PHYSICAL REVIEW D VOLUME 52, NUMBER 2 15 JULY 1995

Preferred basis in quantum theory and the problem of classicalization of the quantum Universe

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(Received 12 November 1992; revised manuscript received 5 January 1995)

We revive an old proposal of Zeh for the preferred basis in the many-worlds interpretation of quantum mechanics. The algorithm for the basis reduces to the eigenvalue problems for density matrices of subsystems forming the whole system under consideration. We generalize this procedure to the case of degenerate eigenvalues of reduced density matrices. A semiclassical calculational method for these eigenvalues is developed and applied to some model problems. The classical properties of elements of the preferred basis are investigated. It is shown that classicality exists only in some part of many-worlds branches. Moreover, it depends crucially on the initial conditions and Hamiltonians and under some circumstances turns out to be a temporary phenomenon. Applications of the preferred-basis proposal to quantum cosmology are discussed. The relation between the preferred-basis approach and quantum-histories approach is discussed.

PACS number(s): 03.65.Bz, 98.80.Hw

I. INTRODUCTION

The development of quantum cosmology in recent years [1] has had an impact on the study of the conceptual foundations of quantum mechanics. The idea of the quantum birth of the Universe [2–5] is naturally combined with the treatment of the Universe as a whole quantum object, which in another context was put forward in the framework of the many-worlds interpretation of quantum mechanics [6–9] pioneered in 1957 by Everett [6]. The present paper is devoted to the problem of the preferred basis in the many-worlds interpretation of quantum mechanics and quantum cosmology [10–18], which is closely related to the problem of explaining the classical behavior of the Universe.

In our opinion the majority of the most important questions in the interpretation of quantum mechanics can be reduced to the problem of a rather intricate interrelation between classical mechanics and quantum mechanics. It makes sense to extract three aspects of this problem. The first aspect can be called the question of priority. Which theory is more fundamental: classical mechanics or quantum mechanics? The answer depends crucially on the interpretation of quantum mechanics we choose. In the framework of different "neoclassical" (or hidden parameters) interpretations [19] the main role is the idea of the priority of classical-type determinism and the treatment of the probabilistic character of quantum mechanics is a secondary phenomenon.

The Copenhagen interpretation of quantum mechanics [20,21] seems to be prevailing among physicists. To a certain extent, in this interpretation classical and quantum mechanics have equal status. Recognizing the fundamental nature of quantum probability, adherents of the Copenhagen interpretation introduce also such notions as a classical realm of physical reality, a classical device, or a classical observer, and so on. This dualism of notions in the Copenhagen interpretation of quantum mechanics manifests itself in the existence of two fundamental processes: unitary evolution of the wave function according to the Schrödinger equation and the so-called reduction of the wave packet in the process of quantum measurement [21]. The meaning of the latter consists in the abrupt and noncausal elimination of that part of the wave function which corresponds to those possible outcomes of quantum measurement that were not realized. It is usual to explain this phenomenon by a special role of a classical device in the quantum measurement or even by a special role of consciousness in our world [21], but in any case this postulate restricts the "sovereignty" of the quantum-mechanical description of reality.

This situation can be regarded as unsatisfactory from the point of view of those who, like Everett, are guided by a desire to reduce to a minimum the number of fundamental principles of the theory [22]. Moreover, when trying to apply quantum mechanics to the consideration of cosmological problems we stumble upon the impossibility of finding the place for a classical realm or external classical observer. These two groups of reasons stimulated the development of the many-worlds interpretation of quantum mechanics. The main idea of this interpretation is the priority of quantum mechanics over classical mechanics and the belief in the objective nature of the wave function. Thus in the framework of the

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many-worlds interpretation there is only one object describing the physical reality, the wave function, and only one dynamical process, the Schrödinger evolution. The postulate of the reduction of the wave packet is given up. Instead we can speak about the simultaneous realization of different outcomes of quantum measurement in different "branches" of the wave function of the Universe or in different "parallel" Everett worlds. This approach to the problems of quantum cosmology seems to be promising and we shall attempt to be consistent in carrying it out in this paper. Perhaps it is worth noticing that now, at least in the community of quantum cosmologists, one can speak about a broad propagation of a "post-Everett" attitude to the interpretation of quantum mechanics [23,24]. However, there are different trends inside this "post-Everett" paradigm. Roughly speaking, one can subdivide here into three main approaches. The first one is the so-called decoherence approach originating from the papers [25–27]. The second approach is based on quantum histories or quantum paths and their decoherence functional [23,24,28–33]. The main difference of this approach from the traditional ones is connected with the idea that it is necessary to consider the quantum history as a whole as a principal object of quantum theory instead of the usual study of quantum states at fixed moments of time. The third approach in the framework of the post-Everett paradigm can be called the "biorthogonal basis approach" or "Schmidt basis approach" [10,16–18]. This approach is the subject of the present paper.

The second aspect of the problem of quantum-classical interrelations can be formulated as a question: What does "classicality" mean? In more definite terms we can ask ourselves how can we define the notion of classical behavior in quantum terms. It is necessary to do it for the explanation of the classical behavior of quantum systems. For these purposes the Wigner function approach is usually used [34–36]. In the framework of this approach one introduces the quasiprobability Wigner function

$$W(p,q,t) = \frac{1}{2\pi\hbar}\int dq' \exp\left(-\frac{ipq'}{\hbar}\right) \times\Psi\left(q-\frac{q'}{2},t\right)\Psi^*\left(q+\frac{q'}{2},t\right).$$

This function has properties close to those of the phase-space probability distribution function in classical statistical physics. It is possible to say that, in the case when

$$W(p,q) \sim f(q)\varphi(p-p_{\rm cl}(q,t)),$$

where $p_{\rm cl}(q,t)$ is the classical mechanical law of motion for the momentum and the dispersion of φ is small, we can treat the behavior of a system as behavior in classical statistical mechanics [37–41]. Usually, the investigation of the Wigner function is combined with studying the properties of a density matrix of the system in the framework of the decoherence approach [25–27,42–55]. The main idea of this approach consists in the construction of the reduced density matrix of the observable subsystem by tracing out degrees of freedom belonging to the environment:

$$\hat\rho_{\rm red} = {\rm Tr}_{\{\rm environment\}}\hat\rho_{\rm system+environment}.$$

The vanishing of the off-diagonal elements of $\hat\rho_{\rm red}$ describing quantum interference can be interpreted as a decoherence phenomenon or as a transition of the quantum system to classical behavior. Thus in this way we can reduce the quantum mechanical behavior to the classical statistical one. However, there is an essential difference between the role of statistical principles in classical and quantum physics. In classical physics the probability is "the measure of our ignorance" of the initial conditions or the details of interaction, while in quantum physics we cannot get rid of the probability even in principle, and there is no analogue to the "Laplace demon" who could calculate everything [56]. This situation is an incentive to develop an alternative approach to the notion of classicality in quantum mechanics which could be treated as a complementary one to the decoherence–Wigner function approach. This approach combines the choice of preferred basis with the subsequent investigation of the dynamics of elements of this basis. We shall compare our preferred-basis branches of the wave function with some "yardstick" states which, with respect to their properties, are treated as maximally close to classical ones. These sample states are some kind of coherent states and this approach to the treatment of classical properties of quantum systems was initially proposed by Schrödinger [57] in 1926. These states were first considered for the harmonic-oscillator Hamiltonian; however, it was shown that one can generalize the definition of the coherent oscillator to a number of more complicated cases [58–61]. Moreover, it was shown that in some potentials these generalized coherent states could be stable and possess other attractive properties. The detailed investigation of the properties of the generalized coherent states in different potentials was carried out in a series of papers [62]. It was shown that the generalized coherent states do not possess all the properties which could be demanded for the states pretending to have classical behavior; however, these minimum-uncertainty coherent states can, at least, be regarded as the best candidates for mediation between classical and quantum ones. Therefore we can use the notion of the coherent state as a model of the classical "yardstick" state in spite of the fact that the general definition for such a classical ideal state has not yet been formulated.

The problem of the definition of classicality in quantum terms is closely connected with the third aspect of quantum-classical interrelations—the cosmological one. This can be expressed in the following question: how does the quantum origin of the Universe lead to the observable classical world at the present time? The prevailing approach to this problem is the decoherence approach [25–27]. In the application to cosmology [37–55] the decoherence approach is usually used in combination with the correlation properties of the Wigner function. Probably, the subtlest question here is the choice of the environment. We cannot find an environment external to the whole Universe and we must treat a part of the degrees of freedom inside the Universe as the environment.

Another problem in the application of the decoherence approach concerns the basis dependence of the behavior of the reduced density matrix elements. It is obvious that the definition of the matrix elements of $\hat{\rho}_{\rm red}$ depends on the choice of the basis for this matrix. Usually it was implicitly or explicitly assumed that the pointer basis of Zurek [25,26] was used. This basis is stable against an interaction between the measuring apparatus and the environment. In treating cosmological problems we are also trying to carry out the preferred-basis approach to the wave function of the Universe implying its decomposition into some set of branches which can be identified with observable worlds. Then we can study the time evolution of these branches and discuss to what extent they have classical properties.

The problem of the preferred basis was discussed in Refs. [10–18]. Our approach was briefly sketched in Ref. [16]. We have revived the old proposal of Zeh [10] and gave somewhat different and more transparent arguments in favor of this proposal. It is necessary to stress also that this basis, which we shall call biorthogonal, was first introduced at the dawn of quantum mechanics (and many years before the many-worlds interpretation came into existence) by Schrödinger for the purpose of describing the quantum correlations between interacting subsystems [63]. An analogous basis was used also by Schmidt in the purely mathematical context [64].

In this paper we give a more extensive version of our approach to the problem of the choice of the preferred basis in quantum mechanics and quantum cosmology and to the question of classicalization of the quantum world. The structure of the paper is the following: Sec. II is devoted to the description of the procedure of choosing the preferred basis in the many-worlds interpretation of quantum mechanics; in Sec. III we construct the discrete preferred basis for the wave function of continuous variables; in Sec. IV we study the dynamics of the preferred basis; in Sec. V we discuss the problem of classicality and consider some simple toy models; in Sec. VI we continue the discussion of a classical behavior in a more complicated toy model; in Sec. VII we consider the connection between the preferred-basis approach and the quantum-histories approach to the interpretation of quantum mechanics; and in Sec. VIII we summarize our results.

II. THE CHOICE OF THE PREFERRED BASIS

The many-worlds interpretation of quantum mechanics resolves some problems and paradoxes arising in other interpretations. At the same time it gives rise to some additional problems. An especially important problem among them is the choice of the preferred basis. To explain the essence of this problem it makes sense to recall the mechanism of the "branching" of the wave function in measurementlike processes. For illustrative purposes we shall consider the well-known example of the Stern-Gerlach experiment.

Let the system consist of a device which is initially in the state $|\Phi\rangle_0$ and is ready to measure the spin z component of an atom, which is in the state

$$(c_1|\uparrow\rangle + c_2|\downarrow\rangle),$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are $s_z = \pm\frac{1}{2}$ orthonormal eigenstates of an atom. The initial state of the system as a whole is

$$|\Psi\rangle_{\rm in} = (c_1|\uparrow\rangle + c_2|\downarrow\rangle)|\Phi\rangle_0. \tag{2.1}$$

We see that before the measurement both the device and atom are in the pure state. Let us introduce now the unitary operator $\hat{U}$ describing the interaction between the atom and the device. The measurement operator $\hat{U}$ satisfies the rule [21]

$$\hat{U}|\uparrow\rangle|\Phi\rangle_0 = |\uparrow\rangle|\Phi_\uparrow\rangle\ ,$$

$$\hat{U}|\downarrow\rangle|\Phi\rangle_0 = |\downarrow\rangle|\Phi_\downarrow\rangle\ , \tag{2.2}$$

where $|\Phi_\uparrow\rangle$ and $|\Phi_\downarrow\rangle$ are the orthonormal eigenstates of the device indicating the fact of measuring, respectively, the values $s_z = \frac{1}{2}$ and $s_z = -\frac{1}{2}$. Under the action of $\hat{U}$ the state (2.1) transforms into

$$|\Psi\rangle_{\rm out} = \hat{U}|\Psi\rangle_{\rm in} = c_1|\uparrow\rangle|\Phi_\uparrow\rangle + c_2|\downarrow\rangle|\Phi_\downarrow\rangle. \tag{2.3}$$

From the point of view of the Copenhagen interpretation the unitary transition from the state (2.1) to the state (2.3) is only the first part of the process of quantum measurement. We know that really we can measure only one value of s_z, but in (2.3) there are two terms corresponding to both possible results of measurement. The second part of the quantum measurement consists in the elimination of one of these terms:

$$|\Psi\rangle_{\rm out} \Rightarrow |\uparrow\rangle|\Phi_\uparrow\rangle$$

or

$$|\Psi\rangle_{\rm out} \Rightarrow |\downarrow\rangle|\Phi_\downarrow\rangle. \tag{2.4}$$

The process (2.4) is nothing else but a reduction of the wave packet. However, in the framework of the many-worlds interpretation we reject the process (2.4) and recognize the simultaneous existence of both terms in the superposition (2.3): the measurement process is reduced to the arrangement of correlations between the atom and the device and both outcomes of the experiment exist in parallel worlds. Thus, instead of the reduction of the wave packet we have the "branching" of our world. From the mathematical point of view the branching is merely defactorization of the wave function of a system with respect to the subdivision of this system into subsystems [see Eq. (2.3)] and these subsystems undergo the transition from pure quantum states to mixed ones.

Generally speaking, every transition of the wave function of a system which consists of two subsystems from the factorized state

$$|\Psi\rangle_{\rm in} = |\varphi\rangle|\chi\rangle \tag{2.5}$$

to the defactorized state

$$|\Psi\rangle_{\rm out} = \sum_{i=1}^{n} |\varphi_i\rangle|\chi_i\rangle, \tag{2.6}$$

where the states $|\varphi\rangle$, $|\varphi_i\rangle$ describe one subsystem and the states $|\chi\rangle, |\chi_i\rangle$ another one and $n > 1$ can be regarded as a process of branching or defactorization of the wave function.

We see that in the decomposition (2.6) every state of one subsystem $|\varphi_i\rangle$ has the uniquely determined counterpart $|\chi_i\rangle$, which is usually called the "relative state" [6]. However, at the same time we can choose another set of basis states $|\tilde{\varphi}_i\rangle$ instead of $|\varphi_i\rangle$. In this case we shall have another set of relative states $|\tilde{\chi}_i\rangle$ and the wave function (2.6) will be written in the form

$$|\Psi\rangle_{\text{out}} = \sum_{i=1}^{n} |\tilde{\varphi}_i\rangle|\tilde{\chi}_i\rangle. \tag{2.7}$$

From the mathematical point of view there is no essential difference between formulas (2.6) and (2.7); they simply describe the same wave function, but written in different bases. But from the physical point of view the choice of different bases implies different decompositions of the wave function into some sets of branches which correspond to different real worlds, which can be perceived experimentally. Thus we have to make a definite choice of some preferred decomposition or preferred basis.

One can say that within the usual "common sense" it is impossible to choose in the case of the Gerlach-Stern experiment another basis but

$$|\uparrow\rangle|\Phi_\uparrow\rangle \text{ and } |\downarrow\rangle|\Phi_\downarrow\rangle \tag{2.8}$$

because our device is constructed and arranged to measure the spin z component of an atom. It is true, but in order to separate other possible choices of the basis we have to resort to some classical properties of the device. However, this deprives us of the main advantage of the many-worlds interpretation—its purely quantum nature. Moreover, common sense is applicable only in simple quantum-mechanical experiments. In a more complicated case of quantum cosmology common sense does not always work. The point is that it is necessary to work out the procedure of constructing the preferred basis originating from purely quantum notions. We can try to extract this procedure from the consideration of rather simple quantum-mechanical problems, generalize it, and then apply it to more general and complicated situations amounting to problems of quantum cosmology.

The construction of the preferred basis, which we would like to advocate, consists of two steps: (i) splitting of the system under consideration into certain subsystems and (ii) choosing the proper basis for one subsystem and the basis of relative states for another subsystem. It is necessary to stress that, from the point of view of our prescription, we can choose any subdivision of the system into subsystems, because for every such subdivision our prescription gives a unique preferred basis. In contrast with the Copenhagen interpretation, where observation and measurement play a special role, in the many-worlds interpretation they are simply interactions between subsystems. Therefore we shall treat our subsystems on an equal footing without indication which of them is the observer and which is the observable. All notation concerning the corresponding subsystem will be marked by subscripts I and II.

It seems rather reasonable to require that the fact that of different terms of the decomposition of the wave function belong to different worlds must be accompanied by their mutual orthogonality; moreover, we require also orthonormality of substates corresponding to our subdivision of the system into subsystems. This leads immediately to the decomposition

$$|\Psi\rangle = \sum_n c_n |n\rangle_{\text{I}} |n\rangle_{\text{II}}, \tag{2.9}$$

where both sets of basis vectors $|n\rangle_{\text{I}}$ and their relative states $|n\rangle_{\text{II}}$ are orthonormal,

$${}_{\text{I}}\langle n|m\rangle_{\text{I}} = \delta_{nm}, \quad {}_{\text{II}}\langle n|m\rangle_{\text{II}} = \delta_{nm}, \tag{2.10}$$

and c_n are some complex coefficients which determine *a priori* probabilities p_n of realizing the nth Everett world:

$$p_n = |c_n|^2 ,$$

$$\sum_n |c_n|^2 = \langle\Psi|\Psi\rangle = 1. \tag{2.11}$$

This prescription for the choice of the preferred basis yields also a constructive algorithm for finding it. Consider the density matrix of the total system $\hat{\rho} = |\Psi\rangle\langle\Psi|$ and the density matrix of the first subsystem I, obtained by tracing out the degrees of freedom of subsystem II:

$$\hat{\rho}_{\text{I}} = \text{Tr}_{\text{II}} |\Psi\rangle\langle\Psi|. \tag{2.12}$$

Substituting the decomposition (2.9) and using the orthonormality condition (2.10) one finds that the latter has the form

$$\hat{\rho}_{\text{I}} = \sum_n |c_n|^2 |n\rangle_{\text{I}}\, {}_{\text{I}}\langle n| \tag{2.13}$$

whence one concludes that the vectors $|n\rangle_{\text{I}}$ of the preferred basis solve the eigenvalue problem for the density matrix $\hat{\rho}_{\text{I}}$:

$$\hat{\rho}_{\text{I}} |n\rangle_{\text{I}} = p_n |n\rangle_{\text{I}},$$

$$p_n = |c_n|^2 \tag{2.14}$$

with eigenvalues which exactly coincide with the probability weights (2.11) of Everett worlds. Similarly to $|n\rangle_{\text{I}}$ the basis vectors $|n\rangle_{\text{II}}$ are the eigenvectors of the density matrix of the second subsystem:

$$\hat{\rho}_{\text{II}} = \text{Tr}_{\text{I}} |\Psi\rangle\langle\Psi|, \quad \hat{\rho}|n\rangle_{\text{II}} = p_n |n\rangle_{\text{II}}. \tag{2.15}$$

Both the density matrices $\hat{\rho}_{\text{I}}$ and $\hat{\rho}_{\text{II}}$ are Hermitian, positive semidefinite and (in view of the normalizability of $|\Psi\rangle$) bounded operators. Therefore they possess a countable eigenvalue spectrum and their orthonormal eigenvectors are unique up to inessential phase factors. The only exceptions are the invariant subspaces of $\hat{\rho}_{\text{I}}$ and $\hat{\rho}_{\text{II}}$ corresponding to possible degenerate eigenval-

ues, in which the eigenvectors can be determined up to unitary rotations. These subspaces can only be finite dimensional, because the sum of eigenvalues over each subspace is less than (or equal to) 1 in view of (2.11).

The basis (2.9), to which we came due to reasoning of the above type, coincides with the biorthogonal basis of Schrödinger [63], introduced for the study of quantum correlations between interacting quantum systems or with the "Schmidt canonical basis" of Zeh [10] (this name originates from the classical paper of Schmidt [64] on the integral equations), used in the context of the many-worlds interpretation. This basis can be obtained by solving two eigenvalue problems (2.13) and (2.14) for the density matrices of relevant subsystems. The nonuniqueness of this solution in the invariant subspaces of the degenerate eigenvalues was also mentioned in Ref. [10], but without a concrete proposal for its resolution. Here we shall go somewhat further and resort to an additional principle in order to fix this basis uniquely. From Eqs. (2.13) and (2.14) it follows that the preferred basis depends on the quantum state $|\Psi\rangle$ of the total system and consequently evolves in accordance with the Schrödinger evolution of $|\Psi\rangle = |\Psi(t)\rangle$. Then it seems natural to require that the decomposition of these invariant Hilbert subspaces into equally probable Everett worlds should be stable against this dynamical evolution. To demonstrate the efficiency of this principle we return to the above-mentioned example of the Gerlach-Stern experiment.

The density matrices of measurement device and atom corresponding to the state of the whole system after measurement (2.3) take the form

$$\hat{\rho}_{\text{device}} = |c_1|^2 |\Phi_\uparrow\rangle\langle\Phi_\uparrow| + |c_2|^2 |\Phi_\downarrow\rangle\langle\Phi_\downarrow| ,$$

$$\hat{\rho}_{\text{atom}} = |c_1|^2 |\uparrow\rangle\langle\uparrow| + |c_2|^2 |\downarrow\rangle\langle\downarrow| ,$$

and according to our algorithm the preferred basis is given in the nondegenerate case $|c_1|^2 \neq |c_2|^2$ by the vectors (2.4).

For $|c_1|^2 = |c_2|^2 = \frac{1}{2}$ this procedure becomes insufficient, because the preferred basis in our two-dimensional Hilbert space is already not unique. For example, instead of (2.4) one can take the basis of vectors $|\rightarrow\rangle|\Phi_\rightarrow\rangle$ and $|\leftarrow\rangle|\Phi_\leftarrow\rangle$,where

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle),$$

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle),$$

$$|\Phi_\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\Phi_\uparrow\rangle + |\Phi_\downarrow\rangle),$$

$$|\Phi_\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\Phi_\uparrow\rangle - |\Phi_\downarrow\rangle). \tag{2.16}$$

For the case $c_1 = c_2 = \frac{1}{\sqrt{2}}$ Eq. (2.15) also solves the eigenvalue problems. The new basis is, however, unstable with respect to the measurementlike interaction between the device and the atom. Indeed, there are two branches of $|\Psi\rangle$ in this basis:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\rightarrow\rangle|\Phi_\rightarrow\rangle + \frac{1}{\sqrt{2}}|\leftarrow\rangle|\Phi_\leftarrow\rangle. \tag{2.17}$$

In contrast with branches (2.15) of the decomposition (2.16) those branches (2.4) of the decomposition (2.3) are not destroyed during the dynamical evolution by the operator $\hat{U}$ and thus form the unique preferred basis which gives the reasonable many-worlds picture of the Gerlach-Stern experiment.

In conclusion of this section we would like to notice that the proposed procedure can be used in the case of more complex processes, when it makes sense to subdivide the system into more than two subsystems. In this case we must choose the decomposition

$$|\Psi\rangle = \sum_M c_M |\alpha_M^{(1)}\rangle|\alpha_m^{(2)}\rangle \cdots |\alpha_M^{(N)}\rangle, \tag{2.18}$$

where superscripts enumerate corresponding subsystems, and for different branches, enumerated by M, the corresponding vectors $|\alpha_m^{(k)}\rangle$ for different M can be either orthogonal or equal. The decomposition (2.18) can be designed by a recurrent procedure. As a first step we can construct the density matrix for the first subsystem, tracing out the rest of the subsystems. As a result, we get the decomposition of $|\Psi\rangle$ in the basis vectors of the first subsystem and corresponding relative states:

$$|\Psi\rangle = \sum_n c_n^{(1)} |\alpha_n^{(1)}\rangle|\alpha_n^{\text{rest}}\rangle. \tag{2.19}$$

Here superscripts (1) in the coefficients $c_n^{(1)}$ indicate the first step of a recurrent procedure. Then we can extract from the rest of the subsystems the second one and repeat the procedure. After this we have

$$|\Psi\rangle = \sum_n c_n^{(1)} |\alpha_n^{(1)}\rangle \sum_k c_{nk}^{(2)} |\alpha_{nk}^{(2)}\rangle|\alpha_{nk}^{\text{rest}}\rangle. \tag{2.20}$$

Repeating this procedure $N-1$ times, we have

$$|\Psi\rangle = \sum_n c_n^{(1)} |\alpha_n^{(1)}\rangle \sum_k c_{nk}^{(2)} |\alpha_{nk}^{(2)}\rangle \cdots \sum_l c_{nk\cdots l} |\alpha_{nk\cdots l}^{(N)}\rangle. \tag{2.21}$$

The decomposition (2.21) can be condensed into (2.18), where coefficients c_m are all possible products of coefficients $c_n^{(1)} \cdots c_{nk\cdots l}^{(N)}$.

III. DISCRETE PREFERRED BASIS FOR THE WAVE FUNCTION OF CONTINUOUS VARIABLES

In quantum mechanics and quantum cosmology we often work with the wave function written in terms of

continuous variables. In this case the preferred basis looks like a system of wave packets. In order to construct this system of wave packets we have to resolve a system of eigenvalue equations which turns into a system of integral equations. Because of the boundedness of the density operators mentioned in the preceding section they have a discrete spectrum. Let us write down the system of integral equations and then consider an exactly soluble example.

We consider the wave function

$$\Psi(x,y) \tag{3.1}$$

depending on two continuous variables x and y, which describe the first subsystem and the second one correspondingly. The density operators for the first and the second subsystems are given by the formulas

$$\rho(x,x') = \int_{-\infty}^{+\infty} dy \Psi^*(x,y)\Psi(x',y),$$

$$\rho(y,y') = \int_{-\infty}^{+\infty} dx \Psi^*(x,y')\Psi(x,y), \tag{3.2}$$

and the eigenvalue equations (2.13) and (2.14) turn into

$$\int_{-\infty}^{+\infty} dx' \rho(x,x')\varphi_n(x') = p_n\varphi_n(x) \tag{3.3}$$

and

$$\int_{-\infty}^{+\infty} dy' \rho(y,y')\chi_n(y') = p_n\chi_n(y). \tag{3.4}$$

Similarly to (2.9) the wave function $\Psi(x,y)$ can be written in the form

$$\Psi(x,y) = \sum_n c_n\varphi_n(x)\chi_n(y), \tag{3.5}$$

where $|c_n|^2 = p_n$.

Generally speaking, the resolution of the integral eigenvalue equations (3.3) and (3.4) is a very difficult technical task. However, there is a class of system in which the eigenvalue problem for the preferred basis can be solved by semiclassical methods. These systems possess a quantum state sharply peaked about some point in the configuration space of the theory, and the semiclassical expansion parameter turns out to be the quantum dispersion in the vicinity of this point. Let us denote this variable by x and write the wave function of the system as

$$\Psi(x,\{f\}) = \varphi(x)\Phi(\{f\},x). \tag{3.6}$$

This means that the strongest dependence of $\Psi(x,\{f\})$ on x is exhausted by the function $\varphi(x)$, while its dependence on all the remaining variables $\{f\}$ is given by the function $\Phi(\{f\},x)$ slowly varying with x. This is a typical situation when one singles out the background of x and the rest of the degrees of freedom $\{f\}$. For example, in quantum cosmology x may play the role of some homogeneous macroscopic variable—the inflaton scalar field, the cosmological scale factor, etc., while $\{f\}$ denotes all the other inhomogeneous quantum fields usually considered in the linearized approximation [65–67].

We now identify the subsystems I and II, respectively, with x and $\{f\}$ and compute the density matrix of the variable x. In view of (3.6) it takes the form

$$\rho(x,x') = \mathrm{Tr}_{\{f\}}|\Psi\rangle\langle\Psi| = \varphi(x)\varphi^*(x')\exp[-W(x,x')], \tag{3.7}$$

where the function $W(x,x')$ contains all the correlations between x and x', which indicate that the subsystem $I = x$ is not in the pure state. Suppose now that the function $\varphi(x)$ is sharply peaked about some point in x space. Then, if the correlation corrections $W(x,x')$ do not smear out this peak, the same will be true for the kernel (3.7): with respect to both arguments x and x' it will be concentrated around its stationary points $x = x_0$ and $x' = x_0^*$. Thus the dominant contribution to $\rho(x,x')$ will give the terms quadratic in $(x - x_0)$ and $(x' - x_0^*)$:

$$\rho(x,x') = N\exp[-a(x-x_0)^2 - a^*(x'-x_0^*)^2 + b(x-x_0)(x'-x_0^*)], \tag{3.8}$$

where N is the normalization factor. Generally a and x_0 are complex numbers, while the parameter b is always real (in consequence of the density matrix Hermiticity) and arises entirely due to the nonvanishing correlator $W(x,x')$ which is responsible for the nontrivial branching of the preferred basis. The requirements of boundedness and positive definiteness of the density matrix impose also the conditions

$$\mathrm{Re}a > 0, \quad b > 0, \quad 2\mathrm{Re}a > b. \tag{3.9}$$

Now we can write down the solutions of the eigenvalue problem (3.3) for the density matrix (3.9):

$$\varphi_n(x) = [2^n n!\sqrt{\pi}(2\mathrm{Re}\alpha)^{-1/2}]^{-1/2} H_n[2\sqrt{\mathrm{Re}\alpha}(x-\bar{x})] \times \exp[-\alpha(x-\tilde{x_0})^2], \tag{3.10}$$

where

$$\alpha = \sqrt{(\mathrm{Re}a)^2 - \frac{b^2}{4}} + i\,\mathrm{Im}a, \tag{3.11}$$

$$\tilde{x_0} = \frac{2x_0(a^*+\alpha) - bx_0^*}{2(a^*+\alpha) - b}, \tag{3.12}$$

$$\bar{x} = \frac{\alpha\tilde{x_0} + \alpha^*\tilde{x_0^*}}{2\mathrm{Re}\alpha}, \tag{3.13}$$

$$p_n = p_0(1-p_0)^n, \quad p_0 = \frac{2\sqrt{2\mathrm{Re}a - b}}{\sqrt{2\mathrm{Re}a - b} + \sqrt{2\mathrm{Re}a + b}}. \tag{3.14}$$

Thus the preferred basis of the Gaussian wave function of two continuous parameters consists of the complete set of the generalized harmonic-oscillator wave functions. They describe the tower of Everett worlds with decreas-

ing probability weights (3.12) and the growing quantum dispersions centered around the point $\bar{x}$:

$$\langle\varphi_n|(x-\bar{x})^2|\varphi_n\rangle = (2n+1)\mathrm{Re}\alpha. \tag{3.15}$$

We can notice that in the case of real parameters a and x_0 the expressions for α and $\bar{x}$ will be much simpler [16]:

$$\alpha = \sqrt{a^2 - \frac{b^2}{4}} \tag{3.16}$$

and

$$\bar{x} = \tilde{x_0} = x_0. \tag{3.17}$$

All this is directly applicable to quantum cosmology. As was shown in Refs. [67–70] the one-loop corrected wave function of the Universe in the Hartle-Hawking "no-boundary" prescription [2,3] can be normalizable depending on the field content of the model. Moreover, it may have a good inflationary peak at some reasonable value of the homogeneous inflaton scalar field. Because of these properties the proposed method can be applied to the quantum Universe for the purpose of its many-worlds analysis.

IV. THE DYNAMICS OF THE PREFERRED BASIS

The quantum state of the system $|\Psi\rangle = |\Psi(t)\rangle$ is subject to the Schrödinger evolution

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle\ , \tag{4.1}$$

which makes the basis $|m\rangle_{\mathrm{I(II)}} = |m(t)\rangle_{\mathrm{I(II)}}$ also evolve with time. Now take into account that the matrices ${}_{\mathrm{I}}\langle m(t_0)|n(t)\rangle_{\mathrm{I}}$ and ${}_{\mathrm{II}}\langle m(t_0)|n(t)\rangle_{\mathrm{II}}$ are unitary in view of the orthonormality conditions (2.10). This means that the evolution of the preferred-basis vectors of both subsystems is unitary and is governed by some effective Hermitian Hamiltonians $\mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$:

$$i\frac{\partial}{\partial t}|m(t)\rangle_{\mathrm{I}} = \mathcal{H}_{\mathrm{I}}|m(t)\rangle_{\mathrm{I}},$$

$$\mathcal{H}_{\mathrm{I}} = i\sum_m \left(\frac{\partial}{\partial t}|m(t)\rangle_{\mathrm{I}}\ {}_{\mathrm{I}}\langle m(t)|\right). \tag{4.2}$$

[Equations for $|m(t)\rangle_{\mathrm{II}}$ and $\mathcal{H}_{\mathrm{II}}$ have the same form.] To derive $\mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ note that the total Hamiltonian H contains the coupling V between subsystems I and II together with the proper Hamiltonians H_{I} and H_{II} of these subsystems:

$$H = H_{\mathrm{I}} + H_{\mathrm{II}} + V. \tag{4.3}$$

One can show that in virtue of (4.1) the nondiagonal elements of $\mathcal{H}_{\mathrm{I}}$ are given by

$$[(\cdots)_{mn} \equiv \langle m|(\cdots)|n\rangle],$$

$$(\mathcal{H}_{\mathrm{I}})_{mn} = (H_{\mathrm{I}})_{mn} - \frac{{}_{\mathrm{I}}\langle m|\mathrm{Tr}_{\mathrm{II}}[V, |\Psi\rangle\langle\Psi|]|n\rangle_{\mathrm{I}}}{p_m - p_n},$$

$$m \neq n\ . \tag{4.4}$$

Analogous equations for $\mathcal{H}_{\mathrm{II}}$ can be obtained by substituting in Eq. (4.4) the subscript II instead of I and vice versa. The diagonal elements of $\mathcal{H}_{\mathrm{I(II)}}$ determine the phase behavior of $|n\rangle_{\mathrm{I(II)}}$ and, consequently, they are not unique, because only the sum of phases of c_n, $|n\rangle_{\mathrm{I}}$, and $|n\rangle_{\mathrm{II}}$ is fixed in the decomposition (2.9), two of them being completely arbitrary. To fix this choice it is natural, for example, to require

$$(\mathcal{H}_{\mathrm{I}})_{mm} = (H_{\mathrm{I}})_{mm},\quad (\mathcal{H}_{\mathrm{II}})_{mm} = (H_{\mathrm{II}})_{mm} \tag{4.5}$$

whence it follows that

$$i\frac{\partial c_n}{\partial t} =_{\mathrm{I}} \langle n|_{\mathrm{II}}\langle n|V|\Psi\rangle. \tag{4.6}$$

Equations (4.4) and (4.5) completely determine the Hermitian Hamiltonians $\mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ which thus turn out to be complicated functionals not only of the fundamental Hamiltonian H, but also of the quantum state $|\Psi\rangle$ of the system.

The most unexpected conclusion from the unitary dynamics of the proposed basis is, however, the following. Since the observer (identified, for example, with the subsystem I) observes and measures only one relative state of the second subsystem II in his many-worlds branch, he finds that this state undergoes a unitary evolution of the above type. This is in spite of the impure nature of this open subsystem II described by a nonfactorizable density matrix. Is this an explanation of why we see the surrounding Universe in unitary evolution despite the enormous amount of uncontrollable interaction between our apparatuses and the whole world? Perhaps this is one more argument in favor of our proposal for the preferred basis.

The second conclusion is that the observer studying the dynamics of his relative state measures the effective Hamiltonian $\mathcal{H}_{\mathrm{II}}$ and not the fundamental Hamiltonian H of the total system. This apparently means that research into nature at the most fundamental levels requires additional efforts in reconstructing the fundamental dynamical laws on the grounds of the observable reality. To what extent such a reconstruction of H is possible is an open question.

It is worth noticing also that according to our definition of different branches their probability weights are changing with time due to evolution of c_n [see Eq. (4.6)]. Is it possible to observe this change of the probability weights experimentally? In other words, this question could be formulated in the following form: is it possible to "live" in one branch without feeling the existence of other branches in the case when the weights of these branches are changing? Moreover, an additional question arises: is the rate of change of these probability weights

important from the viewpoint of the self-identification of the observer with one of the branches in the preferred basis? One can choose two different answers to this question. First, we can postulate that after the splitting of the wave function into branches the observer (or, to be more precise one of the "incarnations" of the observer) who identifies himself with one of them cannot perceive the presence of other Everett worlds and consequently cannot find the change of c_n and p_n. Second, we can consider (see Refs. [17,18]) the smallness of $|\frac{dc_n}{dt}|$ as the condition of good definiteness of different branches in our basis and treat the rapid changes of probability weights of branches as an indication of their inseparability.

We are inclined to choose the first answer to the above question, but at the same time we are ready to recognize that we do not see decisive arguments in favor of the first or the second approach to the dynamics of the probability weights. Maybe the resolution of this question can be found along another line of reasoning connected with the discussion of the most adequate choice of splitting a system into subsystems, but this question will be considered elsewhere [71].

V. THE PROBLEM OF CLASSICALITY AND SIMPLE MODELS

We have mentioned in the Introduction that our approach to the notion of classicality consists in studying the properties of individual quantum states. Thus we take our preferred-basis states and compare them with some standard quantum states modeling the classical ones. We shall consider wave functions depending on the continuous variables studied in Sec. III and take coherent states as quantum states, simulating the classical properties. We have already mentioned in the Introduction that such a treatment of coherent states is to some extent tentative. However, it seems to us promising enough because the notion of the coherent state manifested its possibility to be generalized to more complicated situations than that of the harmonic oscillator [58–61]. (Here, it is necessary to stress that these generalized coherent states will not always follow classical trajectories [72].)

The notion of coherent states was introduced by Glauber [73] to study the properties of electromagnetic field oscillators. These states coincide with Gaussian wave packets in configuration space, which were also used by Schrödinger [57] at the dawn of quantum mechanics to study the connection between the quantum and classical descriptions.

What features distinguish the coherent states from other Gaussian wave packets? First, for the coherent states the product of dispersions of coordinate and momentum is minimal,

$$\delta x \delta p = \frac{\hbar}{2}\,, \tag{5.1}$$

and, secondly, these uncertainties expressed in terms of dimensionless values are given by

$$\frac{\delta x}{l} = \frac{1}{\sqrt{2}}, \quad \frac{l\delta p}{\hbar} = \frac{1}{\sqrt{2}}, \tag{5.2}$$

where

$$l = \left(\frac{\hbar}{m\omega}\right)^{1/2} \tag{5.3}$$

is the amplitude of the ground-state oscillations of a harmonic oscillator (or for a more complicated system l is some length characterizing the quantum-size properties of this system).

Let us now consider the Gaussian wave packet in one-dimensional configuration space:

$$\Psi(x) = N \exp(-\alpha x^2 + \beta x), \tag{5.4}$$

where N is a normalization factor. The dispersions of x and p are

$$\sigma_x = \frac{1}{\sqrt{2\mathrm{Re}\alpha}}, \tag{5.5}$$

$$\sigma_p = \frac{\hbar|\alpha|}{\sqrt{2\mathrm{Re}\alpha}}. \tag{5.6}$$

Thus the product of these uncertainties (dispersions) is

$$\sigma_x \sigma_p = \frac{\hbar|\alpha|}{2\mathrm{Re}\alpha}. \tag{5.7}$$

It is not difficult to see that (5.7) has the minimal value $\frac{\hbar}{2}$ at real α. To provide the coherence of the state with real α it is necessary also to require that this α has a special value, which can be obtained by comparison of (5.2) and (5.5):

$$\alpha = \frac{1}{l}. \tag{5.8}$$

At real α different from (5.8) we have the so-called squeezed states [74,75] because the dispersion for the coordinate or momentum can be very small, while the dispersion of the conjugated variable is very large. In the case of complex α we have the so-called correlated states [76] for which we have instead of minimization of the product of uncertainties the equality (5.7) which can be rewritten in the form

$$\sigma_x \sigma_p = \frac{\hbar|\alpha|}{2\mathrm{Re}\alpha} = \frac{\hbar}{2\sqrt{1-r^2}},$$

$$r = \frac{|\mathrm{Im}\alpha|}{|\alpha|}, \tag{5.9}$$

where r is a correlation coefficient which can generally be represented as

$$r = \frac{\frac{1}{2}\langle\hat{x}\hat{p} + \hat{p}\hat{x}\rangle - \langle\hat{x}\rangle\langle\hat{p}\rangle}{\sigma_p \sigma_x}, \tag{5.10}$$

where $\langle\hat{A}\rangle$ denotes the quantum average of the opera-

tor $\hat{A}$. The relation (5.9) represents a generalization of the Heisenberg uncertainty relation and is called the Schrödinger uncertainty relation [77].

Thus, in studying the classical properties of quantum states we have to compare them with the coherent quantum states which represent Gaussian wave packets with certain real coefficients in the quadratic form of the exponential.

Now returning to the tower of Everett worlds considered in Sec. III we can say that only one of them corresponding to the Gaussian wave packet without Hermite polynomial has the chance to be a classical world. All other worlds described by higher oscillator wave functions by no means can be treated as classical.[1] However, this fact should not confuse us, because for the explanation of the classical properties in our world it is enough to have at least one branch in the wave function of the Universe which possesses good classical properties. It is necessary to stress that the true wave function of the Universe is much more complex than all our models and due to this complexity can have a lot of different classical branches.

The question of special interest is the study of the tendencies in the dynamical behavior of the preferred basis for simple model systems. Here we investigate toy models of two coupled subsystems which show that both the decrease and growth of quantum uncertainty can take place depending on their interaction Hamiltonian V (for simplicity, we disregard their free Hamiltonians, which corresponds to the case of two coupled particles of very large masses).

Consider the system with two observables x and y, having as an initial quantum state the Gaussian packet

$$\Psi(x,y,t) = \frac{1}{\sqrt{\pi\sigma\gamma}} \exp\left(-\frac{x^2}{2\sigma^2} - \frac{y^2}{2\lambda^2}\right). \tag{5.11}$$

For the Hamiltonian $V = kxy$ we can calculate dynamically evolving wave function $\Psi(x,y,t)$ and then obtain density matrices $\rho_{\mathrm{I}}(x,x')$ and $\rho_{\mathrm{II}}(y,y')$ of the form (3.8) with time-dependent parameters a and b. According to Eqs. (3.11) and (3.13) we get the quantum dispersions

$$\sigma_x = \frac{\sigma}{(1+k^2t^2\sigma^2\lambda^2)^{1/4}},$$

$$\sigma_y = \frac{\lambda}{(1+k^2t^2\sigma^2\lambda^2)^{1/4}}, \tag{5.12}$$

tending for large times to zero (here we put $\hbar = 1$). However, the decrease of the coordinate widths is accomplished by the infinite growth of momentum dispersions σ_{p_x} and σ_{p_y} inverse to σ_x and σ_y. Therefore the interaction $V = kxy$ does not make the subsystems more classical.

Consider now the binomial interaction containing both pairs of canonical operators:

$$V = \epsilon(xp_y + yp_x). \tag{5.13}$$

(Remember that the Hamiltonian of the form xp_y was introduced by von Neumann [21] for simulating the measurement processes in quantum mechanics.) In this case all the dispersions

$$\sigma_x = \frac{1}{2\sigma_{p_x}} = \sqrt{\sigma\lambda}\left(\frac{\lambda^2\sinh^2\epsilon t + \sigma^2\cosh^2\epsilon t}{\sigma^2\sinh^2\epsilon t + \lambda^2\cosh^2\epsilon t}\right)^{1/4} \tag{5.14}$$

and $\sigma y = \frac{1}{2\sigma_{p_y}}$ (the latter equals σ_x with interchanged σ and λ) tend for large times to finite values and, moreover, become equal for both correlating subsystems $\sigma_x = \sigma_y = \sqrt{\sigma\lambda}, t \to \infty$. This is an example when a sharply peaked state of one subsystem (say y) dynamically reduces the initially large dispersion $\sigma \gg \lambda$ of its counterpart x to the value $\sqrt{\sigma\lambda} \ll \sigma$ and thus renders it more classical (certainly at the price of its own quantum spreading).

Thus we see that the classical properties of the preferred basis states are very sensitive to the choice of the interaction Hamiltonian and the emergence of such properties in the process of dynamical evolution depends crucially on the structure of the initial wave function and the structure of the interaction as well. In the next section we shall consider a more complicated toy model which will give us the opportunity to learn a number of new things concerning classical behavior in the framework of the many-worlds interpretation.

VI. MORE COMPLICATED TOY MODEL

We consider the wave function depending on two continuous variables which has the following form at the initial moment of time $t = 0$:

$$\Psi(x,y,0) = \frac{1}{\sqrt{\pi\Delta\delta}} \exp\left(-\frac{x^2}{2\Delta^2} + ipx - \frac{y^2}{2\delta^2}\right). \tag{6.1}$$

The evolution of this wave function is governed by the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2M} + kxp_y \tag{6.2}$$

[1] It is interesting to mention here that, for example, in the hydrogen atom, it is the lowest modes that are very nonclassical and the highly excited modes with the high order Hermite polynomials are regarded as approximately classical. Thus the picture appears to contradict our notion of classicality. The point is, however, that states of excited electrons are not classical from our point of view because their dispersions are very big. When people speak about the classicality of exited levels of the atom they mean that the distances between these levels become smaller, the spectrum becomes almost continuous, and electromagnetic radiation can be described in terms of classical electrodynamics. To translate such a notion of classicality into our language it is necessary to consider the whole system which consists of electrons, photons, and nuclei and to invent criteria connecting dispersions of different particles including ultrarelativistic ones. Apparently, it is a very difficult task.

including free Hamiltonians $\frac{p_x^2}{2m}$ and $\frac{p_y^2}{2M}$ together with the von Neumann–type interaction term kxp_y which corresponds to the measurement of the coordinate x by the device y [21].

The evolution operator e^{-itH} can be factorized into the product of the exponents of monomial operators due to the termination of the Campbell-Hausdorf series [78]:

$$\exp(\hat{A})\exp(\hat{B}) = *\exp\Bigg(\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}] + \frac{1}{12}[[\hat{A},\hat{B}],\hat{B}] + \frac{1}{12}[\hat{A},[\hat{A},\hat{B}]] + \text{higher commutators}\Bigg). \tag{6.3}$$

It is not difficult to verify that in the case when

$$\hat{A} = -it\left(\frac{p_x^2}{2m}+\frac{p_y^2}{2M}\right), \tag{6.4}$$

$$\hat{B} = -itkxp_y \tag{6.5}$$

only commutators $[\hat{A},\hat{B}]$ and $[[\hat{A},\hat{B}],\hat{B}]$ have nonzero values while $[\hat{A},[\hat{A},\hat{B}]]$ and the higher-order commutators in (6.3) are equal to zero. Thus using the Campbell-Hausdorf formula (6.3) we have

$$\exp\Big(-it\Big[\frac{p_x^2}{2m}+\frac{p_y^2}{2M}\Big]\Big)\exp(-itkxp_y) = \exp\Big(-it\Big[\frac{p_x^2}{2m}+\frac{p_y^2}{2M}+kxp_y\Big] + \frac{it^2kp_xp_y}{2m} - \frac{it^3k^2p_y^2}{12m}\Big). \tag{6.6}$$

After some calculations we can rewrite Eq. (6.6) in a form convenient for our purposes

$$\exp(-itH) = \exp\Big(-it\Big[\frac{p_x^2}{2m}+\frac{p_y^2}{2M}+kxp_y\Big]\Big) = \exp\Big(-it\frac{p_x^2}{2m}\Big)\exp\Big(-it\frac{p_y^2}{2M}\Big) \times \exp\Big(\frac{it^3k^2p_y^2}{3m}\Big)\exp(-itkxp_y) \times \exp\Big(-\frac{it^2kp_xp_y}{2m}\Big). \tag{6.7}$$

Now to calculate $\Psi(x,y,t) = e^{-itH}\Psi(x,y,0)$ we must act by operator (6.7) on the initial wave function (6.1). Taking into account that $p_x = -i\frac{\partial}{\partial x}$ and $p_y = -i\frac{\partial}{\partial y}$ we can reduce the calculation of $\Psi(x,y,t)$ to a set of consecutive operations of two kinds: $e^{a\frac{\partial}{\partial x}}$ and $e^{a\frac{\partial^2}{\partial x^2}}$. The first one is a well-known displacement for which

$$e^{a\frac{\partial}{\partial x}}f(x) = f(x+a). \tag{6.8}$$

We also have to calculate the result of acting by the operator $e^{a\frac{\partial^2}{\partial x^2}}$ on the function e^{bx^2+c}:

$$\begin{aligned} e^{a\frac{\partial^2}{\partial x^2}}e^{bx^2+c} &= \int d\alpha\delta\left(\alpha-\frac{\partial}{\partial x}\right)e^{a\alpha^2}e^{bx^2+c} \\ &= \frac{1}{2\pi}\int\int d\alpha d\beta e^{i\beta(\alpha-\frac{\partial}{\partial x})}e^{a\alpha^2+bx^2+cx} \\ &= \frac{1}{2\pi}\int\int d\alpha d\beta e^{i\beta\alpha+a\alpha^2+b(x-i\beta)^2+c(x-i\beta)} \\ &= \frac{1}{\sqrt{1-4ba}}\exp\left(\frac{x^2b+cx+ac^2}{1-4ba}\right). \end{aligned} \tag{6.9}$$

Here we used the formula (6.8) and the integral representation for the δ-function:

$$\delta(\alpha) = \frac{1}{2\pi}\int d\beta e^{i\beta\alpha}.$$

Thus, using the formulas (6.8) and (6.10) we have

$$\Psi(x,y,t) = \frac{1}{\sqrt{\pi\delta\Delta D}}\exp\Big\{\frac{1}{D}(-Ax^2-By^2+Cxy+Ex+Fy+G)\Big\}, \tag{6.10}$$

where

$$D = 1+it\left(\frac{1}{M\delta^2}+\frac{1}{m\Delta^2}\right)-\frac{t^2}{Mm\Delta^2\delta^2}+\frac{it^3k^2}{3m\delta^2} - \frac{t^4k^2}{12m^2\Delta^2\delta^2}, \tag{6.11}$$

$$A = \frac{1}{2\Delta^2}\left(1+\frac{it}{M\delta^2}+\frac{t^2k^2\delta^2}{\delta^2}+\frac{it^3k^2}{3m\delta^2}\right), \tag{6.12}$$

$$B = \frac{1}{2\delta^2}\left(1+\frac{it}{m\delta^2}\right), \tag{6.13}$$

$$C = \frac{tk}{\delta^2}+\frac{it^2k}{2m\delta^2\Delta 2}, \tag{6.14}$$

$$E = ip-\frac{pt}{M\delta^2}+\frac{pk^2t^3}{m\delta^2}, \tag{6.15}$$

$$F = -\frac{pt^2k}{2m\delta^2}, \tag{6.16}$$

$$G = \frac{p^2t}{2m}\left(-i+\frac{t}{M\delta^2}+\frac{t^3k^2}{12m\delta^2}\right). \tag{6.17}$$

Substituting (6.10) into (3.2) we can obtain

$$\rho(x,x',t) = N\exp(-ax^2 - a^*x'^2 + bxx' + cx + c^*x' + d), \tag{6.18}$$

where

$$N = \frac{1}{\sqrt{\pi}\Delta\delta\sqrt{B+B^*}}, \tag{6.19}$$

$$a = \frac{A}{D} - \frac{C^2D^*}{4D(BD^*+B^*D)}, \tag{6.20}$$

$$b = \frac{CC^*}{2(BD^*+B^*D)}, \tag{6.21}$$

$$c = \frac{F}{D} + \frac{CD^*F}{BD^*+B^*D}, \tag{6.22}$$

$$d = \frac{G}{D} + \frac{G^*}{D^*} + \frac{F^2DD^*}{BD^*+B^*D}. \tag{6.23}$$

The expression for $\rho(y,y',t)$ can be obtained from (6.18)–(6.23) by substituting B instead of A and vice versa and F instead of E and vice versa.

Now, having the explicit expressions for $\rho(x,x',t)$ and $\rho(y,y',t)$, we can write down the preferred-basis vectors in the way described in Sec. III. We are interested in the time dependence of dispersions of the wave packets in our preferred basis. Substituting the expressions (6.20), (6.21), and (6.11)–(6.14) into (3.11) we have

$$\sigma_x = \Delta\sqrt[4]{\frac{1+\frac{2t^2}{m^2\Delta^4}+\frac{k^2t^4}{2m^2\Delta^2\delta^2}+\frac{t^4}{m^4\Delta^8}+\frac{k^2t^6}{2m^4\Delta^6\delta^2}+\frac{k^4t^8}{16m^4\Delta^4\delta^4}}{1+\frac{\Delta^2k^2t^2}{\delta^2}+\frac{k^2t^4}{4m^2\delta^2\Delta^2}}} \tag{6.24}$$

and

$$\sigma_y = \frac{\delta\sqrt{1+\frac{t^2}{M^2\delta^4}+\frac{k^2\Delta^2t^2}{\delta^2}+\frac{k^2t^4}{4m^2\Delta^2\delta^2}-\frac{k^2t^4}{3Mm\delta^4}+\frac{k^4t^6}{36m^2\delta^4}}}{\sqrt[4]{1+\frac{\Delta^2k^2t^2}{\delta^2}+\frac{k^2t^4}{4m^2\delta^2\Delta^2}}}. \tag{6.25}$$

We see that σ_x and σ_y have a rather complicated time dependence. Both of them tend to infinity at $t\to\infty$ (spreading of the wave packets) but at the beginning of the time evolution it is possible to observe the shrinkage of wave packets in the configuration space of x provided $\frac{k^2\Delta^2}{\delta^2} > \frac{2}{m^2\Delta^4}$. At some stage of evolution the squeezing of σ_y is also possible but under very special fine tuned conditions and only at a certain stage of the time development of the system. Thus the classicality, defined as a definite property of a wave packet (the vicinity of the standard coherent state), is not only a phenomenon which is inherent to some special branches of the wave function but also a very unstable phenomenon. It is possible to say that at least in our model classicality can arise as rather specific properties which are only typical for certain intermediate stages of time evolution. We must confess that such a treatment of classicality contradicts the dominant opinion that the classical behavior is the final goal of dynamical evolution. However, it is worth mentioning Ref. [50], where classical properties of the Universe were investigated in the framework of the decoherence approach and the decoherence length was shown to oscillate in time in some circumstances. This phenomenon can also be treated as a manifestation of the nontrivial dependence of classical properties on time.

It is interesting to investigate also the behavior of the correlation coefficient r (5.10) which is responsible for the deviation of the product of dispersions $\sigma_x\sigma_y$ from its minimal value [see formulas (5.9)]. First of all, we must calculate $\mathrm{Im}\alpha$. Substituting formulas (6.20) and (6.11)–(6.14) into (3.11) we have

$$\mathrm{Im}\alpha = \frac{K}{L}, \tag{6.26}$$

where

$$K = \frac{4t}{\delta^2}\Big(\frac{2}{M\delta^2\Delta^2}+\frac{1}{m\Delta^4}-\frac{t^2}{M^2m\Delta^4\delta^4}+\frac{2t^2}{Mm^2\delta^2\Delta^6}+\frac{t^2}{m^3\Delta^8}+\frac{13k^2t^2}{6m\Delta^2\delta^2}+\frac{2k^2t^2}{M\delta^4}+\frac{11k^2t^4}{12Mm^2\Delta^4\delta^4}+\frac{2k^4t^4}{3m\delta^4}$$
$$-\frac{t^4}{M^2m^3\Delta^8\delta^4}+\frac{13k^2t^4}{6m^3\Delta^6\delta^2}-\frac{k^2t^4}{2M^2m\Delta^2\delta^6}-\frac{4k^2t^6}{3M^2m^3\Delta^8\delta^6}-\frac{k^2t^6}{2M^2m^3\Delta^6\delta^6}+\frac{5k^4t^6}{4m^3\Delta^4\delta^4}+\frac{k^4t^6}{6m^2M\Delta^2\delta^6}$$
$$-\frac{k^4t^8}{144m^5\Delta^8\delta^4}-\frac{5k^4t^8}{16Mm^4\Delta^6\delta^6}+\frac{k^6t^8}{9m^3\Delta^2\delta^2}+\frac{k^2t^6}{4m^4M\Delta^8\delta^4}-\frac{k^6t^{10}}{288m^5\Delta^6\delta^6}\Big) \tag{6.27}$$

and

$$L=\frac{8}{\delta^2}\Big(1+\frac{t^2}{M^2\delta^4}+\frac{2t^2}{m^2\Delta^4}+\frac{3k^2t^4}{4m^2\Delta^2\delta^2}+\frac{2k^2t^4}{3Mm\delta^4}+\frac{2t^4}{M^2m^2\Delta^4\delta^4}+\frac{t^4}{m^4\Delta^8}+\frac{k^2t^4}{4m^2M^2\Delta^2\delta^6}+\frac{5k^2t^6}{6Mm^3\Delta^4\delta^4}$$
$$+\frac{3k^2t^6}{4m^4\Delta^6\delta^2}+\frac{k^4t^6}{9m^2\delta^4}+\frac{t^6}{M^2m^4\Delta^8\delta^4}+\frac{35k^4t^8}{144m^4\Delta^4\delta^4}+\frac{k^2t^8}{6Mm^5\Delta^8\delta^4}+\frac{k^4t^8}{6Mm^3\Delta^2\delta^6}+\frac{k^2t^8}{4M^2m^4\Delta^6\delta^6}$$
$$+\frac{k^4t^{10}}{144m^6\Delta^8\delta^4}+\frac{k^4t^{10}}{24Mm^3\Delta^6\delta^6}+\frac{k^6t^{10}}{36m^4\Delta^2\delta^6}+\frac{k^6t^{12}}{576m^6\Delta^6\delta^6}\Big). \tag{6.28}$$

We can see that at $t=0$ $\mathrm{Im}\alpha=0$; then, at small t,

$$\mathrm{Im}\alpha(t)\approx\left(\frac{1}{M\delta^2\Delta^2}+\frac{1}{2m\Delta^4}\right)>0. \tag{6.29}$$

In the limit $t\to\infty$, we have

$$\mathrm{Im}\alpha(t)\approx-\frac{m}{t}<0 \tag{6.30}$$

and $\mathrm{Im}\alpha$ tends to zero from below. Thus, at some intermediate point t^*, $\mathrm{Im}\alpha(t^*)=0$ and the product of uncertainties is minimal. It follows from Eq. (6.26) that, at $t\to\infty$,

$$\mathrm{Re}\alpha(t)\approx\frac{m\delta}{k\Delta t^2}. \tag{6.31}$$

Comparing (6.31) and (6.30) it is easy to see that at $t\to\infty$ $\mathrm{Re}\alpha(t)$ decreases faster than $\mathrm{Im}\alpha$. Hence, according to (5.9) $r\to1$ and the product of uncertainties grows to infinity $\sigma_x\sigma_{p_x}\to\infty$. Thus we see that a good classical behavior in our model can be observed at some intermediate stage of time evolution while at $t\to\infty$ we have the simultaneous spreading of the wave packet in configuration space and in momentum space as well.

VII. QUANTUM-TRAJECTORIES APPROACH TO QUANTUM MECHANICS AND PREFERRED BASIS

In recent years the quantum-histories approach to the interpretation of quantum mechanics became very popular [23,24,28–33]. In the framework of this approach one considers the quantum trajectories or histories rather than quantum states at fixed moments of time. The probability weights are ascribed to these histories and instead of density matrices one can consider the density functionals on these histories and investigate their decoherence properties. We shall try in this section to investigate the connection between the preferred-basis approach and quantum histories. For this purpose we shall use the approach of quantum trajectories recently proposed by Griffiths [33]. In the framework of this approach one considers the quantum trajectories as objects belonging to the Hilbert space of quantum states in contrast to quantum histories connected with the usual classical trajectories in spacetime. Thus it is convenient enough to analyze quantum trajectories compared to the preferred-basis elements at different moments of time. Let us briefly review the definition of quantum trajectories from Ref. [33].

We consider the time-dependent wave function

$$|\Psi(t)\rangle=U(t)\,|\Psi(0)\rangle.$$

Then we take a set of times $t_1<t_2<\cdots<t_n$ and for each moment choose an orthonormal basis $\{|\Phi_j^\alpha\rangle\}$ of the Hilbert space, where α enumerates the basis vectors. One can construct the trajectory graph in which all the basis vectors at a particular moment of time are represented by nodes placed in a vertical column, and lines are drawn between the nodes (j,α) and $(j+1,\alpha')$ if and only if

$$\langle\Phi_{j+1}^{\alpha'}|U(t_{j+1})-t_j)|\Phi_j^\alpha\rangle \tag{7.1}$$

is not zero. A path on this graph is a succession of consecutive nodes $(j,\alpha),(j+1,\alpha_{j+1}),\ldots,(j+k,\alpha_{j+k})$ connected by lines. Provided any pair of nodes at different times are connected by at most one path, we shall say that the graph, or equivalently the choice of basis, satisfies the noninterference condition and individual paths will be called quantum trajectories [33]. A single quantum trajectory may be regarded as a generalization to a sequence of times of the notion of a pure quantum state at a single time: it gives the most precise description which quantum mechanics can provide about the state of the system at the chosen set of times.

Let us choose as a basis $\{|\Phi_j^\alpha\rangle\}$ at any time of our set $t_1<t_2<\cdots<t_n$ our preferred biorthogonal basis which corresponds to some splitting of the system under consideration into subsystems. What is the situation with the fulfillment of the noninterference condition introduced above [33] in this case? Generally, one can say that the noninterference is a property of the graph and depends on the choice of the basis on one hand and on the choice of the succession of times on the other. Increasing the intervals between chosen moments of time, we can turn a graph not satisfying the noninterference condition into a satisfying one. We can also make more "coarse grained" our subdivision of the system into subsystems and hence decrease the number of different basis vectors (we mean here the basis vectors having nonzero probability weights). Such a simplification of the basis also can help us to get rid of the paths breaking the noninterference condition.

Now, we shall discuss these questions in more detail. To begin with we consider two extremal cases. First, if we have only two moments of time, initial t_1 and final t_2, the noninterference condition is satisfied automatically. Second, if we consider our system as a whole without splitting it into subsystems, our preferred basis at every moment consists of the quantum state of the system itself and an arbitrary set of other orthonormal states guaranteeing the completeness (as was explained in Sec. II). Thus we have only one quantum path in our graph and it

does not depend on the succession of times (it could even be continuous). Thus the trivial set of times or triviality (purity) of the quantum structure automatically lead to the noninterference condition.

Let us examine now a more general situation when there are some branches, elements of the preferred basis, and the number of moments of time exceeds 2. If the weights of these branches are constant in time [see Eq. (4.8)] then the only nonvanishing matrix elements (7.1) are those for which $|\Phi^{\alpha'}_{j+1}\rangle$ and $|\Phi^{\alpha}_{j}\rangle$ represent the same branch at different times. Thus in this case the noninterference condition is satisfied independently of the chosen succession of times. However, if the coefficients c_n do depend on time [which is equivalent to the fact that the effective Hamiltonians (4.2) are nontrivial], then there are nonzero matrix elements (7.1) between different branches of the preferred biorthogonal basis and the noninterference condition is violated. One can restore the noninterference condition in two different ways. First, we can choose a more coarse-grained succession of times in such a way that we use only those intermediate time moments t_k for which the weight coefficients are again constant. Then the noninterference condition will be restored. Second, one can choose a more coarse-grained subdivision of the system into subsystems and correspondingly lower the number of different branches in the decomposition of the wave function. If we manage to do it in such a manner that the interaction of different branches is excluded, then all "dangerous" matrix elements (7.1) will disappear and the noninterference condition will be restored. Thus one can say that there is some kind of complementarity between the structure of the system and the structure of the chosen set of times. We can provide the satisfaction of the noninterference condition by coarse-graining either the structure of the system or the structure of time parametrization of the quantum trajectory.

VIII. CONCLUSION

We studied the properties of the preferred basis in quantum mechanics and analyzed the properties of the elements of this basis from the point of view of their classicality. Under classicality we understand the closeness of the quantum states under consideration to some "yardstick" states which are treated as the quantum counterparts of classical states. As a model of such "yardstick" states we choose coherent states. As one knows, these states defined for harmonic-oscillator-type systems [57,73] can be extended to more complicated cases [58–62].

The analysis of some models shows that classicality is a rather subtle phenomenon. First, it is typical not for all Everett worlds but only for a part of them; secondly, it depends crucially on the initial state of the system (Universe) and on the choice of interaction Hamiltonian; thirdly, the classicality can appear in some cases as a temporary phenomenon which exists only at some stages of the dynamical evolution. This last conclusion is, probably, the most unexpected one and contradicts the usual idea that classicalization is the final goal of evolution. The above assumptions seem to be counterintuitive because they contradict our everyday experience, which confirms that classicality is a quite robust phenomenon at least in the realm of macroscopic phenomena [the exceptions such as superconducting quantum interface devices (SQUID'S) or magnetic domain walls and grains at low temperature are not typical]. Moreover, the implementation of the biorthogonal basis seems also counter-intuitive because this basis in many cases is unstable against time evolution. What can we say in defense of our approach? The main advantage of the biorthogonal basis is its consistently quantum origin. In contrast with the Copenhagen paradigm of quantum mechanics where classical and quantum theory are treated on an equal footing, the very spirit of the Everett interpretation consists in the belief that the quantum description of the world is primary and the classical physics is secondary. In accordance with such a viewpoint it is not so surprising that the quantum theory can allow various types of behavior including those having nothing to do with classical physics. It was our task here to try to understand in what conditions classical behavior can be extracted from the abundance of different quantum possibilities. Concerning the question why we see only classical phenomena in the world surrounding us, in spite of its fundamentally quantum character and the quantum origin of our Universe as a whole, one can try a tentative answer resorting to the famous anthropic principle [79] which briefly can be formulated in the following way: we see the world as it stands because otherwise we could not have existed. Naturally, this anthropic principle is to some extent a speculative idea and can be used for justifying almost anything; however, we would like to say that the combination of this vague principle with a quantitative algorithm for the preferred basis can endow this principle with a constructive content.

It seems to us that the classical world can be described by the comparatively small part of the branches in the wave function of the Universe having a tiny weight in the corresponding Hilbert space. We can juxtapose this phenomenon with that habitual fact that biological life occupies a very small part of our usual three-dimensional space. The most important thing is not the volume of the space occupied by life in the Universe or by the classical realm in Hilbert space but the very possibility for their existence.

ACKNOWLEDGMENTS

We would like to thank R. Laflamme, V.F. Mukhanov, D.N. Page, V.N. Ponomariov, and B.S. DeWitt for helpful discussions. One of us (A.Yu.K.) thanks also A. Albrecht, H.F. Dowker, A.A. Grib, J.J. Halliwell, Yu.A. Kostikov, S. Lloyd, and A.A. Starobinsky for useful conversations. A.Yu.K. also thanks the U.S. National Academy of Sciences and Academy of Sciences of the U.S.S.R. for sponsoring of the Soviet-American Cosmology Workshop and Aspen Center for Physics Summer Institute and NASA/Fermilab Astrophysics Center for kind hospitality. A.Yu.K. is grateful to the Russian Foundation for Fundamental Researches for support under Grant

No 94-02-03850-a. A.O.B. is grateful for the financial support of this work provided by the NSERC grant at the University of Alberta, Grant No. INTAS-93-493, and the Russian Foundation for Fundamental Researches under Grant No. 93-02-15594. The research described in this publication was also made possible in part by Grants No. MAE300 and No. MQY300 from the International Science Foundation and Russian Government. The authors are grateful to the Russian Research Project "Cosmomicrophysics" for partial financial support.

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