1994.