Reducing S duality to T duality

Jeffrey A. Harvey

Enrico Fermi Institute, University of Chicago, 5640 Ellis Avenue, Chicago, Illinois 60637

Gregory Moore

Department of Physics, Yale University, New Haven, Connecticut 06520

Andrew Strominger

Department of Physics, University of California, Santa Barbara, California 93106-9530

The infrared limit of $D = 4$, $N = 4$ Yang-Mills theory with compact gauge group G compactified on a two-torus is governed by an effective superconformal field theory. We conjecture that this is a certain orbifold involving the maximal torus of G. Yang-Mills S duality makes predictions for all correlators of this effective conformal field theory. These predictions are shown to be implied by the standard T duality of the conformal field theory. Consequently, the Montonen-Olive duality between electric and magnetic states reduces to the standard two-dimensional duality between momentum and winding states.

Seventeen years ago Montonen and Olive [1] made a bold conjecture: Yang-Mills theory with gauge group G and coupling constant e is *identical* to a gauge theory based on the dual gauge group[1] G^v [2] with coupling $4\pi/e$. The identification involves a relabeling of states and operators, interchanging particles with solitons, and electric charges with magnetic charges. It was quickly realized [3,4] that the conjecture is viable only for $N = 4$ supersymmetric Yang-Mills theory. When a θ angle is included, 2π shifts of θ together with the $\mathbb{Z}_2$ symmetry of Montonen and Olive generate an $SL(2,\mathbb{Z})$ symmetry which acts on the complex coupling constant

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi}{e^2} i \equiv \frac{\theta}{2\pi} + \frac{i}{\alpha} \tag{1}$$

as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{2}$$

with $ad - bc = 1$, $a, b, c, d \in \mathbb{Z}$. This $SL(2,\mathbb{Z})$ symmetry on the space of theories is known as S duality. S duality originated in the study of lattice models [5,6] but has come to play a prominent role in recent speculations concerning the structure of both $N = 4$ Yang-Mills theory and string theory [7] as reviewed in [8].

Initial evidence in favor of S duality for $N = 4$ Yang-Mills theory was provided by the exact agreement [3,4] of the (calculable) masses of the stable elementary particles and solitons with those predicted by the $\mathbb{Z}_2$ symmetry of Montonen and Olive. More recently the masses of some bound states have been demonstrated to be in agreement with the full S duality [8], and the predictions of S duality for a topologically twisted version of the theory on a more

---

[1] We recall the definition of G^v below.

general four-manifold have been tested [9].

A skeptic could remain unconvinced by this evidence. It concerns only zero-momentum, supersymmetric, or topological properties of the theory. Such properties are highly constrained by the powerful symmetries of theory, especially by the $N = 4$ supersymmetry. Thus, a true skeptic may argue that the evidence to date all follows from the known symmetries of the theory in some delicate way. A more reasonable skeptic might argue that S duality is indeed nontrivial, but only holds for the supersymmetric or BPS saturated states of the theory. If the Montonen-Olive conjecture is correct, the theory and its S dual must agree on much more than this. In particular, all finite-momentum correlation functions must agree. This is clearly *not* implied by the known symmetries: the addition of higher dimension operators to the theory can change the correlation functions without affecting the topological quantities. It is also clear that a two-particle state with nonzero center-of-mass momentum is not Bogomolny-Prasad-Sommerfield (BPS) saturated even if the individual one-particle states are. Thus evidence for S duality at nonzero momentum necessarily involves evidence for S duality away from the supersymmetric subspace of the theory.

In this paper, we will propose and confirm, with some assumptions, a finite-momentum test of S duality, albeit in a very special limit. The idea is to compactify four-dimensional $N = 4$ Yang-Mills theory with group G to two dimensions on a torus. At distances large compared to the size of the torus, the effective theory must reduce to a conformal field theory. We conjecture and give plausibility arguments that this takes a particular form involving the maximal torus $T \subset G$ (and some antisymmetric tensor fields if θ is nonzero). S-duality transformations involve no dimensionful parameters, and therefore commute with scale transformations. S duality of $D = 4$, $N = 4$ Yang-Mills theory therefore makes a definite prediction of an exact duality symmetry of the effec-

tive conformal field theory which must act on all finite-momentum correlation functions. This prediction will indeed be confirmed in the following: S duality reduces to the well-known "T duality" of conformal field theory in which tori are interchanged with their duals. The interchange of electric and magnetic charges effected by S duality is essentially the familiar interchange of momen-

tum and winding modes effected by T duality.

Consider compactification of $D = 4, N = 4$ supersymmetric Yang-Mills theory with compact gauge group G. For simplicity we first consider only the case for which the simply connected covering group $\tilde{G}$ associated to G is SU(n). The general case is treated in the Appendix.

The bosonic part of the $D = 4, N = 4$ action is

$$I^{\text{bosonic}} = -\frac{1}{4\pi}\frac{1}{\alpha}\int d^4x\sqrt{-g^{(4)}}\text{Tr}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \sum_{I=1}^{6}D_\mu\phi^I D^\mu\phi^I + \sum_{1\le I<J\le 6}([\phi^I,\phi^J])^2\right] - \frac{\theta}{8\pi^2}\int \text{Tr}F \wedge F , \quad (3)$$

where all fields take values in the Lie algebra $\underline{g}$ of G, and Tr is a nondegenerate bilinear form on $\underline{g}$. We normalize Tr so that Euclidean instantons with integral winding number k have action $2\pi ik\tau$. If $\underline{g} = \mathbf{su(n)}$ is identified with the Lie algebra of $n \times n$ anti-Hermitian matrices the metric is[2]

$$(a,b)_{\mathbf{su(n)}} \equiv -\text{Tr}_{\mathbb{C}^n}(ab) . \quad (4)$$

We now compactify the theory by taking spacetime to be $\sum \times T^2$ where the internal space T^2 is a small two-torus of volume L^2. The line element is

$$ds^2 = -(d\sigma^0)^2 + (d\sigma^1)^2 + \frac{L^2}{\rho_2}|dx^2 + \rho dx^3|^2 , \quad (5)$$

where $0 < x^2, x^3 \le 1$, and $\rho = \rho_1 + i\rho_2$ is a modular parameter for the internal torus. Define fields $X^I \in \underline{g}, I = 1,8$ by

$$\begin{aligned} X^I &\equiv L\phi^I , \quad I = 1,6 , \\ X^7 &\equiv A_2 , \\ X^8 &\equiv A_3 . \end{aligned} \quad (6)$$

The effective action at length scales much greater than L then reduces to

$$I^{\text{bosonic}} = -\frac{1}{4\pi}\frac{1}{\alpha}\int_\Sigma d^2\sigma\text{tr}\left[\frac{L^2}{2}F^2 + \sum_{1\le I,J\le 8}G_{IJ}D_\mu X^I D^\mu X^J + \frac{1}{L^2}\sum_{I<J}[X^I,X^J][X^K,X^L]G_{IK}G_{JL}\right]$$
$$- \frac{\theta}{4\pi^2}\int_\Sigma \text{Tr}\left[F_{01}[X^7,X^8] - D_0 X^7 D_1 X^8 + D_1 X^7 D_0 X^8\right] d\sigma^0 \wedge d\sigma^1 , \quad (7)$$

where $\mu,\nu = 0,1$, and the metric G_{IJ} is

$$G_{IJ} = \delta_{IJ}^{(6)} \oplus \frac{1}{\rho_2}\begin{pmatrix} |\rho|^2 & -\rho_1 \\ -\rho_1 & 1 \end{pmatrix} . \quad (8)$$

There are in addition fermionic terms whose form is fixed by the extended supersymmetry but these will not be needed.

The action (7) contains terms of dimension not equal to two and so does not represent the infrared limit of the theory. Our conjecture is that in the infrared limit it flows to a conformal theory with bosonic action:

$$I^{\text{bosonic}} = -\frac{1}{4\pi}\frac{1}{\alpha}\int d^2\sigma\text{Tr}\left[\sum_{1\le I,J\le 8}G_{IJ}\partial_\mu X^I \partial^\mu X^J\right] - \frac{\theta}{4\pi^2}\int \text{Tr}\left[-\partial_0 X^7 \partial_1 X^8 + \partial_1 X^7 \partial_0 X^8\right]d\sigma^0 \wedge d\sigma^1 . \quad (9)$$

---

[2]The normalization for arbitrary simple G is given in Eq. (A1).

In contrast with (7), the fields $X^I, I = 1, 8$, now take values in the Cartan subalgebra $\underline{t}$. Moreover, they are subject to important global identifications discussed below.

A very naive argument leading to (9) is the following. The third term in (7) is a potential term for the X's. It is relevant and grows in the infrared. At low energies X is thus restricted to values for which the potential vanishes, namely the Cartan subalgebra $\underline{t}$. For X in $\underline{t}$, the charge current of the X's vanish. The gauge fields (which have no local dynamics in two dimensions) may then be completely decoupled from the X's in light cone gauge. Their action is quadratic and they may be integrated out.

This argument is too naive for several reasons. Consider[3] an $N = 2$ Landau-Ginzburg model with a ϕ^n potential. The potential is relevant and for $n = 2$ one indeed finds that the infrared limit is the (trivial) theory with ϕ restricted to the minimum of the potential, in accord with the preceding paragraph. However for $n > 2$ there is no mass gap and the infrared limit is a minimal model determined by n. So in general it is not correct simply to restrict the fields to the minimum of the potential. For our model, if we denote by Y fields orthogonal to the Cartan subalgebra and by Z fields in the Cartan subalgebra, then in the infrared limit $L \to 0$ (7) contains large quartic interactions of the form Y^4 and Z^2Y^2. In four spacetime dimensions these terms give a mass to the Y fields at generic points in the moduli space of $D = 4, N = 4$ vacua, and the IR limit is just a theory of the Z's (that is, an Abelian gauge theory). The action (9) is just the dimensional reduction of this Abelian theory. Unfortunately, the compactification along $\Sigma \times T^2$ is not so simple because the large wavelength fluctuations of the Z fields explore all of the moduli space. Near $Z \sim 0$ the Y fields are light and must be taken into account. As $L \to 0$ the region in Z field space for which the Y fields are light becomes vanishingly small. Thus in the infrared limit we might be able to restrict attention to the Z fields with the Y fields set to zero, and the two-dimensional gauge fields decoupled. If the Y fields do appear in some

way in the infrared limit then the two-dimensional gauge fields do not decouple and must be dealt with. Clearly a more careful analysis, perhaps using the $N = 4$ supersymmetric nonrenormalization theorems, is required to see if the above assumption are justified. Despite these misgivings, we strongly suspect that (9) is at least part of the answer. We henceforth proceed on that assumption.

We now discuss global identifications of the fields $X^I \in \underline{t}$ appearing in (9). Two types of identifications arise because choosing the Cartan subalgebra $\underline{t}$ does not completely fix the gauge freedom.[4] First of all, we must take into account gauge transformations of the form

$$g(x_2, x_3) = \exp[2\pi x_2 A + 2\pi x_3 B] , \qquad (10)$$

where $A, B \in \underline{t}$ must satisfy

$$\exp(2\pi A) = \exp(2\pi B) = 1 \qquad (11)$$

in order that (10) is single valued as a nontrivial loop is traversed in the internal torus. The set of such Lie algebra elements forms the coweight lattice $\Lambda_{\text{coweight}}(G) \subset \underline{t}$:

$$\Lambda_{\text{coweight}}(G) \equiv \{A \in \underline{t} : \exp(2\pi A) = 1\} \subset \underline{t} . \qquad (12)$$

$\Lambda_{\text{coweight}}(G)$ is the dual to the weight lattice $\Lambda_{\text{weight}}(G)$ of G. A gauge transformation of the form (10) shifts A_2 and A_3 by A and B. We must therefore identify

$$X^7 \sim X^7 + 2\pi v , \qquad (13)$$

$$X^8 \sim X^8 + 2\pi v' , \qquad (14)$$

where $v, v' \in \Lambda_{\text{coweight}}(G)$ are coweight vectors. There are no such identifications of $X^1, ..., X^6$ since they descend from $D = 4$ scalars rather than gauge fields. Secondly, we must identify all fields by the action of $W(G)$, the Weyl group of G. We conclude that the proper domain for $(X^1, ..., X^6; X^7, X^8)$ is the orbifold:

$$\left\{\underline{t}^6 \times [\underline{t}/2\pi\Lambda_{\text{coweight}}(G)] \times [\underline{t}/2\pi\Lambda_{\text{coweight}}(G)]\right\} \Big/ W(G) , \qquad (15)$$

where $W(G)$ is the Weyl group, acting diagonally on all X^I.

Finally, there are $8\times$ rank (G), real, left and right-moving fermions. The dimensional reduction of a $D = 4, N = 4$ theory yields a $D = 2, N = 4$ theory,[5] so we arrive at the result: The leading $L \to 0$ behavior of the $D = 4, N = 4$

[4]Related observations were made long ago in [10].
[5]The target space (15) is naturally a hyper-Kähler manifold since it can be written as

$$\left[\underline{t}^4 \times T^*\{(\underline{t} \otimes \mathbb{C})/(2\pi[\Lambda_{\text{coweight}}(G) + i\Lambda_{\text{coweight}}(G)])\}\right] \Big/ W(G) .$$

The $N = 4$ in $D = 4$ actually leads to $N = 8$ in $D = 8$, but the (4,4) superconformal algebra is the largest one that can act irreducibly.

SYM theory is governed by the $D = 2, \hat{c} = 8 \times \text{rank}(G)$, (4,4) superconformal field theory of an orbifold with target space defined by (15).

To be quite explicit, take $\theta = \rho_1 = 0, \rho_2 = 1$, and use the metric on $\underline{\mathbf{g}} = \mathbf{su(n)}$ given in (4) and an orthonormal basis T^j to identify $\underline{\mathbf{t}}$ with $\mathbb{R}^{n-1}$ and the metric (4) with Euclidean metric:

$$\underline{\mathbf{t}} \cong \mathbb{R}^{n-1} : \sum_{j=1}^{n-1} x_j T^j \to (x_1, ..., x_{n-1}) ,$$

$$(X^{I=1,6}, X^7, X^8) \to (\vec{X}^{I=1,6}, \vec{X}^7, \vec{X}^8) , \vec{X}^J \in \mathbb{R}^{n-1}, J = 1, 8 . \tag{16}$$

The action for the conformal field theory is then given by

$$I^{\text{bosonic}} = \frac{1}{4\pi\alpha} \int d^2\sigma \sum_{J=1}^{8} [\partial_0 \vec{X}_j \cdot \partial_0 \vec{X}^J - \partial_1 \vec{X}^J \cdot \partial_1 \vec{X}^J] , \tag{17}$$

where $\vec{X}^J \in \mathbb{R}^{n-1}$ is identified by

$$\vec{X}^7 \sim \vec{X}^7 + 2\pi\vec{u} ,$$
$$\vec{X}^8 \sim \vec{X}^8 + 2\pi\vec{u}' , \quad \vec{u}, \vec{u}' \in \Lambda_{\text{coweight}}(G) , \tag{18}$$
$$(\vec{X}^1, ..., \vec{X}^8) \sim (w \cdot \vec{X}^1, ..., w \cdot \vec{X}^8) , \quad w \in W(G) .$$

S duality exchanges a gauge group with its dual G^v, the magnetic group of Goddard, Nuyts, and Olive (GNO) [2,11].[6] The global structure of a simple compact Lie group G is given by specifying either its weight lattice or coweight lattice. GNO noticed that the coweight lattice of G is always the weight lattice of a dual group G^v. In the case where G and G^v have the same simply connected universal cover, $\tilde{G} = \text{SU}(n)$, the dual group may be defined in terms of the original groups as follows. We must use a metric to identify $\underline{\mathbf{t}} \cong \underline{\mathbf{t}}^*$. For $\text{SU}(n)$, with the metric (4) we have

$$\Lambda_{\text{coweight}}(G^v) \equiv \Lambda_{\text{weight}}(G) = [\Lambda_{\text{coweight}}(G)]^* . \tag{19}$$

Specifically, $\text{SU}(nm)/\mathbb{Z}_n$ is dual to $\text{SU}(nm)/\mathbb{Z}_m$.

Let us now consider the predictions of S duality. Invariance under transformations of the type $\tau \to \tau + 1$, i.e., $\theta \to \theta + 2\pi$ in (1), are obviously symmetries. The other generator of S duality $\tau \to -1/\tau$ acts less trivially. S duality predicts that the theory defined as the Weyl-group orbifold of the free field theory (9) with the identifications (15) for the group G is equivalent to the theory with G replaced by its dual G^v and τ replaced by $-1/\tau$. We now show that this is identical to T duality of the conformal field theory in (9).

First, let us recall the conventions for T duality. Suppose $\Lambda, \Lambda^* \subset \mathbb{R}^d$ are dual lattices, where $\mathbb{R}^d$ has the Euclidean metric. Then standard T duality states that the theory

$$I^1 = \frac{1}{4\pi\alpha} \int d^2\sigma [\partial_0 \vec{X} \cdot \partial_0 \vec{X} - \partial_1 \vec{X} \cdot \partial_1 \vec{X}] ,$$

$$\vec{X} \sim \vec{X} + 2\pi\vec{v} , \quad \vec{v} \in \Lambda \tag{20}$$

is equivalent to the theory

$$I^2 = \frac{\alpha}{4\pi} \int d^2\sigma [\partial_0 \vec{X} \cdot \partial_0 \vec{X} - \partial_1 \vec{X} \cdot \partial_1 \vec{X}] ,$$

$$\vec{X} \sim \vec{X} + 2\pi\vec{v} , \quad \vec{v} \in \Lambda^* . \tag{21}$$

Since the G and G^v theories reduce to the supersymmetric orbifolds based on the lattices

$$G \text{ theory} : \quad \Lambda_{\text{coweight}}(G) \oplus \Lambda_{\text{coweight}}(G) , \tag{22}$$

$$G^v \text{ theory} : \quad \Lambda_{\text{coweight}}(G^v) \otimes \Lambda_{\text{coweight}}(G^v) ,$$

it is now manifest that S duality at $\rho_1 = \theta = 0, \rho_2 = 1$ follows from (19). For other values of ρ, θ the "quadratic form" defining the action of the Gaussian model is given by the matrix $E = B + G$ with[7]

---

[6]G^v is also known as the "Langlands dual" in the mathematics literature.

[7]In *Minkowskian* signature there is no relative factor of i between the kinetic and topological terms.

$$E = [\delta_{IJ}^{(6)} \oplus q(\tau,\rho)] \otimes \text{Tr} , \qquad (23)$$

where $q(\tau,\rho)$ is the 2×2 matrix,

$$q(\tau,\rho) = \frac{1}{\alpha}\sqrt{h}h_{ij} + \frac{\theta}{2\pi}\epsilon_{ij}$$

$$(24)$$

$$= \begin{pmatrix} \frac{\tau_2}{\rho_2}|\rho|^2 & -\tau_2\frac{\rho_1}{\rho_2} + \tau_1 \\ -\tau_2\frac{\rho_1}{\rho_2} - \tau_1 & \frac{\tau_2}{\rho_2} \end{pmatrix} ,$$

and h is the two-metric in (5). Thus, the torus of the sigma model has complexified Kähler form and complex structure given by τ and ρ, respectively. Since

$$q(-1/\tau,\rho) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} q(\tau,\rho)^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad (25)$$

we see that the S-duality transformation $\tau \to -1/\tau$ follows from the T-duality transformation $E \to E^{-1}$ together with the rotation

$$\begin{pmatrix} \vec{X}^7 \\ \vec{X}^8 \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \vec{X}^7 \\ \vec{X}^8 \end{pmatrix} . \qquad (26)$$

It is useful to consider the simple example $G = \text{SU}(2)$ in more detail. We can identify the coweight lattice of SU(2) with the root lattice Λ_{root} of su(2), which we will take to be $\sqrt{2}$ times the integers. The weight lattice of SU(2), $\Lambda_{\text{weight}}[\text{SU}(2)]$ is then the integers divided by $\sqrt{2}$, and $[\Lambda_{\text{weight}}]^* = \Lambda_{\text{root}}$. The dual of SU(2) is $G^v = \text{SO}(3)$. After reduction the SU(2) theory contains "momentum states" created by the vertex operator

$$\cos\left\{\frac{n}{\sqrt{2}}(X_L^8 + X_R^8)\right\} , \qquad (27)$$

where $X_{L,R}^8$ are the left and right moving parts of X^8. The time derivative of X^8 acting on such a state is nonzero. From the four-dimensional point of view, since $X^8 = A_3$ this means that there is electric flux winding around the x^3 direction of the internal torus. Under T duality this is mapped to the winding state

$$\cos\left\{\frac{n}{\sqrt{2}}(X_L^8 - X_R^8)\right\} . \qquad (28)$$

Now one finds that the spatial derivative (in the σ^1 direction) of X^8 is nonzero. Thus there is a magnetic flux in the x^2 direction. Note that this is not quite in accord with S duality. To recover S duality on the states we must compose T duality with the $\mathbb{Z}_2$ transformation (26), in accord with (25).

If the world sheet is a torus, $\Sigma_1 = T^2$ (or, more generally, has $\pi_1 \neq 0$), and X^7, X^8 are in winding number sectors,

$$\vec{X}^7 = 2\pi\sigma^0\vec{v}_0 + 2\pi\sigma^1\vec{v}_1 ,$$

$$(29)$$

$$\vec{X}^8 = 2\pi\sigma^0\vec{w}_0 + 2\pi\sigma^1\vec{w}_1 ,$$

with $v,w \in \Lambda_{\text{coweight}}(G)$ and $0 < \sigma^0, \sigma^1 < 1$, then the θ-dependent part of the action is

$$\theta[(v_0,w_1) - (v_1,w_0)] , \qquad (30)$$

where we use the metric on $\mathbf{t}$ in (4). Thus, one can accordingly map winding number sectors to instanton number sectors. Field configurations with windings defining nontrivial elements of $\Lambda_{\text{weight}}/\Lambda_{\text{root}}$ for both X^7, X^8 satisfy 't Hooft-type boundary conditions.

We now explore an interesting parallel structure between four-dimensional (4D) gauge theories and 2D conformal field theory.[8] A beautiful and famous phenomenon in conformal field theory is the Frenkel-Kac construction, i.e., the existence of enhanced current algebra symmetries in special Gaussian models. Our results suggest a 4D analog. It is natural to conjecture that $N = 4$ Yang-Mills theory in $D = 4$ has enhanced symmetries when the SL(2,$\mathbb{Z}$) action on the coupling is not free, i.e., at $\tau = i, e^{\pi i/3}$. At these points the theory is strongly coupled and $\theta = 0, \pi$. For gauge group $G = \text{SU}(2)$, the conformal field theory we have described has enhanced Kac-Moody symmetries of SU(2)$\times$SU(2) or SU(3) at these points, before dividing by the Weyl group. A surviving SO(2) Kac-Moody symmetry in the orbifold theory has a simple 4D interpretation. At $\tau = i, e^{\pi i/3}$ the theory is self-dual and the gauge bosons are *degenerate* with spin-one monopoles. The isotropy group of the classical SL(2,$\mathbb{R}$) symmetry acting on τ is the classical SO(2) electric-magnetic rotation. Apparently, this *continuous* symmetry survives in the quantum theory at $\tau = i, e^{\pi i/3}$. As for the other currents projected out of the orbifold theory we may remark that, in general, if a theory A can be embedded in a theory B with symmetries it can happen that the symmetries of B strongly constrain the amplitudes of A. Perhaps the $D = 4$ theory has such hidden symmetries at the strong-coupling points $\tau = i, e^{\pi i/3}$. Clearly, this is an interesting topic for further work.

There are several other lines of investigation worth pursuing. The infrared limit of our dimensional reduction should be studied more carefully. Generalizations to other two-manifolds besides the torus are of interest. Much stronger tests of S duality might be obtained by considering the L^2 corrections to our leading result. Even more ambitiously, perhaps a perturbative proof of S duality might be achieved by analyzing the expansion to all orders. Finally, the generalization of these remarks to theories with $N = 2$ (or fewer) supersymmetries promises to be fascinating.

Note added. We would like to draw the reader's attention to Ref. [12] where issues similar to the above are discussed.

_____

[8]The connection between 4D supersymmetric Yang-Mills theory and conformal field theory described in this paper is probably unrelated to the connection uncovered in [9].

APPENDIX A: GENERALIZATION TO ARBITRARY COMPACT GROUPS

The generalization of the above discussion to the case of G, an arbitrary compact group, is straightforward. Locally, G can be written as the product $A \times K$ where A is a torus and K is semisimple. S duality for the part of the theory associated with the Abelian factors of G is elementary, so we focus on K. Quotients by finite subgroups living in different factors yield orbifold versions of the conformal field theories derived below, so, henceforth, we take G to be connected and simple.

1. Normalization of the action

We normalize the action so that anti-self-dual instantons in the simply connected covering group $\tilde{G}$ associated to G have action $2\pi i k \tau$, with the instanton number k taking on all integral values. This gives the normalization

$$\text{Tr}(ab) = \tfrac{1}{2} B_{\mathbf{g}}(v, v) \Phi_{\mathbf{g}}(a, b) , \qquad (A1)$$

where $\Phi_{\mathbf{g}}(\cdot, \cdot)$ is the Killing form on $\mathbf{g}$, $B_{\mathbf{g}}(\cdot, \cdot)$ is the induced form on $\mathbf{g}^*$, and v is the highest root[9] of $\mathbf{g}$.

2. Review of the dual group

We briefly review the general definition of the magnetic group G^v dual to a compact Lie group G [2,11]. We first distinguish two kinds of quantum numbers in a gauge theory with unbroken gauge group G: electric and magnetic. These are defined by representation theory and topology, respectively, as follows.

Electric quantum numbers are given by representations of G. Representations are determined by characters χ. By conjugation, χ is completely determined by its restriction to the maximal torus $T \subset G$. Thus, the electric quantum numbers live on the lattice $\hat{T} \equiv \text{Hom}(T, \text{U}(1))$. Using the exponential map we may think of this lattice as being in $\mathbf{t}^*$: $\Lambda_{\text{weight}}(G) \subset \mathbf{t}^*$.

Magnetic quantum numbers are related to G bundles over S^2. These are determined by the equatorial tran-

sition function $g(\phi) : S^1 \to G$. By conjugation, $g(\phi)$ can be taken to be in T. Thus, the magnetic quantum numbers live on the lattice $\check{T} \equiv \text{Hom}((1), T)$. Using the exponential map we may think of this lattice as being in $\mathbf{t}$: $\Lambda_{\text{coweight}}(G) \subset \mathbf{t}$.

Notice that since $\text{Hom}(\text{U}(1), \text{U}(1)) = \mathbb{Z}$ the weight and coweight lattices of G, hence electric and magnetic quantum numbers, are *canonically* dual:

$$\Lambda_{\text{coweight}}(G) = [\Lambda_{\text{weight}}(G)]^* . \qquad (A2)$$

This is the Dirac quantization condition.[10]

Physically we may define the dual group as follows. Given any compact Lie group G, the dual group G^v is the group for which the electric and magnetic lattices are exchanged [2]. It is a nontrivial fact that G^v exists for every compact group G. Mathematically, the dual group is best understood by thinking of a Lie algebra as defined by its root system R, following [13,14]. We assume $\mathbf{g}$ is semisimple. Let V be a vector space. A finite subset $R \subset V$ is a root system if it satisfies certain axioms [13,14]. One key axiom states that for all $\alpha \in R \exists! \alpha^v \in V^*$ with $\langle \alpha, \alpha^v \rangle = 2$. The axioms are completely symmetric between $R \subset V$ and the set $R^v \subset V^*$. Now, to a root system R (and a choice of simple roots) we associate a Lie algebra $\mathbf{g}(R)$ defined by the Serre presentation. Sine root systems come in pairs R, R^v we get two dual Lie algebras[11] $\mathbf{g}(R)$ and $\mathbf{g}(R^v)$. Furthermore, by construction, we have *canonically*:

$$V = \mathbf{t}^*(R) = \mathbf{t}(R^v) ,$$
$$\qquad (A3)$$
$$V^* = \mathbf{t}(R) = \mathbf{t}^*(R^v) .$$

Finally, we can define the dual groups G, G^v. These have simply connected covers corresponding to $\mathbf{g}(R)$ and $\mathbf{g}(R^v)$, respectively, and have global structure such that

$$\begin{array}{ccc}
\Lambda_{\text{coweight}}(G^v) & = & \Lambda_{\text{coweight}}(G)^* , \\
\cap & & \cap \\
\mathbf{t}(R^v) & = & \mathbf{t}^*(R) ,
\end{array} \qquad (A4)$$

so the lattice of magnetic quantum numbers of G becomes the lattice of electric quantum numbers of G^v, and vice versa.

3. Checking S duality

The Gaussian model in conformal field theory is defined by a triple of data (V, Q, Λ), where V is a vector

---

[9] In this paper V^* indicates the dual of a vector space, i.e., the space of linear functionals on V. A quadratic form Q on V canonically defines a form on Q^* on V^*. If we choose bases the two forms are inverse matrices.

[10] We adopt standard notation whereby, if V is a vector space, the reciprocal lattice $\Lambda^* \subset V^*$ to the lattice $\Lambda \subset V$ is the lattice of vectors with integer pairings: $\Lambda^* = \{v \in V^* : \forall w \in \Lambda, \langle v, w \rangle \in \mathbb{Z}\}$.

[11] The Cartan matrices are related by transposition, except for G_2 where one must reorder the simple roots.

space with nondegenerate quadratic form Q and $\Lambda \subset V$ is a lattice. The action is

$$I = \frac{1}{4\pi} \int d^2\sigma [Q(\partial_0 \vec{X}, \partial_0 \vec{X}) - Q(\partial_1 \vec{X}, \partial_1 \vec{X})] \,,$$

$$\text{(A5)}$$

$$\vec{X} \sim \vec{X} + 2\pi\vec{v} \,, \quad \vec{v} \in \Lambda \,.$$

T duality is the equivalence of the triples (V, Q, Λ) and (V^*, Q^*, Λ^*).

We can now show that T duality implies S duality restricted to the effective conformal field theory. From (A4) we see that the vector spaces and lattices are naturally dual. The metric associated to G follows from (A1). Moreover for a dual pair of groups $G = G(R)$,

$G^v = G(R^v)$ associated to root systems R, R^v we have $\Phi^*_{\mathbf{g}(R)} = B_{\mathbf{g}(R^v)}$ and a simple identity [13]

$$\frac{B_{\mathbf{g}(R^v)}(v, v)}{2} \Phi_{\mathbf{g}(R^v)}(\cdot, \cdot) = \frac{2}{B_{\mathbf{g}(R)}(v, v)} B_{\mathbf{g}(R^v)}(\cdot, \cdot) \quad \text{(A6)}$$

shows that the metrics are inversely related. Nonzero values of ρ_1, θ are handled in the same way as in the $SU(n)$ case explained above. Finally, G and G^v have canonically isomorphic Weyl groups $W(G) = W(G^v)$, so the T-dual orbifold of the dimensionally reduced G theory is identical to the dimensional reduction of the S-dual G^v theory.

[1] C. Montonen and D. Olive, Phys. Lett. **72B**, 117 (1977).

[2] P. Goddard, J. Nuyts, and D. Olive, Nucl. Phys. **B125**, 1 (1977); P. Goddard and D. Olive, *ibid.* **B191**, 528 (1981).

[3] E. Witten and D. Olive, Phys. Lett. **78B**, 97 (1978).

[4] H. Osborne, Phys. Lett. **83B**, 321 (1979).

[5] J. Cardy and E. Rabinovici, Nucl. Phys. **B205**, 1 (1982); J. Cardy, *ibid.* **B205**, 17 (1982).

[6] A. Shapere and F. Wilczek, Nucl. Phys. **B320**, 669 (1989).

[7] A. Font, L. Ibanez, D. Lust, and F. Quevedo, Phys. Lett. B **249**, 35 (1990); S. J. Rey, Phys. Rev. D **43**, 526 (1991); A. Strominger, Nucl. Phys. **B343**, 167 (1990); C. Callan, J. Harvey, and A. Strominger, Nucl. Phys. **B359**, 40 (1991); M. Duff, Class. Quantum Grav. **5**, 189 (1988); M. Duff and J. Lu, Nucl. Phys. **B354**, 129 (1991); **B357**, 354 (1991); Phys. Rev. Lett. **66**, 1402 (1991); Class. Quantum Grav. **9**, 1 (1991); M. Duff, R. Khuri, and J. Lu, Nucl. Phys. **B377**, 281 (1992); J. Dixon, M. Duff, and J. Plefka, Phys. Rev. Lett. **69**, 3009 (1992); I. Pesando and A. Tollsten, Phys. Lett. B **274**, 374 (1992); J. Harvey, J. Gauntlett, and J. Liu, Nucl. Phys. **B409**, 363 (1993); R. Khuri, Phys. Lett. B **259**, 261 (1991); J. Gauntlett and J. Harvey, Report No. hep-th/9407111 (unpublished); J.

H. Schwarz and A. Sen, Nucl. Phys. **B404**, 109 (1993).

[8] A. Sen, Int. J. Mod. Phys. A **9**, 3707 (1994); Phys. Lett. B **329**, 217 (1994).

[9] C. Vafa and E. Witten, Nucl. Phys. **B431**, 3 (1994); see also L. Girardello, A. Giveon, M. Porrati, and A. Zaffaroni, Phys. Lett. B **334**, 331 (1994).

[10] G. 't Hooft, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft *et al.* (Plenum, New York, 1980); E. Witten, Nucl. Phys. **B202**, 253 (1982).

[11] D. Olive, in *Monopoles in Quantum Field Theory: Proceedings of the Monopole Meeting*, Trieste, Italy, 1981, edited by N. S. Craigie, P. Goddard, and W. Nahm (World Scientific, Singapore, 1982).

[12] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa, "Topological Reduction of 4D SYM to 2D σ-Models," Report No. hep-th/9501096 (unpublished); L. Girardello, A. Giveon, M. Porrati, and A. Zaffaroni, "S-Duality in $N = 4$ Yang-Mills Theories with General Gauge Groups," Report No. hep-th/9502057 (unpublished).

[13] N. Bourbaki, *Groupes et Algebres de Lie* (Masson, 1981), Chap. 6.

[14] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer, New York, 1972).