Static axisymmetric approach for the head-on collision of two black holes

M. E. Araújo*
Departamento de Matemática, Universidade de Brasília, 70.919-000 Brasília DF, Brazil

S. R. Oliveira†
Departamento de Física, Universidade de Brasília, 70.910-900 Brasília DF, Brazil

(Received 9 December 1994)

This paper presents a semianalytical approach to the interaction of two (originally) spherically symmetric black holes through a head-on collision process. It is shown that an expression for the rate of emission of gravitational radiation can be derived from the so-called Weyl potential. The total output of gravitational wave energy released is then calculated and the results are compared to recent numerical investigations of this problem.

PACS number(s): 04.25.−g, 04.30.−w, 04.70.Bw, 97.60.Lf

I. INTRODUCTION

The two-body problem is one of the most outstanding problems in general relativity. There are no exact solutions to Einstein's field equations describing the spacetime geometry for such configurations. So far, because of the difficulty in treating the field equations analytically, the problem has mainly been treated either by numerical methods (see [1,2] and references therein) or by post-Newtonian approximations [3].

In this paper we give a semianalytical approach to the head-on collision problem. We begin by considering an exact solution of Einstein's field equations found by Weyl [4]. This solution describes a geometry that can be interpreted as a static axisymmetric spacetime with two black holes plus a conical singularity between them. It is possible to obtain the "force" of attraction between the holes and consequently their "acceleration" [5]. Using this approach, at least the instantaneously static acceleration is being taken care of. Furthermore, it has been shown, in a perturbation theory context [6], that the gravitational wave emission of boosted perturbed black holes is very similar to that of the static ones.

We then use these expressions to calculate the rate of emission and the total amount of gravitational wave energy emitted in the process using the standard quadrupole formulas and assuming that Newtonian mechanics rules the black hole motion. Our results are in good agreement with those recently obtained by Anninos *et al.* using a numerical approach [7]. This agreement seems to point out that the static acceleration contribution is the most important one for this head-on collision problem. We emphasize that this is *a posteriori* justification to the series of simplifying assumptions, yet physically plausible, used in this paper.

*Electronic address: marcelo@mat.unb.br
†Electronic address: samuel@fis.unb.br

The plan for this paper is as follows. In Sec. II we present the Weyl solution. In Sec. III the equation of motion for the holes is obtained. In Sec. IV an expression for the rate of emission of gravitational radiation is derived and the total output of gravitational wave energy is calculated. In Sec. V we discuss our results.

II. WEYL STATIC SPACETIME

In 1917 Weyl published one of the first exact solutions of Einstein's field equations. This solution is a particular case of the class of static axisymmetric vacuum spacetimes that nowadays is known as Weyl class [8]. Other exact solutions of Weyl's class followed, in particular the 1924 Curzon solution [9]. This solution caused some controversy because it was interpreted as a vacuum static metric having two fixed bodies on the axis of symmetry as its source. Silberstein [10] promptly pointed out the problem and naively claimed that general relativity was an unphysical theory. Then Einstein himself and Rosen [11] solved the puzzle showing that the Curzon's solution actually does not have a vacuum between the bodies because there is an angle deficit when one goes around the axis of symmetry in a plane perpendicular to the line linking the bodies. Thus, in general relativity, a two-body system in static equilibrium must have some sort of strut between them, in this case, a conical singularity. Therefore, in principle, one could look for a solution that may be interpreted as two black holes with a "strut" holding them apart.

The line element of a vacuum, static axisymmetric space-time, in "isothermal and conformal" coordinates, is given by [12–14]

$$ds^2 = e^{2\psi}\, dt^2 - e^{-2\psi}\, \left[e^{2\nu}\, (d\rho^2 + dz^2) + \rho^2 d\phi^2\right], \quad (1)$$

where $\psi = \psi(\rho, z)$, $\nu = \nu(\rho, z)$, $0 < \rho < \infty$, $-\infty < z < \infty$, and $0 < \phi < 2\pi$ [4]. Einstein's field equations are

$$\nabla^2\psi \equiv \psi_{\rho\rho} + \frac{\psi_\rho}{\rho} + \psi_{zz} = 0, \quad (2)$$

$$\nu_\rho = \rho(\psi_\rho^2 - \psi_z^2), \tag{3}$$

$$\nu_z = 2\rho\psi_\rho\psi_z \,, \tag{4}$$

where partial derivatives are denoted by subscripts. One can first solve Laplace's equation (2) for ψ and then solve the quadratures (3) and (4) for ν. Asymptotically ψ plays the role of a Newtonian potential.

Since Eq. (2) is elliptic, if one imposes the boundary conditions $\psi \to 0$ and $\nu \to 0$ for $(\rho, z) \to (\infty, \pm\infty)$, which corresponds to saying that the spacetime is asymptotically flat, then the trivial solution is the only possibility. But one can get a nontrivial solution when Laplace's equation (2) is replaced by a Poisson equation with some localized source on the right-hand side of (2). We refer to it as the Newtonian image of the sources of the Weyl spacetime [15,16].

It can be shown that there is a solution in Weyl's class that is isometric to the Schwarzschild solution. That is, the single black hole spacetime is included in Weyl's class [8].

We start by noting that the Newtonian potential for a finite rod of length L located on the z axis and centered at the origin is given by

$$\psi = \frac{m}{L} \ln\left[\frac{R_+ + R_- - L}{R_+ + R_- + L}\right], \tag{5}$$

where

$$R_\pm = \sqrt{\rho^2 + \left(z \mp \frac{L}{2}\right)^2}\,, \tag{6}$$

and m is the mass of this Newtonian bar. Note that ψ is a solution of Laplace's equation (2) subject to the condition $R_+ + R_- - L > 0$. So the solutions to the vacuum Einstein's equations (2), (3), and (4) are subjected to the condition that ρ and z satisfy this inequality.

From (3), (4), and (5) we obtain

$$\nu = 2\left(\frac{m}{L}\right)^2 \ln\left[\frac{(R_+ + R_-)^2 - L^2}{4R_+R_-}\right]. \tag{7}$$

Setting $L = 2M$ and performing the coordinate transformation

$$\rho^2 = \left(r^2 - L\, r\right)\sin^2\theta\,, \tag{8}$$

$$z = (r - L/2)\cos\,\theta \tag{9}$$

puts the line element into the standard Schwarzschild form [5]. Note that in these coordinates the horizon is given by $\rho = 0, -M \le z \le M$ and also that this coordinate transformation applies only to the static region of the Schwarzschild solution.

Since Laplace's equation (2) is linear, one can superpose two solutions, each of them corresponding to the potential of a Newtonian bar. Having done that, one can then solve Eqs. (3) and (4) to obtain what can be interpreted as a two-black-hole solution. In what follows we explicitly show how this superposition can be achieved.

Let m and m' be the masses of two Newtonian bars of lengths $2m$ and $2m'$ sitting on the z axis and centered at $Z_0 + m$ and $-Z_0 - m'$, respectively. The Newtonian potential for this configuration is a solution of Laplace's equation (2) given by

$$\psi = \frac{1}{2} \ln\left[\frac{R_+ + R_- - 2m}{R_+ + R_- + 2m}\right] + \frac{1}{2} \ln\left[\frac{R'_+ + R'_- - 2m'}{R'_+ + R'_- + 2m'}\right], \tag{10}$$

where

$$R_\pm = \sqrt{\rho^2 + [z - (Z_0 + m) \mp m]^2}\,, \tag{11}$$

$$R'_\pm = \sqrt{\rho^2 + [z + (Z_0 + m') \mp m']^2}. \tag{12}$$

Substituting (10) into (3) and (4) and integrating them yields

$$\nu = \nu[\psi] + \nu[\psi'] + 2\sigma[\psi, \psi'], \tag{13}$$

where ψ, $\nu[\psi]$ and ψ', $\nu[\psi']$ are solutions to the single-bar problem and the interaction term is given by

$$\sigma[\psi, \psi'] = \tfrac{1}{2}\ln\left[\frac{(m'+Z_0)R_+ + (m+m'+Z_0)R_- - mR'_-}{Z_0R_+ + (m+Z_0)R_- - mR'_+}\right] + \tfrac{1}{2}\ln\left[\frac{Z_0}{Z_0+m'}\right]. \tag{14}$$

The last term is an integration constant that has been chosen so that $\nu(\rho, z)$ tends to zero far away from the bars.

Note that at $\rho = 0$ the interaction term (14) does not vanish between the two bars, being given by

$$\nu(0, z) = 2\sigma(0, z) = \ln\left[1 - \frac{m\, m'}{(Z_0 + m)(Z_0 + m')}\right] \tag{15}$$

for $|z| < Z_0$. This term generates a conical singularity [17,18] which can be interpreted as a cosmic-string-like object [19] (a strut) that holds the two bodies apart [5].

The conical singularity appearing in the plane orthogonal to the z axis can be obtained through the application of the Gauss-Bonet theorem to the two-dimensional hypersurface obtained by fixing t and z such that $|z| < Z_0$. It has been shown [17] that the Gaussian curvature of this two-dimensional hypersurface is given by

$$K = 2\pi\left(1 - e^{-\nu(\rho,z)}\right)\delta_2(\rho)\,, \tag{16}$$

where $\delta_2(\rho)$ is defined implicitly by

$$\int_0^\infty \int_0^{2\pi} \delta_2(\rho) e^{2\psi - \nu} \rho d\rho d\phi = 1. \tag{17}$$

Since the metric in the two-dimensional manifold considered above is diagonal, the only nonvanishing components of the Ricci tensor are given by [20,21]

$$R^\rho_\rho = R^\phi_\phi = K = 2\pi(e^{-\nu(\rho,z)} - 1)\delta_2(\rho). \tag{18}$$

Assuming that in the four-dimensional case these components of the Ricci tensor are also the only nonvanishing ones and using Einstein's field equations

$$R^{\mu}_{\nu} - \frac{1}{2} g^{\mu}_{\nu} R = -8\pi T^{\mu}_{\nu}, \tag{19}$$

we obtain the following nonvanishing components of the energy-momentum tensor:

$$T^t_t = T^z_z = \frac{e^{-\nu(\rho,z)} - 1}{4} \delta_2(\rho). \tag{20}$$

The compression "force" on a plane perpendicular to the z axis can be calculated [22] by integrating T^z_z on the disk centered at $\rho = 0$; that is,

$$F_z = \int \int_{\text{disk}} T^z_z d\sigma = \frac{1}{4}(e^{-\nu(0,z)} - 1). \tag{21}$$

Substituting (15) into (21) yields the tension on the strut by the two black holes [5]: namely,

$$F_z = \frac{m\, m'}{d^2 - (m + m')^2}, \tag{22}$$

where $d \equiv 2Z_0 + m + m' > m + m'$ is the coordinate distance between the centers of the black holes. We observe that the force is divergent if the horizons touch each other ($Z_0 = 0$) and also that for large distances, i.e., $d \gg m + m'$, (22) takes the form of the Newtonian force between two point particles.

III. EQUATIONS OF MOTION

We now investigate the time evolution of a two-body system interacting through the Weyl force (22). We make the assumption that Newton's equation of motion with Weyl's force is adequate to describe the black holes motion. We then write

$$m_a \frac{d^2\mathbf{r}_a}{dt^2} = -\frac{m_a m_b}{r^2_{ab} - (m_a + m_b)^2} \frac{\mathbf{r}_{ab}}{r_{ab}}, \tag{23}$$

$$m_b \frac{d^2\mathbf{r}_b}{dt^2} = +\frac{m_a m_b}{r^2_{ab} - (m_a + m_b)^2} \frac{\mathbf{r}_{ab}}{r_{ab}}, \tag{24}$$

where $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$ and $\mathbf{r}_a$, $\mathbf{r}_b$ are the position vectors for the particles with masses m_a, m_b, respectively.

We now have a Newtonian two-body problem that can be transformed into a central force problem in the usual way. Then Eqs. (23) and (24) are transformed into

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{M}{r^2 - M^2} \frac{\mathbf{r}}{r} \tag{25}$$

and $d^2\mathbf{R}/dt^2 = 0$, where $M\mathbf{R} = m_a\mathbf{r}_a + m_b\mathbf{r}_b$, $\mathbf{r} \equiv \mathbf{r}_{ab}$ and $M = m_a + m_b$. Integrating (25) yields

$$\frac{1}{2}\frac{d\mathbf{r}}{dt}\frac{d\mathbf{r}}{dt} + \frac{1}{2}\ln\frac{r - M}{r + M} = \epsilon, \tag{26}$$

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}, \tag{27}$$

where ϵ and $\mathbf{h}$ are the constants of motion energy and angular momentum per unit of mass, respectively. What we have done in this case is equivalent to a replacement of the Newtonian potential $V(r) = -\frac{M}{r}$ by the Weyl potential

$$V(r) = \frac{1}{2}\ln\left[\frac{r - M}{r + M}\right]. \tag{28}$$

For a head-on collision we set $\mathbf{h} = \mathbf{0}$ and let $\mathbf{r} = x\mathbf{i}$. We may also set M to 1 by rescaling space and time coordinates appropriately. So the equation of motion we are going to deal with is

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\ln\left[\frac{x - 1}{x + 1}\right] = \epsilon, \tag{29}$$

where $\dot{x} = dx/dt$.

In the next section we use this equation of motion to obtain the amount of gravitational radiation emitted in a head-on collision process. For comparison we show the same calculations for the Newtonian gravitational potential instead of the Weyl potential.

IV. EMISSION OF GRAVITATIONAL RADIATION

One can use the quadrupole formula and the dynamics given by the Weyl potential to calculate the generation of gravitational waves.

The mass traceless quadrupole moment is given by [23]

$$\mathcal{I}_{jk} \equiv \sum_A m_A \left[x^A_j x^A_k - \frac{1}{3}\delta_{jk}\left(x^A\right)^2\right]. \tag{30}$$

Now, for a two-particle system separated by the distance x the only nonvanishing components of $\mathcal{I}_{jk}$ are

$$\mathcal{I}_{xx} = \frac{2}{3}\mu x^2, \quad \mathcal{I}_{yy} = \mathcal{I}_{zz} = -\frac{1}{3}\mu x^2, \tag{31}$$

where $\mu = m_a m_b$ is the reduced mass. Since the total mass M was set to 1, the reduced mass is constrained to be less than 1. Note that the NCSA [7] notation M indicates the mass of one black hole instead of the total mass as we use in this paper.

Consequently, the gravitational luminosity is given by

$$\frac{dE_g}{dt} = \frac{1}{5}\left\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}_{jk}\right\rangle = \frac{6}{45}\mu^2\left\langle\left(\dddot{x^2}\right)^2\right\rangle, \tag{32}$$

where the angular brackets mean an average over several wavelengths.

Since we are mainly interested in the energy radiated along the collision process, we use the following expression in what follows:

$$\frac{dE_g}{dx} = \frac{dE_g}{dt}\frac{1}{\dot{x}} \tag{33}$$

and x is decreasing with time t.

For a head-on collision of two bodies with equation of motion (29) the quadrupole formula (32) yields

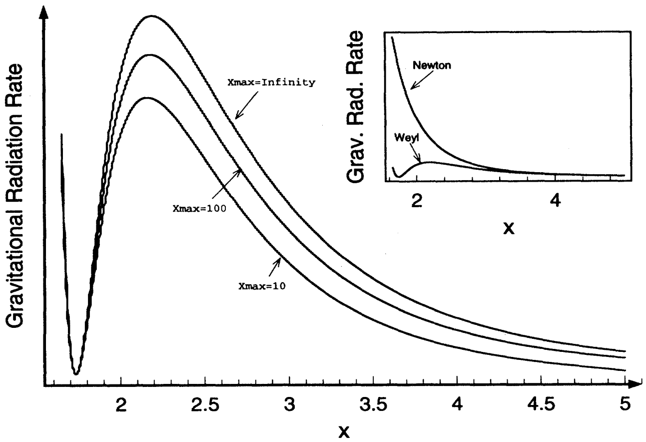

FIG. 1. Gravitational wave energy emission rate as a function of the distance between the holes for three initial conditions. The top curve has an initial condition such that $\epsilon = 0$ (that condition is satisfied, for instance, if the holes are released from infinity at rest) and the others below have $\epsilon < 0$. The inserted graphic shows a comparison of the curve obtained with the use of the Newtonian potential with one of the curves above.

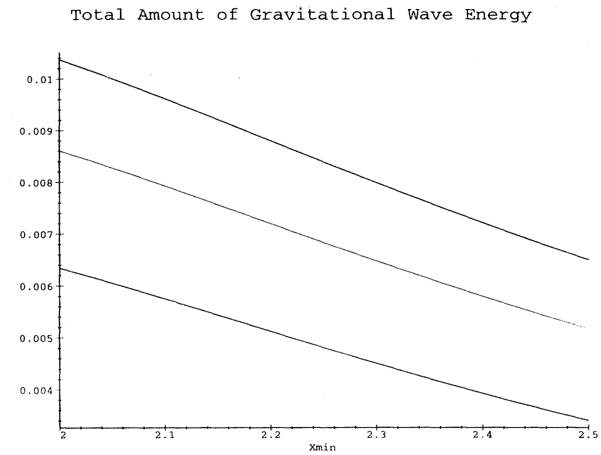

FIG. 2. Total amount of gravitational wave energy emission as a function of $x_{\rm min}$ for some values of $x_{\rm max}$. From top to bottom the values of $x_{\rm max}$ are infinity, 10, and 5, respectively.

$$\frac{dE_g}{dx} = \frac{8\mu^2}{15}\frac{(x^2-3)^2}{(x^2-1)^4}\sqrt{2\epsilon + \ln\left[\frac{x+1}{x-1}\right]}\ . \tag{34}$$

Note that $\frac{dE_g}{dx}$ diverges for $x \mapsto 1$, that is, when the two black hole horizons touch each other. Of course, one has to be careful when using this approach for when the black holes are very close to each other, strong distortions of their horizons and their coalescences could not be taken in account using Weyl's potential alone. Furthermore, the black holes would reach the speed of light at $x = x_{\rm light} \equiv (e^{1-2\epsilon}+1)/(e^{1-2\epsilon}-1) > 1$, that is, before the horizons touch. So the description of the head-on collision given by this approach is valid only for $x > x_{\rm light}$. One can note that the rate of emission reaches a maximum somewhere in the range $x_{\rm light} < x_{\rm peak} < \sqrt{5}$ and vanishes at $x = x_0 \equiv \sqrt{3}$ (see Fig. 1).

For comparison, it is interesting to mention that the use the Newtonian potential gives

$$\frac{dE_g}{dx} = \frac{8\mu^2}{15}\frac{1}{x^4}\sqrt{2\epsilon + \frac{2}{x}}, \tag{35}$$

which steadily increases as x decreases, diverges at $x = 0$, and at $x = 2/(1-2\epsilon)$ the holes reach the speed of light.

The total amount of energy radiated through gravitational waves is given by

$$\Delta E_g(x_{\rm min}, x_{\rm max}) \equiv \int_{x_{\rm min}}^{x_{\rm max}} \frac{dE_g}{dx}\, dx\ , \tag{36}$$

where $x_{\rm max}$ is the relative coordinate distance between the holes at the moment of release. Our approach clearly breaks down when the holes get too close. So we have to chose $x_{\rm min}$ appropriately. There are two possibilities: It can be either $x_{\rm light} = 2.1640$, namely, the point where

TABLE I. The total amount of gravitational energy radiated away as a function of $x_{\rm max}$. The value of $x_{\rm min}$ used is indicated in brackets; that is, $\Delta E_g[x_{\rm min}]$ is the total amount of energy radiated in the head-on collision when the holes are released at rest from $x_{\rm max}$ up to the respective $x_{\rm min}$. (Note that $x_{\rm peak}$ depend on $\epsilon = (1/2)\ln[(x_{\rm max}-1)/(x_{\rm max}+1)]$ through the transcendental equation $4x(x^2-5)\{2\epsilon + \ln[(x+1)/(x-1)]\} + x^2 - 3 = 0$). The values are compared to the calculation obtained from Newton's force and to the NCSA's [24] numerical relativity results. The two numbers on the first lines of NCSA's column reveal different values for different numerical techniques for initial conditions in which the holes are very close to each other.

$x_{\rm max}$	$x_{\rm peak}$	$\Delta E_g[x_{\rm peak}]$	$\Delta E_g[2.1640]$	$\Delta E_{\rm Newton}[2]$	$\Delta E_{\rm NCSA}[2]$
3	2.1159	0.00230	0.00207	0.00649	0.00407 $(3.5 \pm 0.2) \times 10^{-4}$
5	2.1621	0.00536	0.00535	0.01244	0.00582 $(5.5 \pm 1.3) \times 10^{-4}$
7	2.1726	0.00656	0.00662	0.01463	0.00654
10	2.1787	0.00735	0.00746	0.01610	0.00715
20	2.1844	0.00817	0.00833	0.01765	0.00801
30	2.1861	0.00842	0.00860	0.01813	0.00831
100	2.1883	0.00875	0.00895	0.01878	0.00874
∞	2.1892	0.00889	0.00909	0.01905	0.00894

the holes reach the speed of light when released from infinity, a choice that can be justified by special relativity; or $x_{\rm peak}$, which is the point where the rate of emission of gravitational radiation reaches its maximum value, a choice that can only be justified *a posteriori* by virtue of its remarkable agreement with the NCSA results (see Table I). However, it is important to point out that $x_{\rm peak}$ is very close to $x_{\rm light}$ and that may well be the reason why the results, using either one of these points as $x_{\rm min}$, are almost exactly the same. One could then argue that $x_{\rm peak}$ is the correct cutoff point, i.e., $x_{\rm min}$, because it is very close to $x_{\rm light}$ (the relative speed of the holes reach about 99% of the speed of light at $x_{\rm peak}$ when released from infinity) and there is a drastically decrease on the rate of emission of gravitational waves energy beyond this point (see Fig. 1). In the Newtonian potential case $x_{\rm min}$ is chosen to be 2, namely, the point where the holes reach the speed of light when released from infinity.

For the sake of completeness, the dependence of ΔE_g on $x_{\rm peak}$ is shown in Fig. 2.

V. CONCLUDING REMARKS

Using the static and axisymmetric Weyl spacetime, an exact solution to Einstein's field equations that can be interpreted as being the geometry due to a configuration of two black holes with a conical singularity between them, we gave a semianalytical treatment to the head-on collision problem.

We obtain the mutual attraction "force" that can be derived from Weyl's potential for this two-black-hole configuration and used Newtonian equations of motion to find the dynamics of the holes. We then calculate the amount of gravitational radiation energy released in the head-on collision process using the quadrupole radiation formula.

Our approach gives results that are in extremely good agreement with other relativistic calculations and computer simulations, which is somehow a surprise because of its simplicity. Note that the relativistic calculation by Davis *et al.* [25] (for a test particle falling radially into a black hole from infinity at rest) yields $\Delta E = 0.0104\mu^2$; the equivalent Newtonian estimation yields $\Delta E = \frac{2}{105}\mu^2 \approx 0.0191\mu^2$; the computer simulation by Smarr [2] gives $\Delta E = 0.0080\mu^2$ within a factor of 2. Finally, the NCSA group [7] gives $\Delta E = 0.0089\mu^2$. On the other hand our approach using the peak value as the stopping point for the radiation process yields the value $\Delta E = 0.00889\mu^2$.

ACKNOWLEDGMENTS

We wish to thank the Conselho Nacional de Desenvolvimento Científico e Tecnológico–CNPq for research grants and Edward Seidel for illuminating discussions.

[1] L. Smarr, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).

[2] L. Smarr, Ann. N.Y. Acad Sci. **302**, 569 (1977).

[3] A. Einstein, L. Infeld, and H. Hoffmann, Ann. Math. **39**, 65 (1938); **41**, 455 (1940); P. Bergmann, *Introduction to the Theory of Relativity* (Dover, New York, 1976), Chap. XV.

[4] H. Weyl, Ann. Phys. (Leipzig) **54**, 117 (1917); **59**, 185 (1919); T. Levi-Civita, Rend. Acc. Lincei **26**, 307 (1917); R. Bach and H. Weyl, Math. Z. **13**, 134 (1922); P. Chou, Am. J. Math. **53**, 289 (1931).

[5] H. Robertson and T. Noonan, *Relativity and Cosmology* (Saunders, London, 1968), pp. 272–278.

[6] A. Abrahams and G. Cook, Phys. Rev. D **50**, R2364 (1994).

[7] P. Anninos, D. Hobill, E. Seidel, L. Smarr, and W. Suen, Phys. Rev. Lett. **71**, 2851 (1993).

[8] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1979), Sec. 18.1.

[9] H. Curzon, Proc. London Math. Soc. **23**, 477 (1924).

[10] L. Silberstein, Phys. Rev. **49**, 268 (1936).

[11] A. Einstein and N. Rosen, Phys. Rev. **49**, 404 (1936).

[12] J. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), p. 311.

[13] R. Wald, *General Relativity* (Chicago University Press, Chicago, 1984), p. 162.

[14] P. Bergmann, *Introduction to the Theory of Relativity* (Dover, New York, 1976), pp. 206–210.

[15] P. Letelier and S. Oliveira, J. Math. Phys. **28**, 165 (1987).

[16] W. Bonnor, Gen. Relativ. Gravit. **20**, 607 (1988); **24**, 551 (1992).

[17] D. Sokolov and A. Starobinskii, Sov. Phys. Dokl. **22**, 312 (1977).

[18] B. Linet, Gen. Relativ. Gravit. **17**, 1109 (1985).

[19] P. Letelier and S. Oliveira, Class. Quantum Grav. **5**, L47 (1988); F. Cooperstock, Phys. Rev. D **10**, 3171 (1974).

[20] A. Taub, J. Math. Phys. **21**, 1423 (1980).

[21] N. Schleifer, Phys. Lett. **112A**, 204 (1985).

[22] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 138.

[23] S. Shapiro and S. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars — the Physics of Compact Objects* (Wiley, New York, 1983), Chap. 16.

[24] P. Anninos, D. Hobill, E. Seidel, L. Smarr, and W. Suen, "The Head-On Collision of Two Equal Mass Black Holes," NCSA Report No. 048, 1994 (unpublished).

[25] M. Davis, R. Ruffini, W. Press, and R. Price, Phys. Rev. Lett. **27**, 1466 (1971); M. Davis, R. Ruffini, and J. Tiomno, Phys. Rev. D **5**, 2932 (1972); F. Zerilli, *ibid.* **2**, 2141 (1970); R. Price and J. Pullin, Phys. Rev. Lett. **72**, 3297 (1994); A. Abrahams, S. Shapiro, and S. Teukolsky, Phys. Rev. D **51**, 4295 (1995).