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Quantum state diffusion, density matrix diagonalization, and decoherent histories: A model

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We analyze the quantum evolution of a particle moving in a potential in interaction with an environment of harmonic oscillators in a thermal state, using the quantum diffusion (QSD) picture of Gisin and Percival. The QSD picture exploits a mathematical connection between the usual Makovian master equation for the evolution of the density operator and a class of stochastic nonlinear Schrödinger equations (Ito equation) for a pure state $|\psi\rangle$, and appears to supply a good description of individual systems and processes. We find approximate stationary solutions to the Ito equation (exact, for the case of quadratic potentials). The solutions are Gaussians, localized around a point in phase space undergoing classical Brownian motion. We show, for quadratic potentials, that every initial state approaches these stationary solutions in the long time limit. We recover the density operator corresponding to these solutions, and thus show, for this particular model, that the QSD picture effectively supplies a prescription for approximately diagonalizing the density operator in a basis of phase space localized states. We show that the rate of localization is related to the decoherence time, and also to the time scale on which thermal and quantum fluctuations become comparable. We use these results to exemplify the general connection between the QSD picture and the decoherent histories approach to quantum mechanics, discussed previously by Diósi, Gisin, Halliwell, and Percival.

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I. INTRODUCTION

One of the basic premises of quantum theory is that the quantum state of a genuinely closed and isolated system evolves according to the Schrödinger equation. Although some systems of interest are approximately closed and isolated, most of the systems we encounter are not, as a result of either purposeful intervention by measuring devices, or unavoidable interaction with the immediate environment. Such systems are said to be open, and are often studied in quantum optics [1], quantum measurement theory [2,3], and in connection with decoherence and emergent classicality [4–6].

An open quantum system is in essence a distinguished subsystem of a large, closed and isolated system in which there is a natural division into subsystem and environment. Although such divisions of the world cannot be explicitly identified in general, they do exist in a wide variety of situations of both experimental and theoretical interest. For example, in quantum optics, the distinguished subsystem is an atom or small collection of atoms, and the environment is the electromagnetic fields in interaction with it. We will in this paper be primarily concerned with that pradigm of open quantum systems, the quantum Brownian motion model, which consists of a large particle coupled to a bath of harmonic oscillators in a thermal state [7,8].

If the state of the total quantum system is described by a density operator ρ_{total} then the state ρ of the subsystem is obtained by tracing it over the environment. An evolution equation for ρ (a master equation) may then be derived. This is in principle obtained quite simply by tracing the unitary evolution equation for ρ_{total} over the environment. In practice, this is hard to carry out with any degree of generality, and has been carried out in detail only in specific examples (see Ref. [9] for example). As an alternative, one can ask for the most general evolution equation for ρ that preserves density operator properties: hermiticity, unit trace, and positivity. These conditions alone do not allow one to say very much about the form of the equation, but if one makes the additional assumption that the evolution is Markovian, then the master equation must take the Lindblad form [10]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] - \frac{1}{2}\sum_{j=1}^{n}(\{L_j^\dagger L_j,\rho\} - 2L_j\rho L_j^\dagger) \ . \quad (1.1)$$

Here, H is the Hamiltonian of the open system in the absence of the environment (sometimes modified by terms depending on the L_j) and the n operators L_j model the effects of the environment. For example, in the quantum Brownian motion model, there is a single non-Hermitian L which is a linear combination of position and momentum operators. The Markovian assumption is not always valid, but is known to be a good approximation for a wide variety of interesting physical situations, e.g., for the quantum Brownian motion model in a high temperature environment.

Density operators evolving according to a master equation (not always of precisely the above form) have been the subject of a number of studies concerned with decoherence and the emergence of classical behavior

[6,9,11–16]. In particular models, it has been shown that the density operator can become approximately diagonal in some basis (sometimes more than one), indicating that interference between the states in that basis is destroyed. This suggests that one has some right to regard the dynamical variable corresponding to the diagonalizing basis as "definite." One may then ask for the probabilities of successive values of these variables, and whether those probabilities are peaked about approximately classical evolution.

This approach to emergent classicality has considerable intuitive appeal, but there are at least two ways in which it could be made more precise.

First of all, the notion of diagonality of the density operator is too vague. In the quantum Brownian motion model, for example, one expects both position and momentum to become reasonably definite. The argument as to how this comes about often goes as follows [13]: The coupling to the environment is typically through position, and the density operator tends to become approximately diagonal in position very quickly. On longer time scales, the Hamiltonian part of the evolution begins to contribute, and the basis of diagonalization is rotated in phase space. As a result of this interplay between the Hamiltonian and the interaction with the environment, the density operator therefore becomes approximately diagonal in a basis of states that are localized in phase space, such as coherent states. It is, however, difficult to see this precisely and with any degree of generality. What is required is an explicit way of exhibiting the diagonality in phase space localized states.

Second, the way in which one attempts to see the emergence of classical behavior for the variables which have become definite is to consider the evolution of states initially localized in phase space. Such states will tend to follow approximately classical trajectories in phase space, with spreading due to quantum and environmentally-induced fluctuations. The emergence of approximately classical trajectories is, however, much harder to see for *arbitrary* initial states, e.g., for superpositions of phase space localized states. The problem, in essence, is that the density operator does not in general correspond to the behavior of an individual trajectory, but to an ensemble. The density operator for an arbitrary initial state will be very spread out in phase space, and it is not at all clear that it corresponds to the intuitive expectation of a statistical mixture of classical trajectories. Clearly what would be very useful in this context is an alternative description of open systems that could give a clearer physical picture of the behavior of an individual system, rather than ensembles.

A recently developed picture of open systems that may be the required precision tool is the quantum state diffusion picture, introduced by Gisin and Percival [17–19]. In this picture, the density operator ρ satisfying (1.1) is regarded as a mean over a distribution of pure state density operators,

$$\rho = M|\psi\rangle\langle\psi| \ , \tag{1.2}$$

where M denotes the mean (defined below), with the pure states evolving according to the nonlinear stochastic Langevin-Ito equation,

$$|d\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle dt + \frac{1}{2}\sum_j(2\langle L_j^\dagger\rangle L_j - L_j^\dagger L_j - \langle L_j^\dagger\rangle\langle L_j\rangle)|\psi\rangle dt + \sum_j(L_j - \langle L_j\rangle)|\psi\rangle d\xi_j(t) \tag{1.3}$$

for the normalized state vector $|\psi\rangle$. Here, the $d\xi_j$ are independent complex differential random variables representing a complex Wiener process. Their linear and quadratic means are

$$M[d\xi_j d\xi_k^*] = \delta_{jk}dt \ , \quad M[d\xi_j d\xi_k] = 0 \ , \quad M[d\xi_j] = 0 \ . \tag{1.4}$$

The master equation (1.1) is invariant under unitary transformations of the Lindblad operator, $L_j \rightarrow \sum_k U_{jk}L_k$, where U_{jk} are the components of a unitary matrix [17]. Physics therefore corresponds to the equivalence class of master equations equivalent under these transformations. Correspondingly, the Ito equation (1.3) is invariant under the same unitary transformations on the L_j's, supplemented by similar transformations on the noise terms, and thus there is an equivalence class of Ito equations also.

The precise mathematical relation between (1.3) and (1.1) is that the class of Ito equations (1.3) is equivalent to the class of maser equations (1.1). Indeed, this connection supplies an alternative algorithm for numerical solution [19]. However, the strength of this picture is that solutions to (1.3) supply an intuitively appealing picture of the expected behavior of individual systems, and have been seen to correspond very closely to individual runs of actual laboratory experiments in quantum optics [20].

The connection between (1.1) and (1.3) is closely analogous to the connection between the Fokker-Planck equation and the Langevin equation in the classical description of Brownian motion. There, one has two completely equivalent mathematical descriptions with very different pictures. The Fokker-Planck equation describes an ensemble of systems evolving deterministically, whilst the Langevin equation describes an individual system evolving stochastically.

The quantum state diffusion picture has much in common mathematically with a variety of recent attempts to modify quantum mechanics at a fundamental level [21–24]. In such attempts, equations of the form (1.3), or similar, are proposed. The difference between quantum state diffusion (QSD) and such alternative formulations is that QSD is regarded as a phenomenological picture, appropriate only under certain conditions, whereas the alternative formulation are taken to be fundamental. Equations (1.1) and (1.3) also arise in descriptions of continuous measurement in standard quantum mechanics [25,26]. This paper is primarily concerned with the mathematical properties of Eq. (1.3), hence the results will be applicable to all of these situations.

Solutions to the Ito equation often have the feature that they settle down to solutions of rather simple behavior after a period of time. This general pattern of behavior is indicated by numerical solutions [19], along with a number of localization theorems, which show that, unlike evolution according to the master equation, the dispersion of certain operators *decreases* as time evolves [18,27]. That is, certain types of variables become more "definite" as time evolves.

A particularly useful example for our purposes was given by Diósi [28], who considered the Ito equation (1.3) with $L = ax$ and $H = p^2/2m$. (This is the quantum Brownian motion model for the free particle neglecting dissipation.) He showed that there exist stationary solutions $|\psi_{pq}\rangle$ to the Ito equation that consist of Gaussian wave packets tightly concentrated about a point in phase space evolving according to the stochastic equations of classical Brownian motion. This is a particularly appealing result. The solutions to the Ito equation correspond very closely to macroscopic observations of an individual particle interacting with an environment.

Given a set of localized phase space solutions $|\Psi_{pq}\rangle$, such as the Diósi solution, the density operator may be reconstructed via (1.2). This, it may be shown, may be written explicitly as

$$\rho = \int dp\, dq f(p,q,t) |\Psi_{pq}\rangle\langle\Psi_{pq}| \ , \tag{1.5}$$

where $f(p,q,t)$ is a non-negative, normalized solution to the Fokker-Planck equation corresponding to the Langevin equation describing the Brownian motion of the center of the stationary solutions.

The crucial point, now, is that the representation (1.5) of the density operator provides the desired improvements of the density operator program described above. First, the stationary states $|\Psi_{pq}\rangle$ are approximately orthogonal (for sufficiently distinct values of their centers, p,q). Equation (1.5) therefore shows explicitly how the density operator may achieve a form in which it is approximately diagonal in a set of phase space localized states. Secondly, each diagonal element corresponds to an individual classical trajectory (with noise). This means that the density operator might reasonably be interpreted as corresponding to a statistical mixture of classical trajectories.

The object of this paper is to demonstrate the above statements in detail, for systems more general than the case considered by Diósi. We will consider an open system consisting of a particle moving in a potential $V(x)$, coupled to an environment described by Lindblad operators in (1.1) which are a linear combination of position and momentum operators. The detailed description of the model is given in Sec. II.

We shall show, in Sec. III, that the Ito equation (1.3) has stationary solutions consisting of Gaussian wave packets concentrated about points in phase space which undergo classical Brownian motion. These solutions are exact for quadratic $V(x)$. The solutions for general potentials $V(x)$ are approximate, and are valid as long as the localization width is much smaller than the length scale on which the potential varies.

We shall then show, in Sec. IV, that every initial state tends towards one of the stationary solutions, for linear systems. In Sec. V, we consider the rate of localization, and show that it is related to the decoherence time, and also to the time scale on which thermal and quantum fluctuations become comparable.

In Sec. VI, we construct the density operator of the form (1.5) explicitly, and discuss its properties.

Arguably the most comprehensive and fundamental approach to the problem of emergent classicality in quantum theory is the decoherent histories approach [4,29–31]. In fact, in Ref. [32], it was argued that there is a close connection between the quantum state diffusion picture and the decoherent histories approach. In Sec. VII, we use the above results to elaborate on this connection.

We summarize and conclude in Sec. VIII.

II. THE MODEL

In this paper, we are concerned with systems described by a master equation of the form (1.1) with a single non-Hermitian operator linear in $\hat{x}$ and $\hat{p}$:

$$L = a\hat{x} + ib\hat{p} \ , \tag{2.1}$$

where a and b are real constants. The unitary transformations under which the master equation is invariant reduce to a simple phase invariance, $L \to e^{i\theta}L$. What follows therefore applies also to L's of the form (2.1) multiplied by a phase. This form of L is sufficient to describe the quantum Brownian motion model (see below), but also includes the cases in which L is taken to be a creation or annihilation operator.

The operator H in (1.1) is taken to be

$$H = \frac{\hat{p}^2}{2m} + V(\hat{x}) + c\{\hat{x},\hat{p}\} = H_0 + c\{\hat{x},\hat{p}\} \ , \tag{2.2}$$

where c is a real constant. The master equation may then be written

$$\begin{aligned}\dot{\rho} = & -\frac{i}{\hbar}\left[H_0 + \left(c - \frac{1}{2}\hbar ab\right)\{\hat{x},\hat{p}\},\rho\right] - iab[\hat{x},\{\rho,\hat{p}\}] \\ & -\frac{1}{2}a^2[\hat{x},[\hat{x},\rho]] - \frac{1}{2}b^2[\hat{p},[\hat{p},\rho]]\end{aligned} \tag{2.3a}$$

or, alternatively,

$$\dot{\rho} = -\frac{i}{\hbar}\left[H_0 + \left(c + \frac{1}{2}\hbar ab\right)\{\hat{x},\hat{p}\},\rho\right] + iab[\hat{p},\{\rho,\hat{x}\}]$$
$$-\frac{1}{2}a^2[\hat{x},[\hat{x},\rho]] - \frac{1}{2}b^2[\hat{p},[\hat{p},\rho]] \ . \qquad (2.3b)$$

Hereafter, we take $c = \frac{1}{2}ab\hbar$. This ensures that the Ehrenfest type result, $\mathrm{Tr}(\hat{p}\rho) = \frac{d}{dt}\mathrm{Tr}(\hat{x}\rho)$, holds.

The corresponding Ito equation is

$$\begin{aligned} |d\psi\rangle = &-\frac{i}{\hbar}\left(H_0 + \frac{1}{2}\hbar ab\{\hat{x},\hat{p}\}\right)|\psi\rangle dt \\ &-\frac{1}{2}[a^2(\hat{x}-\langle x\rangle)^2 + b^2(\hat{p}-\langle p\rangle)^2 \\ &+2iab(\hat{x}-\langle x\rangle\hat{p}) - \hbar ab]|\psi\rangle dt \\ &+[a(\hat{x}-\langle x\rangle) + ib(\hat{p}-\langle p\rangle)]|\psi\rangle d\xi \ . \end{aligned} \qquad (2.4)$$

We are particularly interested in the quantum Brownian motion model, for which the Lindblad operator is as above, but with

$$a = (2D)^{-1/2} \ , \quad b = (2D)^{1/2}\frac{\gamma}{\hbar} \ , \quad c = \frac{1}{2}\gamma \ . \qquad (2.5)$$

Here, $D = \hbar^2/(8m\gamma kT)$, where γ is the dissipation of the environment and T is its temperature. The master equation in this particular case may then be written

$$\dot{\rho} = -\frac{i}{\hbar}[H_0,\rho] - \frac{i}{\hbar}\gamma[\hat{x},\{\rho,\hat{p}\}] - \frac{2M\gamma kT}{\hbar^2}[\hat{x},[\hat{x},\rho]]$$
$$-\frac{\gamma}{8MkT}[\hat{p},[\hat{p},\rho]] \ . \qquad (2.6)$$

This does not, in fact, completely agree with the master equation given in a number of previous papers on quantum Brownian motion. In particular, the master equation given by Caldeira and Leggett [7] does not involve the term $[\hat{p},[\hat{p},\rho]]$. This difference is due to the fact the above master equation, by design, respects the positivity of the density operator, while the Caldeira-Leggett equation is known to violate it on short times scales [33]. This difference is not important, since we expect the Markovian approximation to hold only for high temperatures, and in this case the extra term is negligible since its coefficient is proportional to T^{-1}. (See Ref. [34] for further discussion, and also Ref. [9] for the derivation of exact master equations.)

Some information on the behavior of the solutions to the Ito equation may be obtained by computing the time evolution of the moments of $\hat{x}$ and $\hat{p}$, and this will be useful in the following sections. For any operator G, the time evolution of its expectation value in the state $|\psi\rangle$ is given by

$$\begin{aligned} d\langle G\rangle &= \langle\psi|G|d\psi\rangle + \langle d\psi|G|\psi\rangle + \langle d\psi|G|d\psi\rangle \\ &= \frac{i}{\hbar}\langle[H,G]\rangle dt - \frac{1}{2}\langle L^\dagger[L,G] + [G,L^\dagger]L\rangle dt \\ &\quad +\sigma(G^\dagger,L)d\xi + \sigma(L,G)d\xi^* \ . \end{aligned} \qquad (2.7)$$

Here, following Percival [27], we have introduced the notation

$$\sigma(B,C) = \langle(B^\dagger - \langle B\rangle^*)(C - \langle C\rangle)\rangle = \langle B^\dagger C\rangle - \langle B\rangle^*\langle C\rangle \qquad (2.8)$$

for the correlation between two operators B,C in the state $|\psi\rangle$.

Setting G equal to $\hat{p}$ and $\hat{x}$ in this equation we obtain the Langevin equations

$$d\langle x\rangle = \frac{\langle p\rangle}{m}dt + \sigma(x,L)d\xi + \sigma(L,x)d\xi^* \ , \qquad (2.9)$$

$$d\langle p\rangle = -\langle V'(\hat{x})\rangle dt - 2\hbar ab\langle p\rangle dt + \sigma(p,L)d\xi + \sigma(L,p)d\xi^* \ . \qquad (2.10)$$

With the choice of parameters (2.5), and for quadratic potentials, these equations describe classical Brownian motion. For more general potentials, this is true only if the state is sufficiently well localized in x for the approximation $\langle V'(\hat{x})\rangle \approx V'(\langle\hat{x}\rangle)$ to be valid (see below).

It is also of interest to compute the mean of the time evolution of higher moments of $\hat{x}$ and $\hat{p}$, and these may again be computed using (2.7). One finds

$$M\frac{d(\Delta x)^2}{dt} = \frac{2R}{m} + 2\hbar ab(\Delta x)^2 + 2b^2\left(\frac{\hbar^2}{4} - R^2\right)$$
$$-2a^2(\Delta x)^4 \ , \qquad (2.11)$$

$$M\frac{d(\Delta p)^2}{dt} = -2\left(\frac{1}{2}\langle\hat{p}V'(\hat{x}) + V'(\hat{x})\hat{p}\rangle - \langle p\rangle\langle V'(\hat{x})\rangle\right)$$
$$-2\hbar ab(\Delta p)^2 + 2a^2\left(\frac{\hbar^2}{4} - R^2\right) - 2b^2(\Delta p)^4 \ , \qquad (2.12)$$

$$M\frac{dR}{dt} = -[\langle\hat{x}V'(\hat{x})\rangle - \langle x\rangle\langle V'(\hat{x})\rangle] + \frac{(\Delta p)^2}{m}$$
$$-2a^2R(\Delta x)^2 - 2b^2R(\Delta p)^2 \ . \qquad (2.13)$$

Here, R is the symmetrized correlation between $\hat{p}$ and $\hat{x}$:

$$R = \frac{1}{2}[\sigma(x,p) + \sigma(p,x)] = \sigma(p,x) + \frac{i\hbar}{2} = \sigma(x,p) - \frac{i\hbar}{2} \ . \qquad (2.14)$$

Also, $(\Delta x)^4$ denotes $\langle(x - \langle x\rangle)^2\rangle^2$, and similarly for $(\Delta p)^4$.

To handle general potentials is too difficult except in special cases, so approximations are required. Under Schrödinger evolution in ordinary quantum mechanics in a wide variety of potentials, there exist approximate solutions consisting of localized Gaussian wave packets concentrated about a classical path [35]. These solutions are possible because a sufficiently localized packet will only "notice" the quadratic approximation to the potential in the neighborhood of the wave packet's center. The solution breaks down after a period of time, however, as a result of spreading of the wave packet.

Similar types of solution to the Ito equation (2.4) are possible, as we shall see in the next section. These have the advantage that wave packets tend to localize with time, rather than spread. We may therefore justifiably approximate the potential-dependent terms in (2.12) and (2.13) by their expansions about the mean values of x and p.

To see this more explicitly, and to assist the estimation of the validity of the approximation, introduce the notation, $\bar{x} = \langle x \rangle$, $\bar{p} = \langle p \rangle$, and then write the potential as

$$V(x) = V(\bar{x}) + (x - \bar{x})V'(\bar{x}) + \frac{1}{2}(x - \bar{x})^2 V''(\bar{x}) + W(x, \bar{x}) \ , \tag{2.15}$$

where

$$W(x, \bar{x}) = \frac{1}{6}(x - \bar{x})^3 V'''(\bar{x}) + \frac{1}{24}(x - \bar{x})^4 V^{(4)} + \cdots \ . \tag{2.16}$$

Then the potential-dependent terms in (2.10), (2.12), and (2.13) become

$$\langle V'(\hat{x}) \rangle = V'(\bar{x}) + \langle W'(\hat{x}) \rangle \ , \tag{2.17}$$

$$\langle \hat{x} V'(\hat{x}) \rangle - \langle x \rangle \langle V'(\hat{x}) \rangle = (\Delta x)^2 V''(\bar{x}) + \langle (x - \bar{x}) W'(\hat{x}) \rangle \ , \tag{2.18}$$

and

$$\frac{1}{2}\langle \hat{p} V'(\hat{x}) + V'(\hat{x})\hat{p} \rangle - \langle \hat{p} \rangle \langle V'(\hat{x}) \rangle = R V''(\bar{x}) + \frac{1}{2}\langle \hat{p} W'(\hat{x}) + W'(\hat{x})\hat{p} \rangle - \langle \hat{p} \rangle \langle W'(\hat{x}) \rangle \ . \tag{2.19}$$

The quadratic approximation to the potential will therefore be valid when the terms involving W may be neglected in the above expressions. This will generally depend on the particular state.

Taking the first few terms in the Taylor expansion of W, Eq. (2.17) for example, implies that the higher order terms may be neglected if

$$|V'(\bar{x})| \gg \frac{1}{2}(\Delta x)^2 |V'''(\bar{x})| \ . \tag{2.20}$$

This is clearly the condition that the width of the state is much less than the length scale on which the potential varies, as one would intuitively expect. The higher order terms in (2.18) and (2.19) also may be neglected if essentially the same type of condition holds.

III. STATIONARY SOLUTIONS TO THE LANGEVIN-ITO EQUATION

We now show how to find stationary solutions to the Langevin-Ito equation, (2.4). It may be written

$$|d\psi\rangle = \hat{u}|\psi\rangle dt + \hat{v}|\psi\rangle d\xi \ , \tag{3.1}$$

where

$$\hat{u} = -\frac{i}{\hbar}H + \frac{1}{2}\hbar ab + iab(\langle x \rangle \hat{p} - \langle p \rangle \hat{x}) - \frac{1}{2}a^2(\hat{x} - \langle x \rangle)^2 - \frac{1}{2}b^2(\hat{p} - \langle p \rangle)^2 \ , \tag{3.2}$$

$$\hat{v} = L - \langle L \rangle \ . \tag{3.3}$$

It is then convenient to rewrite the Ito equation in the exponential form

$$|\psi\rangle + |d\psi\rangle = \exp(\hat{u}dt + \hat{v}d\xi)|\psi\rangle \ . \tag{3.4}$$

The Diósi stationary solution has the feature that under time evolution, its shape is preserved and the only things that change are $\langle \hat{p} \rangle$ and $\langle \hat{x} \rangle$ (and possibly a phase) [28]. Our approach to the search for stationary solutions to our more general equation is to require that the solution have this property. We therefore look for solutions to (3.1) satisfying the condition

$$|\psi\rangle + |d\psi\rangle = \exp\left(\frac{i}{\hbar}\hat{x} d\langle p \rangle - \frac{i}{\hbar}\hat{p} d\langle x \rangle + \frac{i}{\hbar} d\phi \right) |\psi\rangle \ . \tag{3.5}$$

This is the statement that the state at time $t + dt$ differs from the state at time t by nothing more than a phase, and a shift of $\langle p \rangle$ and $\langle x \rangle$ along the classical Brownian path described by (2.9), (2.10). Clearly (3.5) will be satisfied for any states of the form

$$|\psi\rangle = \exp\left(\frac{i}{\hbar}\hat{x}\langle p \rangle - \frac{i}{\hbar}\hat{p}\langle x \rangle \right) |\chi\rangle \ , \tag{3.6}$$

where $|\chi\rangle$ is an arbitrary fiducial state. These are generalized coherent states [36].

We will solve (3.4) and (3.5) by first combining them to yield

$$\exp(\hat{u}dt + \hat{v}d\xi)|\psi\rangle = \exp\left(\frac{i}{\hbar}\hat{x} d\langle p \rangle - \frac{i}{\hbar}\hat{p} d\langle x \rangle + \frac{i}{\hbar} d\phi \right) |\psi\rangle \tag{3.7}$$

and later confirm that the solution satisfies (3.5).

Taking the operator on the right-hand side of (3.7) over to the left-hand side, and combining the exponentials using the Baker-Campbell-Hausdorff formula, one obtains

$$\exp\biggl(-\frac{i}{\hbar}\hat{x}d\langle p\rangle+\frac{i}{\hbar}\hat{p}d\langle x\rangle-\frac{i}{\hbar}d\phi+\hat{u}dt+\hat{v}d\xi -\frac{i}{2\hbar}[\hat{x},\hat{v}]d\langle p\rangle d\xi+\frac{i}{2\hbar}[\hat{p},\hat{v}]d\langle x\rangle d\xi\biggr)|\psi\rangle=|\psi\rangle\ . \tag{3.8}$$

Inserting the explicit expressions for $d\langle p\rangle$, $d\langle x\rangle$, $\hat{u}$, and $\hat{v}$, and writing $d\phi=\phi_0dt+\phi_1d\xi+\phi_1^*d\xi^*$ (where ϕ_0 is real), this equation becomes

$$\exp(\hat{A}dt+\hat{B}d\xi+\hat{C}d\xi^*)|\psi\rangle=|\psi\rangle\ , \tag{3.9}$$

where

$$\hat{A}=\hat{u}+\frac{i}{\hbar}[\langle V'(\hat{x})\rangle+2\hbar ab\langle p\rangle]\hat{x} +\frac{i}{\hbar}\frac{\langle p\rangle}{m}\hat{p}+\frac{1}{2}\sigma(L,L)-\frac{i}{\hbar}\phi_0\ , \tag{3.10}$$

$$\hat{B}=\frac{i}{\hbar}[-\sigma(p,L)\hat{x}+\sigma(x,L)\hat{p}-\phi_1]+L-\langle L\rangle\ , \tag{3.11}$$

$$\hat{C}=\frac{i}{\hbar}[-\sigma(L,p)\hat{x}+\sigma(L,x)\hat{p}-\phi_1^*]\ . \tag{3.12}$$

Expanding the exponential in (3.9), it follows that the state must obey the three equations

$$\hat{A}|\psi\rangle=0\ , \tag{3.13}$$

$$\hat{B}|\psi\rangle=0\ , \tag{3.14}$$

$$\hat{C}|\psi\rangle=0\ . \tag{3.15}$$

Equation (3.14) and (3.15) will be satisfied if

$$\phi_1=\sigma(x,L)\langle p\rangle-\sigma(p,L)\langle x\rangle \tag{3.16}$$

and if the wave function is

$$\langle x|\psi\rangle=N\ \exp\biggl(-\beta(x-\langle x\rangle)^2+\frac{i}{\hbar}\langle p\rangle x\biggr) \tag{3.17}$$

for some constant β, to be determined. The solution satisfies

$$\langle x|\psi\rangle+\langle x|d\psi\rangle=N\ \exp\biggl(-\beta(x-\langle x\rangle-d\langle x\rangle)^2 +\frac{i}{\hbar}(\langle p\rangle+d\langle p\rangle)x\biggr)\ . \tag{3.18}$$

This is clearly a generalized coherent state, and thus satisfies Eq. (3.5).

An equation for β may be obtained by inserting (3.17) in (3.13). On obtains the purely algebraic equation

$$4\left(b^2+\frac{i}{m\hbar}\right)\hbar^2\beta^2+4\hbar ab\beta-\left(a^2+\frac{i}{\hbar}V''(\langle x\rangle)\right)=0\ , \tag{3.19}$$

where we have neglected terms higher than quadratic in the potential, as described in the previous section.

It is of course possible to write down the explicit solution for β, but it will generally be more useful in what follows to proceed differently. We have the uncertainty relation

$$(\Delta x)^2(\Delta p)^2-R^2\geq\frac{\hbar^2}{4} \tag{3.20}$$

with equality if and only if the state is of the form (3.17) [37]. Let us denote the values of the variances and correlation of the stationary state (3.17) by σ_x^2, σ_p^2, and R_0. Then

$$\sigma_x^2\sigma_p^2-R_0^2=\frac{\hbar^2}{4} \tag{3.21}$$

and

$$\beta=\frac{1-2iR_0/\hbar}{4\sigma_x^2}\ . \tag{3.22}$$

Since, from (3.19), β is a constant [to the extent that the approximation (2.20) holds] the stationary values of the variances and correlation must be those for which the right-hand sides of (2.11)–(2.13) vanish. That is,

$$\frac{R_0}{m}+hab\sigma_x^2+b^2\left(\frac{\hbar^2}{4}-R_0^2\right)-a^2\sigma_x^4=0\ , \tag{3.23}$$

$$-V''(\bar{x})R_0-\hbar ab\sigma_p^2+a^2\left(\frac{\hbar^2}{4}-R_0^2\right)-b^2\sigma_p^4=0\ , \tag{3.24}$$

$$-\sigma_x^2V''(\bar{x})+\frac{\sigma_p^2}{m}-2a^2R_0\sigma_x^2-2b^2R_0\sigma_p^2=0\ . \tag{3.25}$$

These will be the most useful equations to work with in the following section.

To see the complete solution in a particular case, let $V(x)=0$ and $b=0$. The solution for β is then

$$\beta=(1-i)\left(\frac{ma^2}{8\hbar}\right)^{1/2}\ , \tag{3.26}$$

where we have chosen the square root so that $\mathrm{Re}\beta>0$, for normalizability of the state. It follows that

$$\sigma_x^2=\left(\frac{\hbar}{2ma^2}\right)^{1/2}\ ,\quad \sigma_p^2=\left(\frac{\hbar^2ma^2}{2}\right)^{1/2}\ ,\quad R_0=\frac{\hbar}{2}\ . \tag{3.27}$$

This a close to minimal uncertainty state, since it satisfies

$$\sigma_p\sigma_x=\frac{\hbar}{\sqrt{2}}\ . \tag{3.28}$$

The solution (3.26)–(3.28) is very similar to the solution obtained by Diósi [28], but differs by some simple numerical factors, e.g., he obtained

$$(\sigma_x^2)_{\text{Diósi}} = \left(\frac{\hbar}{4ma^2}\right)^{1/2} , \tag{3.29}$$

This difference is due to the fact that Diósi used an Ito equation with a single real Wiener process, whereas the Wiener process used here is complex.

The Diósi solution is also discussed in Ref. [38]. Some stationary solutions to (1.3) for the harmonic oscillator have also been found by Salama and Gisin [39], but their choice of Lindblad operator differs from that used here.

Approximate stationary solutions to the Ito equation (2.4), for general potentials, are currently being studied by Brun *et al.* [40].

IV. A LOCALIZATION THEOREM

We now show that all solutions to the Ito equation tend towards the stationary solution in the long-time limit. The demonstration applies primarily to the case of linear systems, but we will work with a general potential in what follows, saving until the end the issue of the extent to which that case is properly covered here.

We have shown that there is a two-parameter family of stationary solutions, parametrized by the centers $\langle x\rangle$, $\langle p\rangle$. To prove that all solutions tend to stationary solution, we will exploit the fact that the stationary solutions are uniquely characterized by the statement that they are the eigenfunctions of the operator

$$A = \hat{p} - 2i\hbar\beta\hat{x} , \tag{4.1}$$

where β is the solution to Eq. (3.19). This means that the stationary solutions are uniquely defined by the statement that $(\Delta A)^2 = 0$. We shall prove the desired result by showing that $(\Delta A)^2$ tends to zero, in the mean.

A number of "localization theorems," showing that the dispersion of certain operators decreases with time, in the mean, have been proved by Gisin and Percival [18] and by Percival [27]. None of these results is applicable to the present case because their assumptions are too restrictive. They assume, for example, that the Hamiltonian is zero (or negligible), or that the Lindblad operators commute with the Hamiltonian. In brief, they assume that the Hamiltonian plays no significant role. An important feature of the case considered in this paper is that the stationary solutions are possible as a result of a balance between the wave packet spreading induced by the Hamiltonian and the localizing effect of the Lindblad operators, and hence the role of the Hamiltonian cannot be ignored. An argument for the local stability of the stationary solution in the free particle case with $b = 0$ was given by Diósi [28], but this proves nothing about arbitrary initial states.

Returning to the case at hand, we have

$$\begin{aligned}(\Delta A)^2 &= \sigma(A, a) \\ &= (\Delta p)^2 + 4\hbar^2|\beta|^2(\Delta x)^2 \\ &\quad -2i\hbar(\beta+\beta^*)R - \hbar^2(\beta+\beta^*) .\end{aligned} \tag{4.2}$$

The rate of change of $(\Delta A)^2$ in the mean, $Md(\Delta A)^2$, is then easily computed from Eqs. (2.11)–(2.13). It is convenient to write

$$(\Delta x)^2 = \sigma_x^2(1+X) , \tag{4.3}$$

$$(\Delta p)^2 = \sigma_p^2(1+Y) , \tag{4.4}$$

$$R = R_0(1+Z) , \tag{4.5}$$

hence the stationary solution is $X = Y = Z = 0$. One then obtains

$$\begin{aligned}M\frac{d(\Delta A)^2}{dt} &= c_1X + c_2Y + c_3Z \\ &\quad -2a^2\left(R_0^2 + \frac{\hbar^2}{4}\right)X^2 - 2b^2\sigma_p^4Y^2 \\ &\quad -2R_0^2\left(a^2 + b^2\frac{\sigma_p^2}{\sigma_x^2}\right)Z^2 \\ &\quad +4a^2R_0^2XZ + 4b^2\frac{\sigma_p^2}{\sigma_x^2}R_0^2YZ ,\end{aligned} \tag{4.6}$$

where

$$c_1 = -\hbar^2a^2 + 2\hbar ab\sigma_p^2 + 2R_0V''(\bar{x}) , \tag{4.7}$$

$$c_2 = -2\hbar ab\sigma_p^2 - \frac{2R_0}{m}\frac{\sigma_p^2}{\sigma_x^2} - \hbar^2b^2\frac{\sigma_p^2}{\sigma_x^2} , \tag{4.8}$$

$$c_3 = \frac{2R_0}{m}\frac{\sigma_p^2}{\sigma_x^2} - 2R_0V''(\bar{x}) , \tag{4.9}$$

and we have used (3.21) to simplify some of these expressions.

The coefficient c_1, c_2, c_3 have a number of useful properties. First, from Eq. (3.24), it is easily seen that

$$c_1 = -\frac{\hbar^2a^2}{2} - 2a^2R_0^2 - 2b^2\sigma_p^4 \tag{4.10}$$

and thus $c_1 < 0$. Second, using Eq. (3.23),

$$c_2 = -2\frac{\sigma_p^2}{\sigma_x^2}\left(a^2\sigma_x^4 + b^2R_0^2 - \frac{\hbar^2b^2}{4}\right) - \hbar^2b^2\frac{\sigma_p^2}{\sigma_x^2} . \tag{4.11}$$

Using (3.21), twice, it then follows that

$$c_2 = -2a^2\sigma_x^2\sigma_p^2 - 2b^2\sigma_p^4 = c_1 . \tag{4.12}$$

Third, c_1 and c_3 are related as follows. From Eq. (3.25), c_3 may be written

$$c_3 = 4R_0^2\left(a^2 + \frac{\sigma_p^2}{\sigma_x^2}b^2\right) = -2\frac{R_0^2}{\sigma_x^2\sigma_p^2}c_1 \tag{4.13}$$

using (3.21) and (4.10). It follows that the linear terms in (4.6) may now be written

$$c_1 X + c_2 Y + c_3 Z = c_1 \left(X + Y - \frac{2R_0^2}{\sigma_x^2 \sigma_p^2} Z \right) . \tag{4.14}$$

Clearly (4.6) is zero at the stationary solution, but it cannot be negative for arbitrary X, Y, and Z, because of the presence of the linear terms. However, X, Y, and Z are not arbitrary but must respect the uncertainty principle [an expression of which is Eq. (3.20), for example]. A convenient way to implement this restriction is to note that

$$0 \leq (\Delta A)^2 = \sigma_p^2 (X + Y) - \frac{2R_0^2}{\sigma_x^2} Z , \tag{4.15}$$

with equality if and only if the state is a general Gaussian, such as the stationary solution. From (4.14), it is clear that

$$c_1 X + c_2 Y + c_3 Z = \frac{c_1}{\sigma_p^2} (\Delta A)^2 . \tag{4.16}$$

Since $c_1 < 0$, the linear terms are negative definite and zero only at the stationary solution.

With some rearrangement of the quadratic terms, and using (3.21),

$$\begin{aligned} M \frac{d(\Delta A)^2}{dt} = {} & \frac{c_1}{\sigma_p^2} (\Delta A)^2 - \frac{\hbar^2 a^2}{2} X^2 - 2a^2 R_0^2 (X - Z)^2 \\ & -2b^2 \sigma_p^4 \left(Y - \frac{R_0^2}{\sigma_p^2 \sigma_x^2} Z \right)^2 - \frac{\hbar^2 b^2 R_0^2}{2\sigma_x^4} Z^2 . \end{aligned} \tag{4.17}$$

We therefore deduce that

$$M \frac{d(\Delta A)^2}{dt} \leq 0 \tag{4.18}$$

with equality if and only if the solution is the stationary solution. This completes the proof of localization.

As stated earlier, the stationary solutions to the Ito equation are valid for general potentials as long as the localization width is much less than the length scale on which the potential varies, i.e., as long as the approximation (2.20) holds. This approximation becomes exact for linear systems.

We have essentially assumed the approximation (2.20) in proving the above localization theorem. This means that the proof is strictly valid only for systems with quadratic potentials. It cannot be valid for general potentials because even if there exist approximate stationary solutions for which the neglect of the higher derivative terms of the potential is valid, there will always be initial states for which (2.20) is not a valid approximation and localization is therefore not guaranteed for these states. For general potentials, therefore, the above proof implies localization only for a rather limited class of initial states, e.g., for states that are already close to the stationary states.

Still, one intuitively expects that when approximate stationary solutions exist for general potentials, there will be situations in which most initial states will tend towards one of those solutions. Consider, for example, the case of a double well potential with minima of distance L apart, and suppose that the initial state has a width greater than L, where L is chosen so that the approximation (2.20) is not valid. Then one can see from Eq. (2.11) that a very large initial width will be reduced very rapidly, in the mean, bringing it into the regime in which the approximation (2.20) is valid. Our localization theorem would then apply. We hope to investigate this point further in a future publication.

Note that the stationary solutions and the localization theorem do not depend on the sign of $V''(\bar{x})$, and therefore will be valid for the upside-down harmonic oscillator (which is sometimes used as a prototype for chaotic systems [41]).

V. LOCALIZATION RATE

It is also possible to estimate the rate of localization. Clearly,

$$M \frac{d(\Delta A)^2}{dt} \leq \frac{c_1}{\sigma_p^2} (\Delta A)^2 \tag{5.1}$$

and thus localization proceeds on a time scale of order $\tau = \sigma_p^2 / |c_1|$. Using (4.12), this becomes

$$\tau = (2a^2 \sigma_x^2 + 2b^2 \sigma_p^2)^{-1} . \tag{5.2}$$

In the quantum Brownian motion model for the free particle with $b = 0$, Eqs. (2.5), (3.27) imply that

$$\tau \sim \left(\frac{\hbar}{\gamma k T} \right)^{1/2} . \tag{5.3}$$

This, as noted previously, is the time scale on which thermal fluctuations become comparable to the quantum ones [42–44].

The above represents the minimum rate of localization. The actual rate can be much higher, e.g., if X is very large. Consider again the free particle with $b = 0$. Suppose the initial state consists of a superposition of wave packets a large distance l apart. Then $(\Delta x)^2 \sim l^2$,

$$(\Delta A)^2 \approx 4\hbar^2 |\beta|^2 (\Delta x)^2 \sim \frac{\hbar^2 l^2}{\sigma_x^4} \tag{5.4}$$

and the dominant contribution to the localization rate is the X^2 term:

$$M \frac{d(\Delta A)^2}{dt} \approx -2a^2 \left(R_0^2 + \frac{\hbar^2}{4} \right) X^2 \sim -\frac{\hbar^2 a^2 l^4}{\sigma_x^4} . \tag{5.5}$$

It follows that, in this case

$$\tau \sim \frac{1}{l^2 a^2} . \tag{5.6}$$

For the quantum Brownian motion model, Eq. (2.5) then implies that

$$\tau \sim \frac{\hbar^2}{l^2 m\gamma kT} \ . \tag{5.7}$$

Both of the time scales (5.3) and (5.7) are typically exceedingly small for macroscopic values of m, γ, and T.

As we shall show in detail in the next section, once the solutions to the Ito equation have become localized, the corresponding density operator has the form (1.5). The localization time scale is therefore the time scale on which the density operator approaches the form (1.5). Since the process of decoherence of density operators is commonly associated with the approach to approximately diagonal form, it is natural to regard the localization time scale as essentially the same thing as the decoherence time scale.

Note, however, that the so-called "decoherence time scale" is sometimes taken to be (5.7) [14,45,15]. What is clear from the above is that the rate of approach to diagonal form depends on initial state, and that (5.7) is appropriate only for initial states with very large $(\Delta x)^2$.

The connections between the time scales of decoherence and thermal fluctuations has certainly been noted before [43,42], but what is new here is the observation that both of these things are in turn related to the time scale of localization in quantum state diffusion.

VI. RECOVERY OF THE DENSITY OPERATOR

We now show how a density operator satisfying the master equation may be recovered from the stationary solutions to the Ito equation.

Each solution to the Ito equation is in general a functional of the noise term $\xi(t)$ over the entire history of the solution's evolution. Equation (2.1) indicates that the density operator is formally recovered from these solutions by averaging $|\psi\rangle\langle\psi|$ over all possible histories of the noise $\xi(t)$, and we write

$$\rho = M|\psi_\xi\rangle\langle\psi_\xi| \ . \tag{6.1}$$

A completely explicit form of this expression may be found in Ref. [32] but it will not be needed here.

When the solutions $|\psi_\xi\rangle$ are the stationary solutions, (3.17), they depend on the noise $\xi(t)$ only through their centers, $\langle x\rangle, \langle p\rangle$, which obey the Langevin equations (2.9), (2.10). We may therefore rewrite (6.1) as

$$\rho = M\int dp\, dq \delta(p-\bar{p})\delta(q-\bar{x})|\psi_{pq}\rangle\langle\psi_{pq}| \ , \tag{6.2}$$

where we have again introduced the notation $\bar{x} = \langle x\rangle$, $\bar{p} = \langle p\rangle$, and $|\psi_{pq}\rangle$ denotes the stationary solution (3.17) with centers p and q. The $\xi(t)$ dependence is now contained entirely in $\bar{p}$ and $\bar{x}$, and Eq. (6.2) may be trivially rewritten

$$\rho = \int dp\, dq f(p,q,t)|\psi_{pq}\rangle\langle\psi_{pq}| \ , \tag{6.3}$$

where

$$f(p,q,t) = M\delta(p-\bar{p})\delta(q-\bar{x}) \ . \tag{6.4}$$

The weight $f(p,q,t)$ is non-negative and satisfies

$$\int dp\, dq f(p,q,t) = 1 \ . \tag{6.5}$$

It is in fact the solution to the Fokker-Planck equation corresponding to the Langevin equations. This Fokker-Planck equation is readily derived as follows. First note that

$$f + df = M\delta(p-\bar{p}-d\bar{p})\delta(q-\bar{x}-d\bar{x}) \ . \tag{6.6}$$

Now expanding the δ functions to second order, we have

$$\begin{aligned} f + df = M\Big(&\delta(p-\bar{p})\delta(q-\bar{x}) - d\bar{x}\,\delta(p-\bar{p})\delta'(q-\bar{x}) - d\bar{p}\,\delta'(p-\bar{p})\delta(q-\bar{x}) \\ &+\frac{1}{2}d\bar{x}^2\delta(p-\bar{p})\delta''(q-\bar{x}) + d\bar{p}\,d\bar{x}\,\delta'(p-\bar{p})\delta'(q-\bar{x}) + \frac{1}{2}d\bar{p}^2\delta''(p-\bar{p})\delta(q-\bar{x})\Big) \ . \end{aligned} \tag{6.7}$$

We may now use the Langevin equations for $\bar{x}$ and $\bar{p}$, and also pull the derivatives outside the mean M: for example,

$$\begin{aligned} M[d\bar{x}\delta(p-\bar{p})\delta'(q-\bar{x})] &= M\left(\frac{\bar{p}}{m}\delta(p-\bar{p})\frac{\partial}{\partial q}\delta(q-\bar{x})\right)dt \\ &= \frac{p}{m}\frac{\partial f}{\partial q}dt \ . \end{aligned} \tag{6.8}$$

We thus obtain the Fokker-Planck equation

$$\begin{aligned} \frac{\partial f}{\partial t} = &-\frac{p}{m}\frac{\partial f}{\partial q} + V'(q)\frac{\partial f}{\partial p} + 2\hbar ab\frac{\partial f}{\partial p} \\ &+|\sigma(p,L)|^2\frac{\partial^2 f}{\partial p^2} + |\sigma(x,L)|^2\frac{\partial^2 f}{\partial q^2} \\ &+2\,\mathrm{Re}[\sigma(x,L)\sigma(L,p)]\frac{\partial^2 f}{\partial p\partial q} \ . \end{aligned} \tag{6.9}$$

The coefficients of the second derivative terms are

$$|\sigma(p,L)|^2 = a^2R_0^2 + b^2\sigma_p^4 - \hbar ab\sigma_p^2 + \frac{\hbar^2 a^2}{4} \ , \tag{6.10}$$

$$|\sigma(x,L)|^2 = b^2 R_0^2 + a^2\sigma_x^4 - \hbar ab\sigma_x^2 + \frac{\hbar^2 b^2}{4} \,, \qquad (6.11)$$

$$2\,\mathrm{Re}[\sigma(x,L)\sigma(L,p)] = 2a^2 R_0\sigma_x^2 + 2b^2\sigma_p^2 - 2\hbar ab R_0 \,. \qquad (6.12)$$

We have $2\hbar ab = 2\gamma$, and for high temperature, the dominant term of the three second derivative terms is the first one, which has a coefficient:

$$|\sigma(p,L)|^2 \approx 2m\gamma kT \,. \qquad (6.13)$$

The resulting Fokker-Planck equation is well known [46]. All solutions (for potentials for which $e^{-V/kT}$ is normalizable) tend towards the stationary solution

$$f(p,q) = N \exp\left(-\frac{p^2}{2mkT} - \frac{V(q)}{kT} \right) \qquad (6.14)$$

like $e^{-\gamma t}$, where N is a normalization factor. For simplicity consider now the harmonic oscillator case $V(q) = \frac{1}{2}m\omega^2 q^2$. Then the integrations over p and q may be done explicitly, with the result

$$\rho(x,y) = \exp\left(-\frac{|\beta|^2}{\Delta}(x-y)^2 - \frac{m\omega^2(\beta+\beta^*)}{2kT\Delta}(x^2+y^2) \right) \qquad (6.15)$$

up to a normalization factor, where

$$\Delta = \frac{m\omega^2}{2kT} + \beta + \beta^* \,. \qquad (6.16)$$

For large temperature, this is readily shown to be a thermal state [47]. Similar results are expected to hold for the case of more general potentials.

To summarize, an initial density operator approaches the form (6.3) on the localization time scale, i.e., typically very quickly. On much longer time scales, it will then relax to an equilibrium density operator, when one exists for the system (it does not for the free particle, for example).

Note that although the above derivation of the asymptotic form (6.3) strictly concerned pure initial states, it is readily extended to mixed initial states by writing the initial state in a diagonal basis,

$$\rho_0 = \sum_n c_n |n\rangle\langle n| \qquad (6.17)$$

and then applying the above to each initial state $|n\rangle\langle n|$. One thus finds that the density operator tends to form (6.3), with $f(p,q,t)$ of the form

$$f(p,q,t) = \sum_n c_n f_n(p,q,t) \,, \qquad (6.18)$$

where $f_n(p,q,t)$ is the solution to the Fokker-Planck equation corresponding to the initial state $|n\rangle\langle n|$.

As a final comment, note that *any* density operator may be written in the form (6.3), for some function $f(p,q)$—this is a property of the coherent states [36]. What is special about the particular function $f(p,q,t)$ derived here is that it is non-negative, and that it obeys the Fokker-Planck equation (6.9). It may therefore reasonably be interpreted as a phase space probability distribution. (See Ref. [48] for related work on this point.)

VII. CONNECTION WITH THE DECOHERENT HISTORIES APPROACH

As shown in Ref. [32], there is a close connection between the quantum state diffusion approach to open systems and the decoherent histories approach. In this section, we use the results of the previous sections to exemplify and amplify this connection.

The primary mathematical aim of the decoherent histories approach is to assign probabilities to the possible histories of a closed system [4,29–31,49]. The approach is, however, applicable to open systems since they may be regarded as subsystems of a large closed system. A quantum-mechanical history is defined by an initial state ρ_0 at time $t = t_0$ together with a string of projection operators $P_{\alpha_1} \cdots P_{\alpha_n}$ acting at times $t_1 \cdots t_n$, characterizing the possible alternatives of the system at those times. The projections are exhaustive, $\sum_\alpha P_\alpha = 1$, and exclusive, $P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha$. Because of interference, most sets of histories for a closed system cannot be assigned probabilities. The interference between pairs of histories in a set is measured by the so-called decoherence functional:

$$D(\underline{\alpha},\underline{\alpha}') = \mathrm{Tr}\big(P_{\alpha_n}(t_n)\cdots P_{\alpha_1}(t_1)\rho P_{\alpha_1'}(t_1)\cdots P_{\alpha_n'}(t_n)\big) \,, \qquad (7.1)$$

where $P_{\alpha_k}(t_k) = \exp(-\frac{i}{\hbar}Ht_k)P_\alpha \exp(\frac{i}{\hbar}Ht_k)$, H is the Hamiltonian of the closed system, and $\underline{\alpha}$ denotes the string $\alpha_1 \cdots \alpha_n$. When

$$D(\underline{\alpha},\underline{\alpha}') \approx 0 \qquad (7.2)$$

for all pairs $\underline{\alpha} \neq \underline{\alpha}'$, interference may be neglected, and the set of histories is then said to be decoherent. One may then assign the probability $p(\underline{\alpha}) = D(\underline{\alpha},\underline{\alpha})$ to the history, which may be shown to obey the sum rules of probability theory.

For a given Hamiltonian and initial state, one's initial aim is to find those histories for which the decoherence condition is satisfied. In general, it is satisfied only by histories which are coarse-grained, which loosely speaking means that the projections at each moment of time give a less than complete description of the system. For open systems, a natural coarse graining is to focus only on the properties of the distinguished system itself, while ignoring the environment. This involves using projections of the form, $P_\alpha \otimes I^{\mathcal{E}}$ at each moment of time, where P_α is a projection onto the distinguished subsystem and $I^{\mathcal{E}}$ denotes the identity on the environment. Assuming that the initial density operator factorizes, the trace over the environment may be carried out explicitly in the decoherence functional (7.1), and, in the regime in which a Markovian approximation holds, it then has the form

$$D(\underline{\alpha}, \underline{\alpha}') = \mathrm{Tr}(P_{\alpha_{n-1}} K^{t_n}_{t_{n-1}} [P_{\alpha_{n-1}} \cdots K^{t_2}_{t_1} (P_{\alpha_1} K^{t_1}_{t_0} [\rho_0] P_{\alpha'_1}) \cdots P_{\alpha'_{n-1}}] P_{\alpha_n}) \ , \tag{7.3}$$

where the trace is now over the distinguished subsystem only. The quantity $K^{t_{k+1}}_{t_k}$ is the reduced density operator propagator associated with the master equation (1.1), $\rho_t = K^t_0[\rho_0]$.

The results of the previous sections have provided us with some information about the density operator propagator, and we can use this information to establish some properties of the decoherence functional (7.3).

We have seen that any density operator will tend, on a typically very short time scale, to the form (6.3), in which it is approximately diagonal in a set of phase space localized states. Once in that form, under further evolution its form will be preserved and the only change will be that the function $f(p, q, t)$ will evolve according to the Fokker-Planck equation (6.9).

Take the projection operators in the decoherence functional to be phase space projectors, of the form

$$P_\alpha = \int_{\Gamma_\alpha} dq\, dq |\psi_{pq}\rangle\langle\psi_{pq}| \ , \tag{7.4}$$

where $|\psi_{pq}\rangle$ are the generalized coherent states (3.17), and are eigenstates of the operator (4.1). These quantities are not exact projection operators, but will be approximate projectors if the phase space region Γ_α is sufficiently large, and if its boundary is sufficiently smooth [30]. They have the property that $P_\alpha|\psi_{pq}\rangle \approx |\psi_{pq}\rangle$, if p, q lie in the phase space cell Γ_α, and $P_\alpha|\psi_{pq}\rangle \approx 0$ otherwise. Again this approximation should be valid if Γ_α is sufficiently large compared to the phase space area occupied by the generalized coherent states (which is of order $\hbar$).

Consider the time evolution from t_0 to t_1 in the decoherence functional. Clearly if this time interval is greater than the localization time it follows from the results of Sec. VI that the density operator will evolve into the form

$$K^{t_1}_{t_0}[\rho_0] = \int dp\, dq f(p, q, t_1) |\psi_{pq}\rangle\langle\psi_{pq}| \ . \tag{7.5}$$

because it is approximately diagonal in the coherent states, it is easy to see that

$$P_{\alpha_1} K^{t_1}_{t_0}[\rho] P_{\alpha'_1} \approx 0 \tag{7.6}$$

if $\alpha_1 \neq \alpha'_1$. Keeping only the diagonal terms, $\alpha_1 = \alpha'_1$, and evolving to time t_2, the (unnormalized) density operator $P_{\alpha_1} K^{t_1}_{t_0}[\rho] P_{\alpha_1}$ should again evolve into approximately diagonal form, and again we get

$$P_{\alpha_2} K^{t_2}_{t_1}[P_{\alpha_1} K^{t_1}_{t_0}[\rho] P_{\alpha_1}] P_{\alpha'_2} \approx 0 \tag{7.7}$$

if $\alpha_2 \neq \alpha'_2$. Continuing in this way for the entire history, it is easy to see that we will have approximate decoherence if the projections at each moment of time are taken to be phase space projectors. We have not estimated the degree of approximate decoherence (and this tends to be rather involved in general), but we expect it to be good if the size of the phase space cells is much larger than $\hbar$, and if the time between projections is longer than the localization time. We therefore find that localization in quantum state diffusion and decoherence of histories in the decoherent histories approach occur in the *same variables.*

This conclusion is in agreement with the general connection between quantum state diffusion and decoherent histories outlined in Ref. [32], but it also extends it somewhat. There, it was argued that localization and decoherence tend to occur in the Lindblad operators. Here, the Lindblad operator is essentially position, but we have actually obtained the stronger conclusion that localization or decoherence occurs in the operator (4.1), and hence, approximately, in both position *and* momentum. (Note that the Lindblad operator has a small momentum part added, but this is not the primary source of momentum localization. Rather, it is the interplay between the position part of the Lindblad operator and the Hamiltonian, as discussed earlier.)

Given approximate decoherence, we now consider the probabilities for histories, given by the diagonal elements of the decoherence functional. From Eq. (7.5), and from the properties of the phase space projections, it follows that

$$P_{\alpha_1} K^{t_1}_{t_0}[\rho_0] P_{\alpha_1} \approx \int_{\Gamma_{\alpha_1}} dp_1 dq_1 f(p_1, q_1, t_1) |\psi_{p_1 q_1}\rangle\langle\psi_{p_1 q_1}| \ . \tag{7.8}$$

Now consider the evolution from t_1 to t_2. We have, from Sec. VI,

$$K^{t_2}_{t_1}[|\psi_{p_1 q_1}\rangle\langle\psi_{p_1 q_1}|] = \int dp_2 dq_2 f(p_2, q_2, t_2 | p_1, q_1, t_1) |\psi_{p_2 q_2}\rangle\langle\psi_{p_2 q_2}| \ , \tag{7.9}$$

where $f(p_2, q_2, t_2|p_1, q_1, t_1)$ is the solution to the Fokker-Planck equation satisfying the initial condition

$$f(p_2, q_2, t_1|p_1, q_1, t_1) = \delta(p_2 - p_1)\delta(q_2 - q_1) \ . \tag{7.10}$$

$f(p_2, q_2, t_2|p_1, q_1, t_1)$ is therefore the Fokker-Planck propagator, i.e., the probability of finding the particle at p_2, q_2 at time t_2, given that it was at p_1, q_1 at time t_1. Assembling (7.8) and (7.9), it follows that

$$\begin{aligned} P_{\alpha_2} K_{t_1}^{t_2}[P_{\alpha_1} K_{t_0}^{t_1}[\rho_0] P_{\alpha_1}] P_{\alpha_2} \approx \int_{\Gamma_{\alpha_2}} dp_2 dq_2 \int_{\Gamma_{\alpha_1}} & dp_1 dq_1 f(p_2, q_2, t_2|p_1, q_2, t_1) \\ & \times f(p_1, q_1, t_1) |\psi_{p_1 q_1}\rangle\langle\psi_{p_1 q_1}| \ . \end{aligned} \tag{7.11}$$

Continuing in this way for the entire history, one finds that

$$\begin{aligned} p(\alpha_1, \ldots, \alpha_n) = \int_{\Gamma_{\alpha_n}} dp_n dq_n \cdots \int_{\Gamma_{\alpha_1}} & dp_1 dq_1 f(p_n, q_n, t_n|p_{n-1}, q_{n-1}, t_{n-1}) \\ & \times \cdots f(p_2, q_2, t_2|p_1, q_1, t_1) f(p_1, q_1, t_1) \ . \end{aligned} \tag{7.12}$$

This is the desired result. Equation (7.12) is the probability that a particle evolving according to the stochastic process described by the Fokker-Planck equation (6.9) will be in the sequence of phase space cells $\Gamma_{\alpha_1} \cdots \Gamma_{\alpha_n}$ at times $t_1 \cdots t_n$.

This result is in agreement with the probabilities one would assign to histories in the quantum state diffusion approach. For there, once the solutions to the Ito equation have become localized, the description of the motion on scales greater than the localization width is classical Brownian motion according to the Langevin equations (2.9),(2.10). This is equivalent to the description in terms of the Fokker-Planck equation (6.9). We have therefore exemplified the second part of the connection between quantum state diffusion and decoherent histories put forward in Ref. [32], that the probabilities assigned to histories in each approach are the same.

A further claim in Ref. [32] is that the degree of localization is related to the degree of decoherence. Although they are clearly related, it is difficult to check this here because, as stated above, explicit computation of the degree of approximate decoherence is quite difficult. This point will be pursued in more detail elsewhere.

Finally, a property of the Fokker-Planck propagator associated with Eq. (6.9) is that it is peaked about classical evolution (with dissipation). It follows that the probability for histories (7.12) will be most strongly peaked when the phase space cells lie along a classical path.

VIII. SUMMARY AND DISCUSSION

Our main results are as follows.

We have found stationary solutions to the Langevin-Ito equation (2.4) which are exact for linear systems, and approximate for nonlinear systems as long as the localization width is much less than the scale on which the potential varies. The solutions consist of localized wave packets concentrated about a point in phase space undergoing classical Brownian motion.

For linear systems, every initial state tends towards one of the stationary solutions. For nonlinear systems, some form of localization is plausible, and will certainly be true in the neighborhood of the stationary solutions, but our investigations on this point are inconclusive.

Localization proceeds on a time scale which is typically very short. It is related to the time scale on which thermal and quantum fluctuations become comparable, and also to the decoherence time scale.

The density operator corresponding to the stationary solutions may be reconstructed and has the form (1.5). It is therefore diagonal on a set of phase space localized states. For linear systems (and plausibly for many nonlinear systems also) any initial density operator approaches this form on the localization time scale. On longer time scales, when dissipation is present, the density operator approaches a thermal state (when it exists) in the long-time limit, as expected on general grounds. These results fulfill the aims set out in the Introduction, concerning the density matrix approach to decoherence.

Our work also has some implications for the question of approximate versus exact density matrix diagonalization. As discussed in the Introduction, it is often held important in the context of decoherence studies to find the basis in which the density matrix is diagonal. This can of course always be done, since the density operator is a Hermitian operator, but the basis in which ρ is exactly diagonal is generally nontrivial, i.e., it does not usually consist of the eigenstates of a simple operator. Furthermore, the basis consists of eigenstates of a different operator at each moment of time.

Here, we have shown that the quantum state diffusion approach naturally leads to a basis in which the density matrix is *approximately* diagonal. The basis states are the eigenstates of a simple operator, the same operator at each moment of time. There therefore appears to be much to be gained by relaxing the condition of *exact* diagonality. Corresponding to these exactly and approximately diagonalizing bases, there will be exactly and approximately decoherent set of histories in the decoherent histories approach. In Sec. VII, we exhibited the approximately decoherent set.

The bases of approximate and exact diagonality do not appear to be "close" in any sense. For example, for a Gaussian density operator (in the position representation), the exactly diagonal basis consists of Hermite polynomials multiplied by Gaussians (similar to energy eigenstates of the harmonic oscillator) [5], whereas the approximately diagonal one consists of phase space localized states. (See also Ref. [50] for examples of different bases in which the density matrix is diagonal.) This suggests that the corresponding exactly decoherent set of histories is not necessarily "close" to the approximately diagonal one, somewhat contrary to the expectation sometimes expresses [51] (although it is not clear whether there are other exactly decoherent sets of histories that *are* close to the approximate one).

The basis of states picked out by the QSD approach appears to be "natural," in the sense that they correspond to the trajectories that would actually be observed in an individual experiment, whereas the exactly diagonal basis does not, in general. Correspondingly, the approximately decoherent set of histories may seem to be more "natural" than the exactly decoherent set. The question of whether one is in any sense preferred over the other is, however, a subtle one. It depends on the sort of predictions one wishes to make, and on the extent to which the simplified situation consisting of a distinguished system coupled to an environment is really part of a much larger universe in which there may be adaptive systems that can measure different properties of the distinguished subsystem [4].

The sum up, the model described in this paper illustrates the connection between the intuitive pictures and physical predictions provided by the quantum state diffusion approach, density matrix approaches, and the decoherent histories approach. In our model, localization in quantum state diffusion, diagonalization in the density matrix approach, and decoherence of histories in the decoherent histories approach all occur under the same conditions and are essentially the same thing, for each is concerned with the conditions under which "definite properties" may be assigned to the system. Furthermore, the probabilities assigned to histories in the quantum state diffusion approach and the decoherent histories approach approximately coincide.

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