Topological G/G WZW model in the generalized momentum representation

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We consider the topological gauged WZW model in the generalized momentum representation. The chiral field g is interpreted as a counterpart of the electric field E of conventional gauge theories. The gauge dependence of wave functionals $\Psi(g)$ is governed by a new gauge cocycle ϕ_{GWZW}. We evaluate this cocycle explicitly using the machinery of Poisson σ models. In this approach the GWZW model is reformulated as a Schwarz-type topological theory so that the action does not depend on the world-sheet metric. The equivalence of this new formulation to the original one is proved for genus one and conjectured for an arbitrary genus Riemann surface. As a by-product we discover a new way to explain the appearance of quantum groups in the WZW model.

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INTRODUCTION

In this paper we investigate the quantization of the gauged (G/G) Wess-Zumino-Witten (WZW) model in the generalized momentum representation. The consideration is inspired by the study of (two-dimensional) Yang-Mills (YM) and BF theories in the momentum representation [1].

The problem of quantization of gauge theories in the momentum representation has been attracting attention for a long time [2],[1] [3]. While in the connection representation the idea of gauge invariance may be implemented in a simple way

$$\Psi(A^g) = \Psi(A) , \qquad (1)$$

we get a nontrivial behavior of the quantum wave functions under gauge transformations in the momentum representation. Indeed, one can apply the following simple argument. The wave functional in the momentum representation may be thought of as a functional Fourier transformation of the wave functional in the connection representation (1):

$$\Psi(E) = \int \mathcal{D}A \, \exp\left(i \int \text{tr} E_i A_i\right) \Psi(A) . \qquad (2)$$

Taking into account the behavior of A and E under gauge transformations,

$$A_i^g = g^{-1}A_i g + g^{-1}\partial_i g, \quad E_i^g = g^{-1}E_i g , \qquad (3)$$

we derive

$$\Psi(E^g) = \exp\left(i \int \text{tr} E_i \partial_i g g^{-1}\right) \Psi(E) . \qquad (4)$$

We conclude that the wave functional in the momentum representation is not invariant with respect to gauge transformations. Instead, it gains a simple phase factor $\phi(E,g)$, which is of the form

$$\phi(E,g) = \int \text{tr} E_i \partial_i g g^{-1} . \qquad (5)$$

The infinitesimal version of the same phase factor,

$$\phi(E,\epsilon) = \int \text{tr} E_i \partial_i \epsilon , \qquad (6)$$

corresponds to the action of the gauge algebra.

It is easy to verify that ϕ satisfies the equation

$$\phi(E,gh) = \phi(E,g) + \phi(E^g,h) . \qquad (7)$$

This property reflects that the composition of two gauge transformations (4) with gauge parameters g and h is the same as a gauge transformation with a parameter gh. Equation (7) is usually referred to as a cocycle condition. It establishes the fact that ϕ is a one-cocycle of

the (infinite dimensional) gauge group. A one-cocycle is said to be trivial, if there exists some $\tilde{\phi}$ such that

$$\phi(E, g) = \tilde{\phi}(E^g) - \tilde{\phi}(E) . \tag{8}$$

In such a case the gauge invariance of the wave function may be restored by the redefinition

$$\tilde{\Psi}(E) = e^{-i\tilde{\phi}(E)} \Psi(E) . \tag{9}$$

An infinitesimal cocycle $\phi(E, \epsilon)$ is trivial, if it can be represented as a function of the commutator $[\epsilon, E]$:

$$\phi(E, \epsilon) = \tilde{\phi}([\epsilon, E]) . \tag{10}$$

It is easy to see that the cocycle (6) is nontrivial. Indeed, let us choose both E and ϵ having only one nonzero component E^a and ϵ^a (in the Lie algebra). Then the commutator in (10) is always equal to zero, whereas the expression (6) is still nontrivial. As a consequence, also the gauge group cocycle (5) is nontrivial.

On the other hand, on some *restricted* space of values for the field E the cocycle may become trivial (generically if we admit nonlocal expressions for $\tilde{\phi}$). This is important to mention as one may rewrite the (integrated) Gauss law (4) as a triviality condition on the cocycle: Let us parametrize Ψ as $\Psi(E) := \exp[i\tilde{\phi}(E)]$, which is possible whenever $\Psi \neq 0$, and insert this expression into (4). The result is precisely (8) with $\phi \equiv \int \mathrm{tr} E_i \partial_i g g^{-1}.$[2] In fact, e.g., in two dimensions the wave functions of the momentum representation are supported on some conjugacy classes $E(x) = g(x) E_0 g^{-1}(x)$ with specific values of E_0. But away from these specific conjugacy classes, and in particular in the original, unrestricted space of values for E, the general argument of the cocycle (8) being nontrivial applies. More details on this issue may be found in Appendix A.

It is worth mentioning that in the Chern-Simons theory a cocycle appears in the connection representation as well:

$$\Psi(A^g) = e^{i\phi(A, g)} \Psi(A) . \tag{11}$$

The cocycle $\phi(A, g)$ is usually called Wess-Zumino action [4]. It is intimately related to the theory of anomalies [5].

Recently, a cocycle of type (5) has been observed in two-dimensional BF and YM theories. In this paper we consider the somewhat more complicated example of the gauged WZW (GWZW) model for a semisimple Lie group. Like the BF theory, it is a two-dimensional topological field theory [6] (for a detailed account see [7]). It has a connection one-form (gauge field) as one of its dynamical variables and possesses the usual gauge sym-

metry. However, there is a complication which makes the analysis different from the pattern (4). In the GWZW model the variable which is conjugate to the gauge field, and which shall be denoted by g in the following, takes value in a Lie group G instead of a linear space. So, we get a sort of curved momentum space. We calculate the cocycle ϕ_{GWZW} which governs the gauge dependence of wave functions in a g representation and find that it differs from the standard expression (5). We argue that while the cocycle (5) corresponds to a Lie group G, our cocycle is related to its quantum deformation G_q. In the course of the analysis we find that the GWZW model belongs to the class of Poisson σ models recently discovered in [8]. This theory provides a technical tool for the evaluation of the cocycle ϕ_{GWZW}.

Let us briefly characterize the content of each section. In Sec. I we develop the Hamiltonian formulation of the GWZW model, find canonically conjugate variables, and write down the gauge invariance equation for the wave functional in the g representation.

Section II is devoted to the description of Poisson σ models. A two-dimensional topological σ model of this class is defined by fixing a Poisson bracket on the target space. Using the Hamiltonian formulation (the topology of the space-time being a torus or cylinder), we prove that the GWZW model is equivalent to a certain Poisson σ model coupled to a "topological" term S_δ that has support of measure zero on the target space of the field theory. The target space of the (coupled) Poisson σ model is the Lie group G. We start from the GWZW action, evaluate the Poisson structure on G and discover its relation to the theory of quantum groups. The origin of the term S_δ is considered in details in Appendix B.

In Sec. III we solve the gauge invariance equation and find the gauge dependence of the GWZW wave functional in the g representation. This provides a new cocycle ϕ_{GWZW}. Calculations are performed for the Poisson σ model without the singular term. In Appendix C we reconsider the problem in the presence of the topological term. It is shown that in the case of $G = \mathrm{SU}(2)$ at most one quantum state is affected. We compare the results with other approaches [9,7].

In some final remarks we conjecture that the Poisson σ model coupled to S_δ gives an alternative formulation of the GWZW model valid for a Riemann surface of arbitrary genus. We comment on the new relation between the WZW model and quantum groups which emerges as a by-product of our consideration.

I. HAMILTONIAN FORMULATION OF THE GWZW MODEL

The WZW theory is defined by the action

$$A_{\mathrm{WZW}}(g) = \frac{k}{8\pi} \int \mathrm{tr} \partial_\mu g g^{-1} \partial^\mu g g^{-1} d^2 x$$
$$+ \frac{k}{12\pi} \int \mathrm{tr} d^{-1}(dg\, g^{-1})^3 , \tag{12}$$

where the fields g take values in some semisimple Lie

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[2]More accurately, one obtains (8) only mod 2π. But anyway this modification of (8) is quite natural in view of the origin of the cocycle within (4). Alternatively one might regard also a multiplicative cocycle $\Phi = \exp(i\phi)$ right from the outset, cf. Appendix A.

group G and indices μ are raised with the standard Minkowski metric. The case of a Euclidean metric may be treated in the same fashion. Some remarks concerning the second term in (12) may be found in Appendix B.

The simplest way to gauge the global symmetry transformations $g \to lgl^{-1}$ is to introduce a gauge field h taking its values in the gauge group; the action

$$A_{\text{GWZW}}(g, h) = A_{\text{WZW}}(hgh^{-1}) \tag{13}$$

is then invariant under the local transformations $g \to lgl^{-1}, h \to hl^{-1}$. With the celebrated Polyakov-Wigmann formula and $a_{\pm} := h^{-1}\partial_{\pm}h$, where $\partial_{\pm} = \partial_0 \pm \partial_1$, A_{GWZW} can be brought into the standard form

$$A_{\text{GWZW}}(g, a_+, a_-)$$
$$= A_{\text{WZW}}(g) + \frac{k}{4\pi} \int \text{tr}[a_+\partial_-gg^{-1} - a_-g^{-1}\partial_+g$$
$$-a_+ga_-g^{-1} + a_+a_-]d^2x . \tag{14}$$

In the course of our construction of A_{GWZW} $a \equiv a_+dx^+ +$

a_-dx^- has been subject to the zero curvature condition $da + a^2 \equiv 0$. This condition results also from the equations of motion arising from (14). So, further on we treat $a_{\pm}$ as unconstrained fields (taking their values in the Lie algebra of the chosen gauge group).

In order to find a Hamiltonian formulation of the GWZW model we first bring (14) into first order form. To this end we introduce an auxiliary field $p(x)$ into the action by the replacement ($\dot{g} \equiv \partial_0 g$)

$$\frac{k}{8\pi}(\dot{g}g^{-1} + a_+ - ga_-g^{-1})^2$$
$$\to p(\dot{g}g^{-1} + a_+ - ga_-g^{-1}) - \frac{2\pi}{k}p^2 . \tag{15}$$

As p enters the action quadratically, it may be eliminated always by means of its equations of motion so as to reproduce the original action (14). In the functional integral approach this corresponds to performing the Gaussian integration over p.

Now the action (14) may be seen to take the form (with $\partial g \equiv \partial_1 g$)

$$A_{\text{GWZW}}(g, p, a_{\pm}) = \frac{k}{12\pi} \int \text{tr}\, d^{-1}(dg\, g^{-1})^3 + \int d^2x\, \text{tr}\left\{ p\dot{g}g^{-1} - a_- \left[g^{-1}pg - p + \frac{k}{4\pi}(g^{-1}\partial g + \partial gg^{-1}) \right] \right.$$
$$\left. -p\partial gg^{-1} - \frac{k}{8\pi}\left(a_+ - a_- - \frac{4\pi}{k}p + \partial gg^{-1} \right)^2 \right\} . \tag{16}$$

This is already linear in time derivatives. After the simple shift of variables

$$a_+ \to \tilde{a}_+ \equiv a_+ - a_- - \frac{4\pi}{k}p + \partial gg^{-1} \tag{17}$$

the last term is seen to completely decouple from the rest of the action. Therefore one can exclude it from the action without loss of information. We can again employ the argument about integration over $\tilde{a}_+$ [or also a_+ in (16)]. So we have introduced one extra variable p and now one variable is found to drop out from the formalism.

After $\tilde{a}_+$ is excluded, the rest of formula (16) provides the Hamiltonian formulation of the model. The first two terms play the role of a symplectic potential, giving rise to the symplectic form[3]

$$\Omega^{\text{field}} = \text{tr} \oint \left[dp\, dg\, g^{-1} \right.$$
$$\left. + \left(p + \frac{k}{4\pi}\partial gg^{-1} \right)(dg\, g^{-1})^2 \right] dx^1 . \tag{18}$$

Here d is interpreted as an exterior derivative on the phase space. It is interesting to note that the nonlocal

term in (16) gives a local contribution to the symplectic form on the phase space. The third term, which includes a_-, represents a constraint:

$$g^{-1}pg - p + \frac{k}{4\pi}(g^{-1}\partial g + \partial gg^{-1}) \approx 0 . \tag{19}$$

The variable a_- is a Lagrange multiplier and the constraint is nothing but the Gauss law of the GWZW model. It is a nice exercise to check with the help of (18) that the constraints (19) are first class and that they generate the gauge transformations. Equation (19) implies

$$\text{tr}\left(g^{-1}pg + \frac{k}{4\pi}g^{-1}\partial g \right)^2 \approx \text{tr}\left(p - \frac{k}{4\pi}\partial gg^{-1} \right)^2$$

and hence

$$\text{tr}[p\partial gg^{-1}] \approx 0 . \tag{20}$$

This permits us to eliminate the Hamiltonian in (16) in agreement with the fact that the model (14) is topological.

Being a Hamiltonian formulation of the GWZW model, the form (16) is not quite satisfactory, if one wants to solve the Gauss law equation (19). We therefore apply here some trick usually referred to as bosonization [10,11]. The main idea is to substitute the Gauss decomposition for the matrix g into the GWZW action:

$$g = g_{\downarrow}^{-1}g_{\uparrow} , \tag{21}$$

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[3]See also Appendix C.

where $g_\downarrow$ is lower triangular, $g_\uparrow$ is upper triangular. In the case of compact group G both of them are elements of the complexification of G. (Note, however, that we do not complexify the target space G here, but only use complex coordinates $g_\uparrow, g_\downarrow$ on it.) If the diagonal parts of $g_\downarrow$ and $g_\uparrow$ are taken to be inverse to each other, this splitting is unique up to sign ambiguities in the evaluation of square roots. Analogously any element of the Lie algebra $\mathcal{G}$ corresponding to G may be split into upper and lower triangular parts according to

$$Y = Y_\downarrow + Y_\uparrow, \quad (Y_\downarrow)_d = (Y_\uparrow)_d = \tfrac{1}{2} Y_d , \qquad (22)$$

where a subscript d is used to denote the diagonal parts of the corresponding matrices.

Observe that the three-form $\operatorname{tr}(dg\, g^{-1})^3$ may be rewritten in terms of $g_\uparrow$ and $g_\downarrow$ as

$$\omega = \frac{k}{12\pi} \operatorname{tr}[(dg\, g^{-1})^3]$$

$$= d\left[\frac{k}{4\pi} \operatorname{tr}(dg_\downarrow g_\downarrow^{-1} \wedge dg_\uparrow g_\uparrow^{-1})\right] + \tilde{\omega} . \qquad (23)$$

Here $\tilde{\omega}$ is a three-form on G supported at the lower dimensional subset of G which does not admit the Gauss decomposition. Now we can rewrite the topological Wess-Zumino term as

$$A_{\mathrm{WZ}}(g) = \frac{k}{12\pi} \operatorname{tr} \int d^{-1}(dg\, g^{-1})^3$$

$$= \frac{k}{4\pi} \operatorname{tr} \int dg_\downarrow g_\downarrow^{-1} \wedge dg_\uparrow g_\uparrow^{-1}$$

$$+ \frac{k}{12\pi} \int d^{-1}\tilde{\omega} . \qquad (24)$$

In this way we removed the symbol d^{-1} in the first term of the right-hand side. The topological term

$$S_\delta = \frac{k}{12\pi} \int d^{-1}\tilde{\omega} \qquad (25)$$

is considered in details in Appendix B. In contrast with the conventional WZ term the new topological term (25) influences the equations of motion only on some lower dimensional subset of the target space.

Let us return to the action of the GWZW model. We make the substitution (21) and introduce a new momentum variable

$$\Pi = \Pi_\uparrow + \Pi_\downarrow$$

$$= g_\downarrow p g_\downarrow^{-1} - \frac{k}{4\pi}(\partial g_\uparrow g_\uparrow^{-1} + \partial g_\downarrow g_\downarrow^{-1}) . \qquad (26)$$

Rescaling a_- according to $\lambda := (k/2\pi)a_-$ we now may rewrite the GWZW action in the form

$$A_{\mathrm{GWZW}}(g, \Pi, \lambda) = S_\mathcal{P}(g, \Pi, \lambda) + S_\delta(g) , \qquad (27)$$

where S_δ has been introduced in (25) and $S_\mathcal{P}$ is given by

$$S_\mathcal{P}(g, \Pi, \lambda) = \int d^2x \operatorname{tr} \left\{ \Pi(\partial_0 g_\uparrow g_\uparrow^{-1} - \partial_0 g_\downarrow g_\downarrow^{-1}) + \lambda \left[g_\uparrow^{-1} \partial_1 g_\uparrow - g_\downarrow^{-1} \partial_1 g_\downarrow + \frac{2\pi}{k}\left(g_\uparrow^{-1}\Pi g_\uparrow - g_\downarrow^{-1}\Pi g_\downarrow \right) \right] \right\} . \qquad (28)$$

In the further consideration we shall disregard the topological term S_δ. In Appendix C we prove that if we take (25) into account, the results will change only for wave functions having support on those adjoint orbits in G [one in the case of $G = $SU(2)] on which the Gauss decomposition breaks down.

For the formulation of a quantum theory in the g representation, the momentum Π should be replaced by some derivative operator on the group. The first term in (28) represents the symplectic potential on the phase space and suggests the ansatz

$$g \to g, \quad \Pi \to -i\left(g_\uparrow \frac{\delta}{\delta g_\uparrow} - g_\downarrow \frac{\delta}{\delta g_\downarrow} \right) . \qquad (29)$$

At this point some remark on the notational convention is in order: On GL(N) coordinates are given by the entire g_{ij} of the matrix representing an element $g \in$ GL(N). The corresponding basis in the tangent space may be arranged into matrix form via

$$\left(\frac{\delta}{\delta g} \right)_{ij} \equiv \frac{\delta}{\delta(g_{ji})} . \qquad (30)$$

With this convention the entries of $g\partial/\partial g$ are seen to be the right translation invariant vector fields on GL(N). Given a subgroup G of GL(N) the trace can be used to

project the translation invariant derivatives from GL(N) to G. In more explicit terms, given an element Y of the Lie algebra of G, a right translation invariant derivative on G is defined by $\operatorname{tr} Yg\partial/\partial g$. The matrix valued derivatives in this paper are to be understood in this sense. In particular, (29) means that the quantum operator associated with $\operatorname{tr} Y\Pi$ is given by

$$\operatorname{tr} Y\Pi \to -i \operatorname{tr} \left(Y_\uparrow g_\uparrow \frac{\delta}{\delta g_\uparrow} - Y_\downarrow g_\downarrow \frac{\delta}{\delta g_\downarrow} \right) . \qquad (31)$$

With this interpretation it is straightforward to prove that commutators of the quantum operators defined in (29) reproduce the Poisson algebra of the corresponding classical observables, as defined by the symplectic potential term in (28).

Let us look for the wave functionals of the GWZW model in the g representation. This means that we must solve the equation

$$(g_\uparrow^{-1}\partial_1 g_\uparrow - g_\downarrow^{-1}\partial_1 g_\downarrow)\Psi(g_\uparrow, g_\downarrow)$$

$$= \frac{2\pi i}{k}\left[g_\downarrow^{-1}\left(g_\downarrow \frac{\delta}{\delta g_\downarrow} - g_\uparrow \frac{\delta}{\delta g_\uparrow} \right) g_\downarrow \right.$$

$$\left. - g_\uparrow^{-1}\left(g_\downarrow \frac{\delta}{\delta g_\downarrow} - g_\uparrow \frac{\delta}{\delta g_\uparrow} \right) g_\uparrow \right] \Psi(g_\uparrow, g_\downarrow) \quad (32)$$

for Ψ being a wave functional; the functional derivatives are understood to act on Ψ only (but not on everything to their right). The problem is clearly formulated, but at first sight it is not evident how to solve Eq. (32). To simplify it we introduce another parametrization of the matrix g:

$$g = h^{-1}g_0 h \ . \tag{33}$$

Here g_0 is diagonal and h is defined up to an arbitrary diagonal matrix which may be multiplied from the left. The part of the operator (32) which includes functional derivatives simplifies dramatically in terms of h. One can rewrite Eq. (32) as

$$\left(g_\uparrow^{-1}\partial_1 g_\uparrow - g_\downarrow^{-1}\partial_1 g_\downarrow + \frac{2\pi i}{k}\frac{\delta}{\delta h}h\right)\Psi[g_0,h] = 0 \ , \quad (34)$$

where $g_\uparrow, g_\downarrow$ are determined implicitly as functions of h and g_0 via

$$g_\downarrow^{-1}g_\uparrow = h^{-1}g_0 h \ . \tag{35}$$

We discuss the interpretation of Eqs. (32) and (34) in Sec. II and solve them efficiently in Sec. III.

II. GAUGED WZW MODEL AS A POISSON σ MODEL

The Gauss law equations of the previous section may be naturally acquired in the theory of Poisson σ models. We start with a short description of this type of topological σ model.

The name Poisson σ model originates from the fact that its target space N is a Poisson manifold, i.e., N carries a Poisson structure $\mathcal{P}$. We denote coordinates on the two-dimensional world-sheet M by x^μ, $\mu = 1, 2$ and coordinates on the target space N by X^i, $i = 1, \ldots, n$. Poisson brackets $\{.,.\}$ on N are defined by specifying its value for some coordinate functions: $\{X^i, X^j\} = \mathcal{P}^{ij}(X)$. Equivalently the Poisson structure may be represented by a bivector

$$\mathcal{P} = \frac{1}{2}\mathcal{P}^{ij}(X)\frac{\partial}{\partial X^i} \wedge \frac{\partial}{\partial X^j} \ . \tag{36}$$

In terms of this tensor the Jacobi identity for $\{.,.\}$ becomes

$$\mathcal{P}^{li}\frac{\partial \mathcal{P}^{jk}}{\partial X^l} + \mathcal{P}^{lk}\frac{\partial \mathcal{P}^{ij}}{\partial X^l} + \mathcal{P}^{lj}\frac{\partial \mathcal{P}^{ki}}{\partial X^l} + 0 \ . \tag{37}$$

For nondegenerate $\mathcal{P}$ the notion of a Poisson manifold coincides with that of a symplectic manifold. In general, however, $\mathcal{P}$ need not be nondegenerate.

In the world-sheet picture our dynamical variables are the X^i's and a field A which is a one-form in both world-sheet and target space. In local coordinates A may be represented as

$$A = A_{i\mu}dX^i \wedge dx^\mu \ . \tag{38}$$

The topological action of the Poisson σ model consists of two terms, which we write in coordinates:

$S_\mathcal{P}(X, A)$

$$= \int_M \left(A_{i\nu}\frac{\partial X^i}{\partial x^\mu} + \frac{1}{2}\mathcal{P}^{ij}A_{i\mu}A_{j\nu}\right)dx^\mu \wedge dx^\nu \ . \tag{39}$$

Here A and X are understood as functions on the world sheet. Both terms in (39) are two-forms with respect to the world sheet. Thus, they may be integrated over M. The action (39) is obviously invariant with respect to diffeomorphisms of the world sheet. It is also invariant under diffeomorphisms of the target space which preserve the Poisson tensor. Equations of motion for the fields X and A are

$$\partial_\mu X^i + \mathcal{P}^{ij}A_{j\mu} = 0 \ ,$$
$$\tag{40}$$
$$\partial_\mu A_{i\nu} - \partial_\nu A_{i\mu} - \frac{\partial \mathcal{P}^{jk}}{\partial X^i}A_{j\mu}A_{k\nu} = 0 \ .$$

Here ∂_μ is the derivative with respect to x^μ on the world sheet.

Let us remark that the two-dimensional BF theory may be interpreted as a Poisson σ model. Indeed, if one chooses a Lie algebra with structure constants $f^{ij}{}_k$ as the target space N and uses the natural Poisson brackets

$$\{X^i, X^j\} = f^{ij}{}_k X^k \ , \tag{41}$$

one reproduces the action of the BF theory

$$A_{\mathrm{BF}}(X, A) = \int_M \mathrm{tr}X(dA + A^2) \ . \tag{42}$$

In the traditional notation X is replaced by B and the curvature $dA + A^2$ of the gauge field is denoted by F. The class of Poisson σ models includes also nontrivial examples of two-dimensional theories of gravity (for details see [12,8]).

We argue that the gauged WZW model is equivalent to a Poisson σ model coupled to the term (25). The target space is the group G, parametrized by $g_\uparrow$ and $g_\downarrow$. The (1,1)-form A is identified readily from (28):

$$A = \Pi(dg_\uparrow g_\uparrow^{-1} - dg_\downarrow g_\downarrow^{-1}) \wedge dx^1$$
$$-\lambda(g_\uparrow^{-1}dg_\uparrow - g_\downarrow^{-1}dg_\downarrow) \wedge dx^0 \ . \tag{43}$$

Here we have interpreted the terms linear in Π and λ.

Then the part of the action quadratic in Π and λ directly determines the Poisson structure. In our formulation of the general Poisson σ model (39) the indices i, μ of A correspond to a coordinate basis dX^i in T^*N and dx^μ in T^*M. In such a formulation we simply have to replace $A_{i\mu}$ by $\partial/\partial X^i$ in the quadratic part of the action to obtain the Poisson bivector (36) as the "coefficient" of the volume-form $dx^\mu \wedge dx^\nu$. Each of the matrix-valued one-forms $dg_\uparrow g_\uparrow^{-1} - dg_\downarrow g_\downarrow^{-1}$ and $g_\uparrow^{-1}dg_\uparrow - g_\downarrow^{-1}dg_\downarrow$ in the present expression (43) for A, however, represents a nonholonomic basis in the cotangent bundle of the target space G. In such a case the corresponding components of A, i.e., Π and λ in our notation, have to be replaced by the respective dual derivative matrices. Applying this simple recipe to the quadratic part of (28) we find the

Poisson bivector on G:

$$\Pi \to \left(g_\uparrow \frac{\partial}{\partial g_\uparrow} - g_\downarrow \frac{\partial}{\partial g_\downarrow}\right), \quad \lambda \to \left(\frac{\partial}{\partial g_\uparrow} g_\uparrow - \frac{\partial}{\partial g_\downarrow} g_\downarrow\right)$$

$$\Rightarrow \mathcal{P} = \frac{4\pi}{k}\mathrm{tr}\left(\frac{\partial}{\partial g_\uparrow} g_\uparrow - \frac{\partial}{\partial g_\downarrow} g_\downarrow\right) \wedge \left[g_\uparrow^{-1}\left(g_\downarrow \frac{\partial}{\partial g_\downarrow} - g_\uparrow \frac{\partial}{\partial g_\uparrow}\right) g_\uparrow - g_\downarrow^{-1}\left(g_\downarrow \frac{\partial}{\partial g_\downarrow} - g_\uparrow \frac{\partial}{\partial g_\uparrow}\right) g_\downarrow\right] . \qquad (44)$$

Using the parametrization (33) and (35) this expression can be formally simplified to

$$\mathcal{P} = \frac{4\pi}{k}\mathrm{tr}\left(\frac{\partial}{\partial g_\uparrow} g_\uparrow - \frac{\partial}{\partial g_\downarrow} g_\downarrow\right) \wedge \frac{\partial}{\partial h} h . \qquad (45)$$

For means of completeness we should check now that this bivector fulfills the Jacobi identity (37). In our context the simplest way to do so is to recall that the constraints of the GWZW model are first class; this suffices, because one can show that the constraints of any action of the form (39) are first class exactly iff $\mathcal{P}^{ij}$ obeys (37). Certainly one can verify the Jacobi identity also by some direct calculation and in fact this is done implicitly when establishing (47) and (50) below.

The above Poisson bracket on G requires further comment. For this purpose it is useful to introduce some new object. We always assume that the group G is realized as a subgroup in the group of n by n matrices. Then the following matrix r acting in $C^n \otimes C^n$ is important for us:

$$r = \frac{1}{2}\sum_i h^i \otimes h_i + \sum_\alpha t_{-\alpha} \otimes t^\alpha . \qquad (46)$$

Here h^i and h_i are generators of dual bases in the Cartan subalgebra, t^α and $t_{-\alpha}$ are positive and negative roots, respectively. The matrix r is usually called classical r matrix. It satisfies the classical Yang-Baxter equation in the triple tensor product which reads

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 . \qquad (47)$$

Here $r_{12} = r \otimes 1$ is embedded in the product of the first two spaces and so on. An important property of the r matrix is

$$\mathrm{tr}_{1,2} r A^1 B^2 = \mathrm{tr}A_\uparrow B_\downarrow + \frac{1}{2}\mathrm{tr}A_d B_d , \qquad (48)$$

where the trace on the left-hand side is evaluated in the tensor product of two spaces and

$$A^1 \equiv A \otimes 1, \quad B^2 \equiv 1 \otimes B . \qquad (49)$$

Now we are ready to represent the bracket (44) in a more manageable way. As the most natural coordinates on the group are matrix elements, we are interested in Poisson brackets of entries of $g_\uparrow$ and $g_\downarrow$. Using shorthand notations (49) and the definition of the r matrix, we arrive at the following elegant result:

$$\{g_\uparrow^1, g_\uparrow^2\} = \frac{4\pi}{k}[r, g_\uparrow^1 g_\uparrow^2] ,$$

$$\{g_\downarrow^1, g_\downarrow^2\} = \frac{4\pi}{k}[r, g_\downarrow^1 g_\downarrow^2] , \qquad (50)$$

$$\{g_\downarrow^1, g_\uparrow^2\} = \frac{4\pi}{k}[r, g_\downarrow^1 g_\uparrow^2] .$$

We omit the calculation which leads to (50), as it is rather lengthy but straightforward. Each equation in (50) provides a Poisson bracket between any matrix element of the matrix with superscript 1 with any matrix element of the matrix with superscript 2. In order to clarify this statement we rewrite the first equation in components:

$$\{g_\uparrow^{ij}, g_\uparrow^{kl}\} = \frac{4\pi}{k}(r_{k\tilde{k}}^{i\tilde{i}} g_\uparrow^{\tilde{i}j} g_\uparrow^{\tilde{k}l} - g_\uparrow^{i\tilde{j}} g_\uparrow^{k\tilde{l}} r_{\tilde{l}l}^{\tilde{j}j}) . \qquad (51)$$

Here summation over the indices with the tilde on the right-hand side is understood. The remaining equations of (50) can be rewritten in the same fashion. The formulas (50) define the Poisson bracket only on the subset of the group G which admits the Gauss decomposition (21). One can easily recover the Poisson brackets of matrix elements of the original matrix g. We leave this as an exercise to the reader. The result may be presented in tensor notation:

$$\{g^1, g^2\} = \frac{4\pi}{k}[g^2 r g^1 - g^1 r' g^2 - r g^1 g^2 + g^1 g^2 r'] . \qquad (52)$$

Here r' is obtained from r by exchanging the two copies of the Lie algebra:

$$r' = \frac{1}{2}\sum_i h^i \otimes h_i + \sum_\alpha t_\alpha \otimes t^{-\alpha} . \qquad (53)$$

The Poisson bracket (52) is quadratic in matrix elements of g and obviously smooth. This means that the bracket (50), which has been defined only on the part of the group G where the Gauss decomposition is applicable, may now be continued smoothly to the whole group; for example, for the case of $G =$SU(2) it is straightforward to establish that the right-hand side of (52), and thus also the smoothly continued Poisson tensor $\mathcal{P}$, vanishes at antidiagonal matrices $g \in$SU(2). The latter represent precisely the one-dimensional submanifold of SU(2) where a decomposition (21) for g does not exist. It is worth mentioning that the bracket (52) appeared first in [13] within the framework of the theory of Poisson-Lie groups.

The group G equipped with the Poisson bracket (52) may be used as a target space of the Poisson σ model. We have just proved that the Hamiltonian formulation (geometry of the world sheet is torus or cylinder) of this Poisson σ model coupled to the topological term (25) coincides with the gauged WZW model.

III. SOLVING THE GAUSS LAW EQUATION

This section is devoted to the quantization of Poisson σ models. More exactly, we are interested in the Hilbert space of such a model in the Hamiltonian picture. This implies that we need a distinguished time direction on the

world sheet and thus we are dealing with a cylinder. The remarkable property of Poisson σ models is that the problem of finding the Hilbert space in this two-dimensional field theory may be actually reduced to a quantum mechanical problem. This has been realized in [8] and here we give only a short account of the argument.[4]

It follows from (39) that in the Hamiltonian formulation the variables X^i and A_{i1} are canonically conjugate to each other (this changes slightly when the Poisson σ model is coupled to the term S_δ, see Appendix C). In the X representation of the quantum theory the X^i act as multiplicative operators and the A_{i1} act as functional derivatives

$$A_{i1} = i\frac{\delta}{\delta X^i} \ . \tag{54}$$

The components A_{i0} enter the action linearly. They are naturally interpreted as Lagrange multipliers. The corresponding constraints look as

$$G^i \equiv \partial_1 X^i + \mathcal{P}^{ij}(X)A_{j1} \approx 0 \ . \tag{55}$$

Combining (54) and (55) one obtains an equation for the wave functional in the X representation

$$\left(\partial_1 X^i + i\mathcal{P}^{ij}(X)\frac{\delta}{\delta X^j}\right)\Psi[X] = 0 \ . \tag{56}$$

Equations (32) and (34) are particular cases of this equation. In order to solve (56) we first turn to a family of finite dimensional systems on the target space N defined by the Poisson structure $\mathcal{P}$.

As the target space of a Poisson σ model carries a Poisson bracket, it may be considered as the starting point of a quantization problem. Namely, one can consider the target space as the phase space of a finite dimensional Hamiltonian system, which one may try to quantize. The main obstruction on this way is that the Poisson bracket $\mathcal{P}$ may be degenerate. This means that if we select some point in the target space and then move it by means of all possible Hamiltonians, we still do not cover the whole target space with trajectories but rather stay on some surface $\mathcal{S} \subset N$. The simplest example of such a situation is a three-dimensional space $N = \mathbb{R}^3$ with the Poisson bracket

$$\{X^i, X^j\} = \epsilon_{ijk}X^k \ . \tag{57}$$

This Poisson bracket describes a three-dimensional angular momentum and it is well known that the square of the length

$$R^2 := \sum_i (X^i)^2 \tag{58}$$

commutes with each of the X^i. So, R^2 cannot be changed by means of Hamiltonian flows and the surfaces $\mathcal{S}$ are two-dimensional spheres.

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[4]But cf. also Appendix C.

If the Poisson bracket $\mathcal{P}$ is degenerate we cannot use N as a phase space. However, if we restrict to some surface $\mathcal{S}$ (these surfaces are also called symplectic leaves), the Poisson bracket becomes nondegenerate and one can try to carry out the quantization program. In the functional integral approach we are interested in the exponent $\exp(i\mathcal{A})$ of the classical action $\mathcal{A}$, being the main ingredient of the quantization scheme. In order to construct the classical action $\mathcal{A}$ we invert the matrix of Poisson brackets (restricted to some particular surface $\mathcal{S}$) and obtain a symplectic two-form

$$\Omega = \tfrac{1}{2}\Omega_{ij}dX^i \wedge dX^j, \quad \Omega_{ik}\mathcal{P}^{kj} = \delta_i^j \ . \tag{59}$$

As a consequence of the Jacobi identity the form Ω is closed

$$d\Omega = 0 \tag{60}$$

and we can look for a one-form α which solves the equation

$$d\alpha = \Omega \ . \tag{61}$$

If Ω belongs to some nontrivial cohomology class, $\alpha \sim p\,dq$ does not exist globally. Still the expression

$$\Psi[X] := \exp\left(i\int d^{-1}\Omega\right) \tag{62}$$

makes sense, if the cohomology class of Ω is integral, i.e., if

$$\oint_\sigma \Omega = 2\pi n, \quad n \in Z \ , \tag{63}$$

for all two-cycles $\sigma \subset \mathcal{S}$; in this case $\mathcal{A} = \int d^{-1}\Omega$ is defined mod 2π and (62) is one-valued [cf. also (64) below]. Alternatively to the functional integral approach we might use the machinery of geometric quantization [14] to obtain condition (63): Within the approach of geometric quantization it is a well-known fact that a Hamiltonian system $(\mathcal{S}, \Omega)$ may be quantized consistently only if the symplectic form Ω belongs to an integral cohomology class of $\mathcal{S}$. In the example of two-dimensional spheres in the three-dimensional target space considered above the requirement of the symplectic leaf to be quantizable, obtained in any of the two approaches suggested above, implies that the radius R of the sphere is either integer or half-integer (for more details confer [15,14]). This is a manifestation of the elementary fact that a three-dimensional spin has to be either integer or half-integer.

After this excursion into Hamiltonian mechanics we return to Eq. (56). It is possible to show that formula (62) provides a solution of Eq. (56). Moreover, any solution of (56) can be represented as a linear combination of expressions (62) corresponding to different integral symplectic leaves [8].

Let us explain this in more detail. The wave functional $\Psi[X]$ of the field theory depends on n functions X^i on the circle. They define a parametrized closed trajectory (loop) in the target space N. Now it is a more

or less immediate consequence of (56) that the quantum constraints of the field theory restrict the support of Ψ to trajectories (loops) $X(x^1)$ which lie completely within a symplectic leaf $\mathcal{S}$ (just use coordinates X^i in the target space adapted to the foliation of N into symplectic leaves). A further analysis, recapitulated in part in Appendix C within the more general framework of a Poisson σ model coupled to a topological term, shows that these leaves have to be quantizable and that admissible quantum states are indeed all of the form (62) or a superposition of such functionals. In the case that $\mathcal{S}$ is simply connected, (62) may be rewritten more explicitly as

$$\Psi[X] \propto \exp[i\mathcal{A}(X)],$$

$$\mathcal{A}(X) = \int_\Sigma \Omega \pmod{2\pi} ,$$
(64)

where the two-dimensional surface Σ is bounded by the closed path $X(x^1)$ lying in some quantizable leaf $\mathcal{S}$.[5] As Ω belongs to an integral cohomology class (by the choice of $\mathcal{S}$), (64) is a globally well-defined functional of $X(x^1)$. As stated already before any such functional solves the quantum constraints (56) and, vice versa, any solution to the latter has to be a superposition of states (64). On the other hand (62) or (64) may be also reinterpreted as exponentiated point particle action. x^1 then is the "time-parameter" of the trajectory $X(x^1)$, which one requires to be periodic in time.

So we obtain the following picture for the relation between the Poisson σ model and finite dimensional quantum mechanics: In order to obtain the Hilbert space of the σ model on the cylinder, one may regard the target space as a phase space of a dynamical system. This space splits into a set of surfaces on which the Poisson brackets are nondegenerate, creating a family of finite dimensional systems. Some of these systems are quantizable in the sense that the cohomology class of the symplectic form is integral. To each quantum system generated in this way corresponds a linearly independent vector in the Hilbert space $\mathcal{H}$ of the σ model. In the case that the respective (quantizable) symplectic leaf $\mathcal{S}$ is not simply connected, however, there is a linearly independent vector in $\mathcal{H}$ for *any* element of $\pi_1(\mathcal{S})$. This idea may be successfully checked for BF theories in two dimensions (for more details confer [8]).

Now we apply the machinery of this section to the GWZW model. First, we should look at the surfaces $\mathcal{S}$ in the group G where the restriction of the Poisson bracket (50) is nondegenerate. For generic leaves this problem has been solved in [13]. In order to make $\mathcal{P}$ nondegenerate, one should restrict to some conjugacy class in the group

$$g = h^{-1} g_0 h .$$
(65)

Each conjugacy class may be used as the phase space of a Hamiltonian system. However, in the case of $G = \mathrm{SU}(2)$ we found that the Poisson bracket vanishes on the subset of antidiagonal matrices. Hence, any antidiagonal matrix represents a zero-dimensional symplectic leaf in $G = \mathrm{SU}(2)$. So, some exceptional conjugacy classes may further split into families of symplectic leaves. This occurs precisely where the Gauss decomposition does not hold.

The form Ω on a generic orbit characterized by g_0 has been recently evaluated in [16] (a presentation more adapted to the physical audience can be found in [17]) and has the form

$$\Omega = \frac{k}{4\pi} \mathrm{tr}[h^{-1} dh \wedge (g_\uparrow^{-1} dg_\uparrow - g_\downarrow^{-1} dg_\downarrow)] ,$$
(66)

where $g_\uparrow, g_\downarrow$, and h are related through (35). The corresponding point particle action or phase factor of the quantum states is

$$\mathcal{A}_{\mathrm{GWZW}}(g)$$
$$= \frac{k}{4\pi} \int d^{-1} \mathrm{tr}[h^{-1} dh \wedge (g_\uparrow^{-1} dg_\uparrow - g_\downarrow^{-1} dg_\downarrow)] . \quad (67)$$

As outlined above quantum states are assigned only to integral symplectic leaves. In Appendix C the corresponding integrality condition (63) is evaluated explicitly for the example of $G = \mathrm{SU}(2)$.

The exceptional conjugacy classes require some special attention. From the point of view of the *pure* Poisson σ model there corresponds a quantum state to any integral symplectic leaf which the respective conjugacy class may contain. For the case of $\mathrm{SU}(2)$, e.g., there is one exceptional (two-dimensional) conjugacy class (65) characterized by $\mathrm{tr} g = 0$. It contains the one-dimensional submanifold $\mathcal{C}$ of antidiagonal matrices in $\mathrm{SU}(2)$. Any point of $\mathcal{C}$ is a zero-dimensional symplectic leaf and, because zero-dimensional leaves are always quantizable, one would be left with a whole bunch of states corresponding to this exceptional conjugacy class.

However, we know that in order to describe the GWZW model in full generality, we need to add the topological term S_δ to the pure Poisson σ part of the action. Also, appropriate boundary conditions of A have to be taken into account at the part of G where the Gauss decomposition breaks down. Whereas S_δ and these boundary conditions may be seen to be irrelevant for the quantum states corresponding to generic conjugacy classes, they decisively change the picture at the exceptional ones; e.g., for $G = \mathrm{SU}(2)$ the net result is that there corresponds only one or even no quantum state to the exceptional conjugacy class defined above, depending on whether k is even or odd, respectively. Further details on this may be found in Appendix C.

From (67) it is straightforward to evaluate the cocycle ϕ_{GWZW} which controls the behavior of the wave functional with respect to gauge transformations; e.g., for the case of infinitesimal transformations

$$\delta g = -[\epsilon, g], \quad \delta h = h\epsilon ,$$
(68)

the new gauge cocycle looks as

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[5]In the language of Appendix C the definition (64) corresponds to the choice of a constant (pointlike) "loop of reference" for Ψ_0.

$$\phi_{\text{GWZW}}(g, \epsilon) = \frac{k}{4\pi} \int \text{tr}\epsilon(g_\uparrow^{-1} dg_\uparrow - g_\downarrow^{-1} dg_\downarrow) . \qquad (69)$$

An integrand of this type has been studied in the framework of Poisson-Lie group theory [18]. However, the gauge algebra interpretation is new.

In order to check that the cocycle ϕ_{GWZW} is nontrivial, it is convenient to use the same trick as the one we applied in the Introduction. Indeed, choose both g and ϵ to be diagonal. Then any trivial cocycle vanishes, but (69) is not equal to zero for generic diagonal g and ϵ.

IV. DISCUSSION

Let us briefly recollect and discuss the results of the paper. Using the Hamiltonian formulations we have shown that the GWZW model is equivalent to a Poisson σ model coupled to the topological term S_δ:

$$A_{\text{GWZW}}(g, A) = S_{\mathcal{P}}(g, A) + S_\delta(g) . \qquad (70)$$

It is natural to conjecture that this equivalence holds true for a surface of arbitrary genus. Let us mention that originally the GWZW is formulated as a Witten-type topological field theory. This means that the action includes the kinetic term and explicitly depends on the world-sheet metric. Then one can use some supersymmetry to prove that in fact the terms including the world-sheet metric do not influence physical correlators. The Poisson σ model provides a Schwarz-type formulation of the same theory. The right-hand side of (70) is expressed in terms of differential forms exclusively and does not include any metric from the very beginning.

At the moment the GWZW model is solved in many ways whereas the general Poisson σ model has not been investigated much. Applying various methods which work for the GWZW to Poisson σ models, one can hope to achieve two goals. First, one can select the methods which work in a more general framework and, hence, which are more reliable. This is especially important when one deals with functional integrals. The other ambitious program is to solve an arbitrary Poisson σ model coupled to a topological term explicitly. Solution should include an evaluation of the partition function and topological correlators in terms of the data of the target space. In this respect an experience of the GWZW model may be very useful.

Another issue which deserves some comment is the relation between quantum groups and WZW models. This issue has been studied much in the literature [19]. The picture of the quantum symmetry in WZW models may be described in short as follows. Separating left-moving and right-moving sectors of the model we add some finite number of degrees of freedom to the system. The quantum group symmetry is a gauge symmetry acting on the left and right movers. The physical fields are invariants of the quantum group action. Usually one can choose some special boundary conditions when separating the sectors in order to make the quantum group symmetry transparent.

Let us compare this picture to the considerations of this paper. The gauged WZW model appears to be equivalent to some Poisson σ model with gauge group G as target space. We derive the Poisson bracket (52) directly from the GWZW action. This bracket is quite remarkable. Quantizing the brackets (52) one gets the generating relations of the quantum group [20]. We have found that the gauge dependence of the wave functional of the GWZW model is described by the classical action defined on the symplectic leaves. This type of action for the bracket (52) has been considered in [17]. It is proved there that such an action possesses a symmetry with respect to the quantum group. So, confirming our expectations, the quantum group governs the nonphysical gauge degrees of freedom of the GWZW model. The new element of the picture is that we do not have to introduce any new variables or choose specific boundary conditions in order to discover the quantum group structure. Let us remark that the treatment may look somewhat more natural for GWZW than for the original WZW model. The reason is that GWZW may be viewed as a chiral theory from the very beginning.[6] The only choice which we make is the way how we bosonize the WZW action. We conclude that the quantum group degrees of freedom are introduced by bosonization. It would be interesting to explore this idea from a more mathematical point of view.

APPENDIX A: GAUGE COCYCLES AND INTEGRAL COADJOINT ORBITS

Here we study in details the one-cocycle

$$\phi(E, g) = \oint \text{tr}E\partial g g^{-1} dx \qquad (A1)$$

of the loop group LG, which plays the role of the gauge group on the circle. Along with the additive cocycle ϕ we consider a multiplicative cocycle

$$\Phi(E, g) = \exp[i\phi(E, g)] . \qquad (A2)$$

The counterparts of the cocycle and coboundary conditions in the multiplicative setting are, respectively,

$$\Phi(E, g_1 g_2) = \Phi(E^{g_1}, g_2)\Phi(E, g_1) , \qquad (A3)$$

$$\Phi(E, g) = \tilde{\Phi}(E^g)\tilde{\Phi}(E)^{-1} . \qquad (A4)$$

Let us observe that one can consistently restrict the

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[6]We are grateful to K. Gawedzki for this remark.

region of definition of E from the loop algebra $l\mathcal{G}$ to any subspace invariant with respect to the action of the gauge group by conjugations. Let us choose such a subset in the form

$$E = h(x)^{-1} E_0 h(x) \tag{A5}$$

for E_0 being a constant diagonal matrix. For fixed x Eq. (A5) defines a conjugacy class in the algebra $\mathcal{G}$ (coadjoint orbit).

The diagonal matrix E_0 may be decomposed using a basis of fundamental weights w_i in the Cartan subalgebra:

$$E_0 = \sum_i E_0^i w_i \ . \tag{A6}$$

In the case of compact groups the cocycle Φ is trivial if and only if all coefficients E_0^i are integer. To demonstrate this, let us present the explicit solution for $\tilde{\Phi}$. It is given by

$$\tilde{\Phi} = \exp\left(i \oint \mathrm{tr} E_0 \partial h h^{-1} dx \right) \ . \tag{A7}$$

It is easy to check that (A7) provides a solution of the coboundary problem. It is less evident that (A7) is well defined. The group element $h(x)$ is defined by $E(x)$ only up to an arbitrary diagonal left multiplier. When coefficients in (A6) are integral, this multiplier does not influence (A7).

For noncompact groups, though, (A7) may turn out to be well defined even for continuously varying choices of E_0.

To establish contact with the presentation in the main text, one may observe that the additive coboundary (generically not well defined)

$$\tilde{\phi} = \oint \mathrm{tr} E_0 \partial h h^{-1} dx \tag{A8}$$

may be reformulated in terms of the (well-defined) Kirillov form on the coadjoint orbit,

$$\Omega = \mathrm{tr} E_0 (dh\, h^{-1})^2 = \tfrac{1}{2} \mathrm{tr} dE \wedge h^{-1} dh \ , \tag{A9}$$

as

$$\tilde{\phi} = \int_\Sigma \Omega, \quad \partial\Sigma = E(x) \ . \tag{A10}$$

The ambiguity in the choice of Σ does not influence the multiplicative cocycle $\tilde{\Phi}$, iff the Kirillov form is integral; i.e., iff Ω satisfies (63).

For compact groups the integrality condition (63) on Ω coincides with the before-mentioned condition on the E_0^i. If (63) is satisfied with $n = 0$ not only the multiplicative but also the additive cocycle ϕ becomes trivial. This occurs, e.g., in the noncompact case $\mathcal{G} = \mathrm{sl}(2, \mathbb{R})$.

It is worth mentioning that (A8) is the action for a quantum mechanical system with the phase space being a coadjoint orbit. We consider a similar system in Sec. III. There the quantum mechanical phase space is a conjugacy class in the group and the analogue of the Kir-

illov form (A9) is (66), the Kirillov form of the quantum group.

We conclude that for certain restricted subspaces of the loop algebra the cocycle $\Phi(E, g)$ may become trivial. In two dimensions the wave functionals in the momentum representation are supported on these special subspaces. The corresponding coboundary $\tilde{\Phi}$ governs the gauge dependence of the wave functionals:

$$\Psi = \tilde{\Phi} \Psi_0 \tag{A11}$$

for Ψ_0 being a gauge independent distribution with support on loops in integral coadjoint orbits.

Let us stress again that the triviality condition (A4) is actually an integrated form of the Gauss law (as shown in the Introduction). Then (A11) provides a universal solution of the Gauss law. In the example which we considered in this appendix we observe a new phenomenon in the theory of gauge cocycles. A nontrivial cocycle may shrink its support in order to become trivial and produce a physical wave functional. This may lead (as in the example of 2D YM theory with compact gauge group) to a discrete spectrum in the momentum representation.

APPENDIX B: TOPOLOGICAL TERM FOR $G = \mathrm{SU}(2)$

The topological Wess-Zumino term in the WZW model is usually represented in the form

$$A_{\mathrm{WZ}}(g) = \frac{k}{12\pi} \mathrm{tr} \int_\Sigma d^{-1} (dg\, g^{-1})^3 \ . \tag{B1}$$

The integration is formally performed over the two-dimensional surface Σ. [Here Σ is the image of the worldsheet M under the map $g(x)$ from $M \to G$.] The symbol d^{-1} has been introduced by Novikov [21] and applied to construct multivalued action functionals in [21,22]. It is understood in the following way. One chooses a three-dimensional submanifold B in the group G so that $\partial B = \Sigma$. The integration over Σ is replaced by an integration over B:

$$A_{\mathrm{WZ}}(g) = \frac{k}{12\pi} \mathrm{tr} \int_B (dg\, g^{-1})^3 \ . \tag{B2}$$

The definition (B2) is ambiguous as B may be chosen in many ways. The possible ambiguity in the definition of $\mathrm{WZ}(g)$ is an integral over the union of two possible B's:

$$\Delta A_{\mathrm{WZ}}(g) = \frac{k}{12\pi} \mathrm{tr} \left(\int_{B'} (dg\, g^{-1})^3 - \int_{B''} (dg\, g^{-1})^3 \right)$$
$$= \frac{k}{12\pi} \mathrm{tr} \int_{B' \cup \bar{B}''} (dg\, g^{-1})^3 \ . \tag{B3}$$

Here we denote by $\bar{B}''$ the manifold B'' with opposite orientation. Let us restrict our consideration to the case of $G = \mathrm{SU}(2)$. The only nontrivial three-dimensional cycle in $\mathrm{SU}(2)$ is the group itself. It implies that the integral (B3) is always proportional with some integer coefficient to the normalization integral

$$I = \frac{k}{12\pi}\mathrm{tr}\int_G (dg\,g^{-1})^3 = 2\pi k \ . \qquad \text{(B4)}$$

Here we used the fact that the volume of the group SU(2) with respect to the form $\mathrm{tr}(dg\,g^{-1})^3$ is equal to $24\pi^2$. This calculation explains why one should choose integer values of the coupling constant k. In this case the Wess-Zumino term WZ(g) is defined modulo 2π and its exponent $\exp[iA_{\mathrm{WZ}}(g)]$ is well defined.

Usually $A_{\mathrm{WZ}}(g)$ is referred to as a topological term because the defining three-form $\mathrm{tr}(dg\,g^{-1})^3$ on the group G is closed and belongs to a nontrivial cohomology class. This implies that the integral (B1) does not change when we fix Σ and vary B in a smooth way. Choosing the proper coefficient $k/12\pi, k \in N$, we get a three-form which belongs to an integer cohomology class. As we have seen this ensures that $\exp(iA_{\mathrm{WZ}}(g))$ is preserved even by a topologically nontrivial change of B.

So the fact that the three-form

$$\omega = \frac{k}{12\pi}\mathrm{tr}(dg\,g^{-1})^3$$

is closed and belongs to integer cohomology of G makes the action WZ(g) well defined. However, it is not true that $A_{\mathrm{WZ}}(g)$ is defined already by the cohomology class of ω. If we choose some other representative in the same class [as, e.g., $\tilde{\omega}$ in Eq. (B10) below], we get a new topological term, which is well defined for the same reason as WZ(g). In fact, the new action will differ from $A_{\mathrm{WZ}}(g)$. The reason is that the integral (B2) is defined over the manifold with a boundary and, hence, it is not defined by the cohomology class of the integrand. It depends on the representative as well.

Now we are prepared to introduce a new topological term for the WZW model. As it was explained in Sec. I we use the Gauss decomposition for the group element g:

$$g = g_{\downarrow}^{-1}g_{\uparrow} \ . \qquad \text{(B5)}$$

Observe that the Gauss decomposition is not applicable for some elements in SU(2). The Gauss components $g_{\downarrow}, g_{\uparrow}$ do not exist on the subset of antidiagonal unitary matrices. In a parametrization

$$g = \begin{pmatrix} z & \sqrt{1-z\bar{z}}\,e^{i\phi} \\ -\sqrt{1-z\bar{z}}\,e^{-i\phi} & \bar{z} \end{pmatrix},$$

$$z \in C, |z| \le 1, \phi \in [0, 2\pi) \quad \text{(B6)}$$

these elements are given by $z = 0$. They form a circle $\mathcal{C}$ parametrized by ϕ.

We can apply the Gauss decomposition on the rest of the group in order to remove the symbol d^{-1} from the topological term ω. Indeed, consider a two-form

$$\gamma = \frac{k}{4\pi}\mathrm{tr}(dg_{\downarrow}g_{\downarrow}^{-1} \wedge dg_{\uparrow}g_{\uparrow}^{-1}) \qquad \text{(B7)}$$

on the compliment of $\mathcal{C}$. It is easy to verify the relation

$$d\gamma = \omega \ . \qquad \text{(B8)}$$

Here we have used the fact that

$$\mathrm{tr}(dg_{\downarrow}g_{\downarrow}^{-1})^3 = \mathrm{tr}(dg_{\uparrow}g_{\uparrow})^3 = 0 \ . \qquad \text{(B9)}$$

which holds since the diagonal parts of $(dM\,M^{-1})^m$ vanish for any triangular matrix M if $m \ge 2$.

We established Eq. (B8) on the part of the group G which admits the Gauss decomposition. It is easy to see that this equation cannot hold true on all of G. Indeed, the left-hand side is represented by the exact form $d\gamma$ whereas the right-hand side belongs to a nontrivial cohomology class. In order to improve (B8) we introduce a correction to it:

$$d\gamma = (\omega - \tilde{\omega}) \ . \qquad \text{(B10)}$$

This equation is to be understood in a distributional sense: The three-form $\tilde{\omega}$ is supported on $\mathcal{C}$. Moreover it is closed and belongs to the same cohomology class as ω.

To determine $\tilde{\omega}$ for $G = $SU(2) we return to the parametrization (B6). In these coordinates (B7) takes the form

$$\gamma = i\left(\bar{z}\,dz - z\,d\bar{z} - 2\frac{dz}{z}\right)d\phi \ . \qquad \text{(B11)}$$

Multiplying γ by test one-forms, the resulting three-forms are integrable on G. So γ is a regular distribution and therefore the derivative $d\gamma$ is also well defined. Using

$$d(dz/z) = \pi\delta[\mathrm{Re}(z)]\delta[\mathrm{Im}(z)]dz\,d\bar{z}$$
$$= : -2\pi i\delta(\mathcal{C}) \ ,$$

where $\delta(\mathcal{C})$ has been introduced to denote the δ two-form supported on the critical circle $\mathcal{C}$ we obtain

$$\tilde{\omega} = k\delta(\mathcal{C})d\phi \ . \qquad \text{(B12)}$$

Let us conclude that the topological Wess-Zumino term may be replaced by the sum of a local term and a topological term supported on the set $\mathcal{C}$ of antidiagonal matrices:

$$\mathrm{WZ}(g) = \frac{k}{4\pi}\mathrm{tr}\int_{\Sigma}(dg_{\downarrow}g_{\downarrow}^{-1} \wedge dg_{\uparrow}g_{\uparrow}^{-1}) + S_{\delta}(g) \ ,$$

$$S_{\delta}(g) = \int_B \tilde{\omega} = k\int_B \delta(\mathcal{C})d\phi \ . \qquad \text{(B13)}$$

The new topological term $S_{\delta}(g)$ depends exclusively on the positions of the points where Σ intersects $\mathcal{C}$. In particular, it vanishes if Σ belongs to the part of the group which admits the Gauss decomposition.

In Sec. II we showed that the local term of (B13) fits nicely into the formalism of Poisson σ models. Coupling of such a model to the topological term S_{δ} is subject of Appendix C.

APPENDIX C: POISSON σ MODEL COUPLED TO A TOPOLOGICAL TERM AND QUANTUM STATES FOR SU(2) GWZW

Within this last appendix we pursue the following three goals: First we investigate the change of a Poisson σ

model

$$S_{\mathcal{P}}(X, A) = \int_M \left(A_{i\nu} \frac{\partial X^i}{\partial x^\mu} + \frac{1}{2} \mathcal{P}^{ij} A_{i\mu} A_{j\nu} \right) dx^\mu \wedge dx^\nu$$

$$\text{(C1)}$$

under the addition of topological term:

$$S(X, A) = S_{\mathcal{P}}(X, A) + S_{\text{top}}(X) . \qquad \text{(C2)}$$

Here $S_{\text{top}}(X)$ is supposed to be given by some closed three-form ω_{top},

$$S_{\text{top}}(X) = \int_B \omega_{\text{top}}, \quad \partial B = \text{Im} M , \qquad \text{(C3)}$$

of (generically) nontrivial cohomology on the target space N of the model (cf. also Appendix B). To not spoil the symmetries of (C1) we further require ω_{top} to be invariant under any transformation generated by vector fields of the form $\mathcal{P}^{ij} \partial_j$. We will focus especially on the change in the Hamiltonian structure that is induced by (C3).

Next we will specify the consideration to the GWZW model. In the main text and the previous appendix we have shown that the (Hamiltonian) GWZW action (16) may be rewritten *identically* in the form (C2) with $\omega_{\text{top}} = \tilde{\omega}$. However, an additional complication arises due to the fact that the matrix-valued one-form

$$\beta \equiv \beta_i dX^i := g_\uparrow^{-1} dg_\uparrow - g_\downarrow^{-1} dg_\downarrow , \qquad \text{(C4)}$$

which we used in the identification (43) for A, becomes singular at the part of G where the Gauss decomposition breaks down. The singular behavior of A has to be taken into account in the variation for the field equations, if we want to describe the GWZW model by means of (C2) globally. We will show that the bulk of the quantum states obtained in the main text remains unchanged by these modifications. The considerations change only for states that have support on loops lying on exceptional conjugacy classes in G.

Finally we will make the considerations more explicit for $G =$SU(2) and compare the resulting picture to the literature.

In the classical Hamiltonian formulation the term (C3) contributes only into a change of the symplectic structure of the field theory. With

$$\omega_{\text{top}} = \frac{1}{6} \omega_{ijk}^{(\text{top})} dX^i \wedge dX^j \wedge dX^k \qquad \text{(C5)}$$

the symplectic structure takes the form

$$\Omega^{\text{field}}(X, A) = \oint_{S^1} dA_{i1}(x^1) \wedge dX^i(x^1) dx^1 + \Omega_{\text{top}}^{\text{field}} \quad \text{(C6)}$$

with the extra piece

$$\Omega_{\text{top}}^{\text{field}} = \frac{1}{2} \oint_{S^1} \omega_{ijk}^{(\text{top})} [X(x^1)] \partial_1 X^i(x^1)$$
$$\times dX^j(x^1) \wedge dX^k(x^1) dx^1 . \quad \text{(C7)}$$

Note that as (if) ω_{top} is nontrivial in cohomology on the target space, Ω^{field} becomes nontrivial as well, i.e., glob-

ally there will not exist any symplectic potential Θ^{field} such that $\Omega^{\text{field}} = d\Theta^{\text{field}}$.

In the case $N = G$ and $\omega_{\text{top}} := \tilde{\omega}$ the symplectic forms (C6) and (18) in the main text coincide. Actually $A_{i1}(x^1)$ and $X^i(x^1)$ are Darboux coordinates of the symplectic form Ω^{field} of the GWZW model. As Ω^{field} has nontrivial cohomology such Darboux coordinates cannot exist globally. The situation may be compared to the one of a sphere with standard symplectic form $\Omega = \sin \vartheta \, d\vartheta \wedge d\varphi$; trying to extend the local Darboux coordinates $\cos \vartheta$ and φ as far as possible, one finds (again in a distributional sense) $\Omega = d(\cos \vartheta d\varphi) + 2\pi \delta^2(\text{"southpole"}) - 2\pi \delta^2(\text{"northpole"})$. Here we used $d(d\varphi) = \sum_{\text{poles}} 2\pi \delta^2$ (pole), resulting from the breakdown of $d\varphi$ as a coordinate differential at the poles while it still remains a regular one-form in a distributional sense. By the way, one may infer Eq. (18) also from (C6) and (C7): Just replace the coordinate basis dX^i by the left-invariant basis $dg \, g^{-1}$ and note that $d(p \, dg \, g^{-1}) = dp \, dg \, g^{-1} + p(dg \, g^{-1})^2$ has to be substituted for $d(A_{i1} dX^i) = dA_{i1} dX^i$.

The classical Gauss law (55), on the other hand, remains unaltered by the addition of a term (C3) to the action. Indeed the constraints $G^i \approx 0$ emerge as the coefficient of A_{i0} within the action $S = S_{\mathcal{P}} + S_\delta$ and S_δ does not depend on A.

Now let us turn to the quantum theory of the coupled model (C2). Again we go into an X representation. In general "wave functions" may be regarded as section of a line bundle, the curvature of which is the symplectic form [14]. In the case that this line bundle is trivial, i.e., when the symplectic form Ω^{field} allows for a global symplectic potential, one may choose a global nonvanishing section in the bundle. The relative coefficient of any other section with respect to the chosen one is then a function, the wave function $\Psi[X]$. This procedure is called trivialization of the line bundle. In the case of prominent interest for us in which ω_{top} and (thus) Ω^{field} belong to some nontrivial cohomology class the quantum line bundle over the loop space will be nontrivial [23]. Sections may be represented by functions $\Psi[X]$ then only within some local charts.

The X^i may still be represented as multiplicative operators. However, the change in the symplectic structure implies that one cannot represent A_{i1} as the derivative operators (54) any more. Indeed the modification $\Omega_{\text{top}}^{\text{field}}$ preserves commutativity of the X^i as well as the commutation relations between the A_{i1} and the X^i; however, the A_{i1} do not commute among each other any longer. The net result of the change in the symplectic structure is that we have to add some X-dependent piece to the operator representation of A_{i1}:

$$A_{i1} = i \frac{\delta}{\delta X^i} + \vartheta_i^{\text{field}}(X) . \qquad \text{(C8)}$$

The new quantity $\vartheta_i^{\text{field}}$ is a symplectic potential to the nontrivial part $\Omega_{\text{top}}^{\text{field}}$ of the symplectic form, i.e.,

$$\vartheta_{\text{t} \cap \text{p}}^{\text{field}} = \oint \vartheta_i^{\text{field}} dX^i(x^1) dx^1 \qquad \text{(C9)}$$

is a solution to the equation

$$\Omega_{\text{top}}^{\text{field}} = d\vartheta_{\text{top}}^{\text{field}} \quad \text{(locally)} . \tag{C10}$$

$\vartheta_{\text{top}}^{\text{field}}$ is not unique and may be chosen in many ways. If ω_{top} belongs to a trivial cohomology class, (C10) may be solved globally. Any choice for $\vartheta_i^{\text{field}}$ then corresponds to the choice of a trivialization of this line bundle. If, on the other hand, ω_{top} belongs to some nontrivial cohomology class, we can speak about a solution to (C10) only locally. Still any choice of a local potential $\vartheta_{\text{top}}^{\text{field}}$ corresponds to a local trivialization of the quantum line bundle within some chart. Within the latter, quantum states may be represented again as ordinary functions $\Psi[X]$ on the loop space and (C8) gives the corresponding operator representation of A_{i1}.

Let us finally write down the new quantum Gauss law. Within a local chart on the loop space it takes the form

$$i(\partial X^i + \mathcal{P}^{ij}\vartheta_j^{\text{field}})\Psi = \mathcal{P}^{ij}\frac{\delta}{\delta X^j}\Psi . \tag{C11}$$

For nonsingular forms $\vartheta_{\text{top}}^{\text{field}}$ these constraints yield a restriction to functionals with support on loops lying entirely within some symplectic leaf again. (This holds true also for a singular $\vartheta_j^{\text{field}}$, as long as its contraction with the Poisson tensor $\mathcal{P}^{ij}$ vanishes.) To see this, just use the first k coordinates X^i to parametrize the symplectic leaves in any considered region of N. Then (C11) yields $\partial X^i \Psi = 0$ for $i = 1, \ldots, k$. So, strictly speaking, the physical wave functionals will be distributions that restrict the loops to lie entirely within symplectic leaves. The remaining $n - k$ equations (C11) then determine the form of Ψ on each leaf.

Let us show this for trivial cohomology of the defining three-form in (C3), i.e., for the special case that

$$\omega_{\text{top}} = d\vartheta_{\text{top}} \tag{C12}$$

globally on N. Then $\vartheta_j^{\text{field}} = \vartheta_{jk}^{(\text{top})}[X(x^1)]\partial_1 X^k(x^1)$ globally on the phase space. To find the form of Ψ on a given symplectic leaf $\mathcal{S}$ we multiply (C11) for $i = k + 1, \ldots, n$ by Ω_{li} from the left [cf. Eq. (59)]. The resulting equation can be integrated easily to yield

$$\Psi = \Psi_0 \exp\left(i \int d^{-1}(\Omega + \vartheta_{\text{top}})\right) , \tag{C13}$$

where Ψ_0 is an integration constant, which, however, may depend on the chosen symplectic leaf (and, if $\mathcal{S}$ is not simply connected, also on the homotopy class of the argument loop of Ψ). Ψ_0 may be regarded as the evaluation of Ψ on some reference loop on $\mathcal{S}$ and the phase is determined by the integration of the two-form $\Omega + \vartheta_{\text{top}}$ over a two-surface that is enclosed between the reference loop and the argument loop of Ψ. Independence of the choice of the chosen two-surface requires, e.g., for a simply connected $\mathcal{S}$:

$$\oint_\sigma \Omega + \vartheta_{\text{top}} = 2\pi n, \quad n \in Z , \tag{C14}$$

for all two-cycles $\sigma \in \mathcal{S}$. This generalizes the integrality condition (63), which corresponds to $\vartheta_{\text{top}} \equiv 0$. (C14) is a well-formulated condition, as the invariance requirement for (C3) under the symmetries of (C1) may be seen to imply that the *restriction* of ϑ_{top} onto any symplectic leaf must be a closed two-form (while, certainly, ϑ_{top} will not be closed in general on all of N).

For a truly *topological* term (C3) Eq. (C12) holds only locally. Still (C13) provides the local solution to the quantum constraints (C11) in the space of loops on $\mathcal{S}$. However, as the form ϑ_{top} is not defined globally on $\mathcal{S}$ in general, the global integrability for (C11) does not have the simple form (C14). Instead the use of various charts in the line bundle over the loop space will be unavoidable to determine integrability of (C11) on a lead and thus the existence of a quantum state located on that leaf. We will not study this problem in full generality here further. Rather we will restrict our attention to the GWZW model in the following.

Everything that has been written above applies to the GWZW model, too, except for one small change: Actually, the correct Gauss law for GWZW is not $G^i \approx 0$, but

$$\beta_i G^i \approx 0 , \tag{C15}$$

where the matrix-valued coefficients β_i have been defined in (C4). To see this we recall that the constraints of the GWZW model, given first in Eq. (19), result from a variation for $\lambda \propto a_-$ within the action. According to (43) A_{i0} differs from (the components of) λ by $A_{i0} = \text{tr}\lambda\beta_i$. So the correct GWZW Gauss law (19) may be rewritten as (C15). For loops inside the Gauss-decomposable region of G this is equivalent to the old form of the constraints $G^i = 0$, since on that part of G the difference corresponds merely to a change of basis in T^*G. However, as β becomes singular at that lower dimensional part of G where the Gauss decomposition breaks down, the constraints (C15) have somewhat different implications $G^i = 0$ in that region.

This consideration applies also to the quantum constraints; we have to multiply (C11) by β_i from the left. The result is

$$(\beta_i\partial X^i + \beta_i\mathcal{P}^{ij}\vartheta_j^{\text{field}})\Psi + i\beta_i\mathcal{P}^{ij}\frac{\delta}{\delta X^j}\Psi = 0 , \tag{C16}$$

or, equivalently,

$$\left(g_\uparrow^{-1}\partial_1 g_\uparrow - g_\downarrow^{-1}\partial_1 g_\downarrow + \beta_i\mathcal{P}^{ij}\vartheta_j^{\text{field}} + \frac{2\pi i}{k}\frac{\delta}{\delta h}h\right)\Psi[g_0, h]$$

$$= 0 . \tag{C17}$$

The part $\beta_i\mathcal{P}^{ij}\vartheta_j^{\text{field}}$, which may be rewritten also as the insertion of the vector field $(2\pi/k)(\delta/\delta h)h$ into the one-form $\vartheta_{\text{top}}^{\text{field}} = \vartheta_\delta^{\text{field}}$, is the new contribution from S_δ that has been dropped in the derivation of (34).

It is not difficult to see that for loops that lie at least partially outside exceptional conjugacy classes ("critical region") in G one may solve (56) instead of (C16) or (C17). Indeed close to any part of the loop outside the critical region we may use (C11) as the quantum constraint, because (C4) is invertible in that part of G.

But as argued above this restricts the loop to lie *entirely* within a symplectic leaf outside the critical region in G. For *such* loops now we may always choose

$$\vartheta_i^{\text{field}} \equiv 0 , \qquad (C18)$$

as $\Omega_\delta^{\text{field}}$ vanishes on that part of the phase space. This justifies that in the main text we dropped the contributions from S_δ (as well as the multiplicative factor β_i) and restricted our attention to the solution of (56). Also we had not to think of a nontrivial quantum line bundle in this way. The main part of the states could be obtained within one local trivialization of the line bundle, given by (C18).

What has to be considered separately only are possible states that have support on loops lying *exactly* within the critical region of G. In this case the full quantum constraints (C16) have to be taken into account. It is also in this region of G, furthermore, where the notion of symplectic leaves and conjugacy classes do not coincide. From (C17) we learn that it is precisely the modifications of (56) that restore the adjoint transformations as symmetries on the quantum level. β diverges precisely where $\mathcal{P}$ vanishes so as to give rise to the finite contribution $(\delta/\delta h)h$ in (C17). As a result there will correspond at most one quantum state to an exceptional conjugacy class, even if the respective orbit splits into several (possibly in part integrable) symplectic leaves.

Let us now specify our considerations to $G = \text{SU}(2)$. In particular we want to determine all quantum states within our approach. For this purpose let us first consider the splitting of $\text{SU}(2) \sim S^3$ into conjugacy classes. Parametrizing conjugacy classes by [cf. also (B6)]

$$\tfrac{1}{2}\text{tr}g = \text{Re}(z) =: \cos\theta = \text{const}, \quad \theta \in [0,\pi] , \qquad (C19)$$

we find that, topologically speaking, these orbits are two-spheres for $\theta \in (0,\pi)$ and points for $\theta = 0,\pi \leftrightarrow z = \pm 1$. Only one of the conjugacy classes is "exceptional" it corresponds to $\theta = \pi/2 \leftrightarrow \text{tr}g = 0$. Parametrizing this critical S^2 by polar coordinates ϕ and $\vartheta := \arccos \text{Im}(z)$, the part $\mathcal{C}$ of $\text{SU}(2)$ on which the Gauss decomposition is not applicable is identified with the equator $\vartheta = \pi/2$ of this two-sphere.

So the picture we obtain is that $N = S^3$ is foliated into two-spheres except for its "poles" $z = \pm 1$. The "equator" of the three-sphere, itself an S^2, is what we called an exceptional conjugacy class. The equator $\mathcal{C} \sim S^1$ of this S^2 is precisely the subset of $N = G$ where the Gauss decomposition breaks down and, correspondingly, where the support of $\omega_{\text{top}} = \tilde{\omega}$ lies. The exceptional conjugacy class splits into several symplectic leaves: the Northern part of the S^2, its Southern part, and the points of the equator $\mathcal{C}$, where $\mathcal{P}$ vanishes. According to our general considerations above, this splitting is, however, irrelevant; there will correspond at most one quantum state to the exceptional conjugacy class.

On the other hand there corresponds precisely one quantum state to any integral (nonexceptional) conjugacy class, as all of these orbits are simply connected. So let us evaluate the integrality condition (63) for the nonexceptional conjugacy classes in $\text{SU}(2)$. From (66) we

find that in the coordinates (B6)

$$\Omega = \frac{ik}{2\pi} \frac{dz}{z} \wedge d\phi . \qquad (C20)$$

In the parametrization (C19) for the adjoint orbits this yields for the integral of Ω over the respective two-spheres

$$\int_{S^2} \Omega = \begin{cases} 2k\theta, & \theta \in [0,\pi/2), \\ 2k(\theta - \pi), & \theta \in (\pi/2,\pi]. \end{cases} \qquad (C21)$$

Here we have taken into account that the imaginary part $\text{Im}(z)$ of z runs only from $-\sin\theta$ to $+\sin\theta$ since $|z| \leq 1$.

For the critical orbit at $\theta = \pi/2$ the symplectic volume (C21) becomes ill defined. This comes as no surprise. Here obviously the choice (C18) does not apply for all loops on the critical conjugacy class. Still the correct integrability condition may be guessed from a simple limiting procedure: From (C21) we obtain

$$\lim_{\theta \to \pm\pi/2} \int_{S^2} \Omega = \pm k\pi . \qquad (C22)$$

For integer values of k this result is unique modulo 2π. It is plausible to assume that the critical orbit will carry a quantum state, iff again (C22) is an integer multiple of 2π [cf. Eq. (63)].

In fact, one can prove that this is indeed correct. To do so one might use two charts in the quantum line bundle. First (C18), which works for all loops that do not intersect the equator $\mathcal{C}$ of the critical conjugacy class. And second,

$$\vartheta_\delta := \frac{k}{2\pi} \left(\frac{1}{z} + i \right) dz \wedge d\phi \to \vartheta_i^{\text{field}}$$
$$= \frac{k}{2\pi} \left(\frac{1}{z} + i \right) (dz\, \partial_1\phi - \partial_1 z d\phi) . \qquad (C23)$$

This second chart is applicable to all loops on the critical conjugacy class that do not touch its "pole" $z = -i$. The solution to the full quantum constraints (C16) has again the form (C13) within the respective domain of definition of the two charts. Now one might regard the value of the wave functional in *both* charts for two small loops close to the pole $z = -i$, one of which with winding number one around this pole, the other one with winding number zero. In the first chart continuity of the wave function implies that the wave functional will have basically the same value for both loops. In the second chart the two loops are separated from each other by a two-surface that encloses basically all of the critical S^2 (since in this chart the first loop may not be transformed into the second one through the pole $z = -i$, but instead one has to move through the other pole $z = i$); this gives a relative phase factor of the wave functions in this chart that may be determined by means of (C13). The corresponding phase need, however, *not* be a multiple of 2π. Instead, the result of chart two has to coincide with the result of chart one only *after* taking into account the transition functions between the two charts (note that both loops lie in both charts). In fact, for the first loop one picks

up a nontrivial contribution to the integrality condition from there. Further details shall be left to the reader. In any case the result coincides with the one obtained from the limit above. So one finds that there exists a quantum state with support on the critical orbit $\theta = \pi/2$ for even values of k and no such a state for odd values of k.

Let us remark here that in the latter case all "physical" quantum states, i.e., all states in the kernel of the quantum constraints, may be described within just one chart of the quantum line bundle [as, e.g., by (C18)]. So, the restriction to physical states may yield the originally nontrivial quantum line bundle of a coupled model (C2) to become effectively trivial.

Summing up the results for $G = $SU(2) we conclude that the integral orbits [i.e., the orbits allowing for nontrivial quantum states of the SU(2)-GWZW model] are given by $\theta = n\pi/k, n = 0, 1, \ldots, k$.

Now we want to compare this result with the current literature. According to [9], there are two different pictures for the space of states of the GWZW model. (In [9] they consider partition functions of the WZW model. However, these two issues may be related using results of [10].) The first picture eventually coincides with our answer. The second one suggests the finite renormalization $k \to k + 2$. In this case the integral orbits are characterized by $\theta = n\pi/(k + 2)$, $n = 0, \ldots, k + 2$. However, in this picture the singular orbits with $n = 0$ and $n = k + 2$, corresponding to the central elements $\pm I \in$SU(2), should be excluded. In [9] it is proved that the two pictures are equivalent. However, it would be interesting to establish this equivalence in the language of Poisson σ models. One motivation is to compare the results with the similar formalism [7]. Also, it seems to be easier to handle the spectrum of the model in the second picture.

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