# One-loop renormalization of black hole entropy due to nonminimally coupled matter

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The quantum entanglement entropy of an eternal black hole is studied. We argue that the relevant Euclidean path integral is taken over fields defined on an $\alpha$-fold covering of the black hole instanton. The statement that the divergences of the entropy are renormalized by the renormalization of gravitational couplings in the effective action is proved for nonminimally coupled scalar matter. The relationship of entanglement and thermodynamical entropies is discussed.

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## I. INTRODUCTION

According to the thermodynamical analogy one can apply the laws of thermodynamics that are valid for large statistical systems to the description of a single black hole [1]. The key idea of this approach is that a black hole has an intrinsic entropy proportional to the surface area of the event horizon $\Sigma$. This idea has found strong support in Hawking's discovery [2] of the thermal radiation of a black hole that allowed one to determine the entropy precisely as $S = A_\Sigma/4G$. However, the microphysical explanation of the black hole entropy as a counting of states is still absent, though many attempts have been made (see recent review [3]). One possible way is to associate the entropy with a thermal bath of fields propagating just outside the horizon [4]. Recently, it also has been proposed to treat black hole entropy as an entanglement entropy [5, 6]. Starting with the pure vacuum state one traces over modes of a quantum field propagating inside the horizon and obtains the density matrix $\rho$. The entropy then is defined by the standard formula $S = -\mathrm{Tr}\rho\ln\rho$. It measures the number of inside modes which are considered as internal degrees of freedom of the hole. In a similar manner, Frolov and Novikov [7] suggested to trace over modes outside the horizon.

Calculations for the Rindler space and black holes [8–16] have shown that entropy is divergent. This is essentially due to the short-distance correlations between the inside and outside modes.

The purpose of this paper is to demonstrate, following previous investigations [10–12], that this divergence is really the ultraviolet one typically appearing in quantum field theory and it can be removed by standard renormalization of the gravitational couplings in the effective action. In an earlier paper [17] we have given a proof of this for a minimally coupled scalar field and noted that the nonminimal coupling needs special consideration. The reason for this is that there exists a $\delta$-like potential in the field operator due to the scalar curvature $R$ behaving as a distribution on a manifold with a conical singularity. Below we consider this in more detail. We start in Sec. II by formulating the Euclidean path integral which is relevant to the calculation of the entanglement entropy of a black hole. In Sec. III we formulate the statement about the renormalization of black hole entropy. The proof of it for a nonminimally coupled scalar field is given in Sec. IV. We conclude in Sec. V with some remarks concerning the relationship of the entanglement (statistical) and thermodynamical entropies. The Appendixes A and B contain basic formulas for curvature tensors and heat kernel expansion on manifolds with conical singularities obtained in a previous study.

## II. EUCLIDEAN PATH INTEGRAL FOR ENTANGLEMENT ENTROPY

The horizon surface $\Sigma$ naturally separates the whole space-time of a static black hole[1] on the regions $R_+$ and $R_-$, the free information exchange between which is impossible. This is obviously due to the fact that the global Killing vector $\xi_t = \partial_t$, generating translations in time, becomes null, $\xi_t^2 = 0$, on the horizon. Therefore, any light signal emitted from any point on the horizon or behind it never can reach an outside observer. So events happening in the part of the space-time beyond the horizon are unobservable for him in principle. This concerns excitations of quantum fields as well. They are naturally separated on "visible" (propagating in the region $R_+$) and "invisible" (propagating in the $R_-$ region) modes. The partial loss of information about the microstates composing the concrete macrostate typically appears in a statistical de-

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[1]We do not consider here the case of a stationary (rotating) black hole where the situation seems to be more complicated.

 

scription of systems with a large number of degrees of freedom. We can see that a similar phenomenon naturally happens for a black hole. This fact certainly lies in the principles of the thermodynamical analogue allowing one to apply the laws of thermodynamics to a hole.

The situation, when a part of the states of the system is unknown, in quantum mechanics is usually described by a density matrix. Assume that the quantum field $\phi$, being considered on the whole space-time, is, in a pure ground state,

$$\Psi_0 = \Psi_0(\phi_+, \phi_-), \tag{2.1}$$

which is a function of both visible ($\phi_+$) and invisible ($\phi_-$) modes. For an outside observer it is in the mixed state described by the density matrix

$$\rho(\phi_+^1, \phi_+^2) = \int [\mathcal{D}\phi_-]\Psi_0^\dagger(\phi_+^1, \phi_-)\Psi_0(\phi_+^2, \phi_-), \tag{2.2}$$

where one traces over all invisible modes $\phi_-$. Then the entropy, defined as

$$S_{\text{geom}} = -\text{Tr}\hat{\rho}\ln\hat{\rho}, \quad \hat{\rho} = \frac{\rho}{\text{Tr}\rho}, \tag{2.3}$$

is the so-called entanglement (or geometric) entropy [5–7].

Applying this construction to a black hole, we identify all the invisible modes with internal degrees of freedom and (2.3) with the entropy of the hole. The ground state of the black hole is given by the Euclidean functional integral [18] over fields defined on a manifold $E'$ which is the half-period part of the black hole instanton with the metric

$$ds^2_{E'} = \beta_H^2 g(\rho) d\varphi^2 + d\rho^2 + r^2(\rho) d^2\Omega, \tag{2.4}$$

where the angle variable $\varphi$ lies in the interval $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$. The inverse Hawking temperature $\beta_H$ is determined by the derivative of the metric function $g(\rho)$ on the horizon $[g(\rho_h) = 0]$, $\beta_H = \frac{2}{g'(\rho_h)}$. The $\phi_+$ and $\phi_-$ which enter as arguments in (2.1) are the fixed values at the boundaries $\phi_+ = \phi(\varphi = \frac{\pi}{2})$, $\phi_- = \phi(\varphi = -\frac{\pi}{2})$, giving the boundary condition in the path integral. The density matrix $\rho(\phi_+^1, \phi_+^2)$ obtained by tracing the $\phi_-$ modes is defined by the path integral over fields on the full black hole instanton $E$ $(-\frac{3}{2}\pi \le \varphi \le \frac{1}{2}\pi)$ with a cut along the $\varphi = \frac{\pi}{2}$ axis and taking values $\phi_+^{1,2}$ above and below the cut. The trace $\text{Tr}\rho$ is obtained by equating the fields across the cut and doing the unrestricted Euclidean path integral on the complete black hole instanton $E$. Analogously, $\text{Tr}\rho^n$ is given by the path integral over fields defined on $E_n$, the $n$-fold cover of $E$. Thus, $E_n$ is the manifold with Abelian isometry (with respect to angle rotation $\partial_\varphi$) with the horizon surface $\Sigma$ as a stationary set. Near $\Sigma$ $E_n$ looks like a direct product $E_n = \Sigma \otimes \mathcal{C}_n$, where $\mathcal{C}_n$ is a two-dimensional cone with an angle deficit $\delta = 2\pi(1-n)$. This construction can be analytically continued to arbitrary (not integer) $n \to \alpha = \frac{\beta}{\beta_H}$.

Define now the partition function

$$Z(\beta) = \text{Tr}\rho^\alpha, \tag{2.5}$$

which is a path integral over fields defined on $E_\alpha$, the $\alpha$-sheeted covering of $E$. Then the geometric entropy (2.3) takes the standard thermodynamical form

$$S_{\text{geom}} = -\text{Tr}(\hat{\rho}\ln\hat{\rho}) = (-\alpha\partial_\alpha + 1)\ln Z(\alpha)|_{\alpha=1}, \tag{2.6}$$

being expressed via the partition function $Z$. We see that $\beta$ plays the role of the inverse temperature. After all calculations one must put $\beta = \beta_H$ in (2.6). Assuming that the dynamics of matter fields is determined by a differential operator $\hat{\Delta}$ we obtain that the relevant partition function (2.5) is given by the determinant

$$Z(\beta) = \det{}^{-1/2}\hat{\Delta} \tag{2.7}$$

considered on $E_\alpha$. It is essential that $E_\alpha$ is a manifold with a conical singularity since, namely, the singularity produces in the effective action $W(\alpha) = -\ln Z(\alpha)$ terms proportional to $(1-\alpha)$ that contribute to the entropy (2.6).

One can see that the partition function (2.5) looks like a thermal one:

$$Z(\beta) = \text{Tr}e^{-\beta\hat{H}}, \tag{2.8}$$

with $\beta$ playing the role of inverse temperature, and $\hat{H}$ being a relevant Hamiltonian. This fact was previously observed in [9] for Rindler space and was supposed to be general. The relevant Euclidean path integral for the entanglement entropy of Rindler space was found in [8]. The exact construction of the wave function of a black hole was proposed in [18]. The thermality of the corresponding density matrix was established in [19].

## III. STATEMENT

As defined in the previous section entanglement entropy is not free from the ultraviolet divergences. They result from the corresponding divergences of the effective action $W(\alpha)$. It was shown for the minimally coupled scalar matter [20] that the divergent part of the effective action on $E_\alpha$ is really sum of volume and surface terms:

$$W_{\text{div}}(\alpha) = W_{\text{div}}^{\text{vol}} + W_{\text{div}}^{\text{surf}}. \tag{3.1}$$

The volume term in (3.1) is a standard one. It is proportional to $\alpha$ not contributing to the entropy. The second term is given by an integral over the singular surface $\Sigma$. It is proportional to $(1-\alpha)$ and hence contributes to the entropy, resulting in its divergence [13]. The origin of these divergences lies obviously in the short-distance correlation between "visible" and "invisible" modes which is concentrated at the surface $\Sigma$ separating regions $R_+$ and $R_-$.

However, it was proposed in a number of papers that the divergences of entropy can be removed by the standard renormalization of the gravitational couplings. Indeed, the higher curvature terms are necessarily generated by quantum corrections . Therefore, such $R^2$ terms must be added from the very beginning with some bare constants $(c_{1,B}, c_{2,B}, c_{3,B})$ (tree level) to absorb the one-loop infinities. The bare (tree-level) gravitational functional thus takes the form[2]

$$W_{\rm gr} = \int \sqrt{g} d^4x \left( -\frac{1}{16\pi G_B} R + c_{1,B} R^2 + c_{2,B} R^2_{\mu\nu} + c_{3,B} R^2_{\mu\nu\alpha\beta} \right). \quad (3.2)$$

The corresponding tree-level entropy can be obtained within the procedure considered in the previous section as replica of the action (3.2) on introducing of the conical singularity. The conical singularity at the horizon $\Sigma$ manifests itself in that a part of a curvature tensor for such a manifold $E_\alpha$ behaves as a distribution having support on the surface $\Sigma$ [21, 22] (see Appendix A). Hence, the action (3.2) being considered on $E_\alpha$ has volume and surface terms:

$$W_{\rm gr}[E_\alpha] = W^{\rm vol}_{\rm gr}[E_\alpha/\Sigma] + W^{\rm surf}_{\rm gr}[\Sigma], \quad (3.3)$$

where the volume part is given by integral (3.2) over the regular part of the manifold $E_\alpha$. This part is obviously proportional to $W^{\rm vol}_{\rm gr} \propto \alpha$. So the whole contribution to the entropy comes from the surface term. Using the formulas of Appendix A [(A2)–(A5)] we obtain finally, for the tree-level entropy [17, 22],

$$S(G_B, c_{i,B}) = \frac{1}{4G_B} A_\Sigma - \int_\Sigma \left( 8\pi c_{1,B} R + 4\pi c_{2,B} R_{ii} + 8\pi c_{3,B} R_{ijij} \right). \quad (3.4)$$

We see that the classical law $S = \frac{1}{4G} A_\Sigma$ gets modified due to $R^2$ terms in the action (3.2). The additional term now depends on both the external and internal geometries of the surface $\Sigma$. It should be noted that (3.4) exactly coincides with entropy computed by the Noether charge method of Wald [23].

The main point now is that the divergent part of the entanglement entropy (2.6) is such that its sum with the tree-level entropy (3.4),

$$S(G_B, c_{i,B}) + S_{\rm div}(\epsilon) = S(G_{\rm ren}, c_{i,\rm ren}), \quad (3.5)$$

takes again the tree-level form $S(G_{\rm ren}, c_{i,\rm ren})$, expressed through the renormalized constants $G_{\rm ren}$, $c_{i,\rm ren}$. They are related with the bare constants by usual equations originated from the one-loop renormalization of the action

$$W_{\rm gr}(G_B, c_{i,B}) + W_{\rm div}(\epsilon) = W_{\rm gr}(G_{\rm ren}, c_{i,\rm ren}), \quad (3.6)$$

being considered on regular space-times without horizons.

Thus, divergences of the entanglement entropy are removed by the standard renormalization of the gravitational couplings. So no special renormalization procedure for entropy is required.

This statement for the Newton constant $G$ has been advocated in [10, 11] which considered divergences of entropy of the Rindler space-time. The necessity of renormalizing also the higher curvature couplings was argued in [12] for the entropy of the Schwarzschild black hole. For minimal coupling this statement was proved in [17] for a general black hole metric. In a recent paper [24] this procedure was checked for the Reissner-Nordström black hole. Below we demonstrate this statement for the nonminimally coupled scalar matter generalizing the result of [17].

## IV. HEAT KERNEL EXPANSION

For a nonminimally coupled scalar field the curvature directly enters into the matter action:

$$W_{\rm mat} = \frac{1}{2} \int [(\nabla\phi)^2 + \xi R \phi^2]. \quad (4.1)$$

Considering (4.1) on manifold $E_\alpha$ we must take into account the $\delta$-like contribution of the curvature coming from the conical singularity [see (A1)] [21, 22]:

$$R = \bar{R} + 4\pi(1-\alpha)\delta_\Sigma, \quad (4.2)$$

where $\bar{R}$ is the regular part of the scalar curvature. Therefore, the quantization of nonminimal matter on $E_\alpha$ forces one to deal with the problem of treating operators with a $\delta$-like potential. Applying (4.2) to the action (4.1) we obtain that

$$W_{\rm mat} = 2\pi(1-\alpha)\xi \int_\Sigma \phi^2 + \frac{1}{2} \int_{E_\alpha} \phi(-\Box_\xi)\phi \quad (4.3)$$

where we denote $\Box_\xi = \Box - \xi\bar{R}$ and assume regularity of the field $\phi$ on the singular surface $\Sigma$.

Then, considering the path integral over the scalar field $\phi$ we get

$$Z = \int [\mathcal{D}\phi] \exp\left( -2\pi\xi(1-\alpha) \int_\Sigma \phi^2 \right) \times \exp\left( -\frac{1}{2} \int_{E_\alpha} \phi(-\Box_\xi)\phi \right). \quad (4.4)$$

Expanding[3] the first factor in (4.4) by powers of $(1-\alpha)$

[2]Of course, we assume an addition to (3.2) due to a classical matter which can be in principle rather complicated.

[3]We proceed with the perturbation expansion with respect to $(1-\alpha)$. The first term of the expansion is well defined [see (4.11)]. The next terms, however, are expected to be ill defined due to contributions such as $\delta^2(0)$. The indication of this can be found in [25]. In principle, we could use some type of regularization similar to that of [22] to give sense to such terms. It should be noted, however, that these terms are irrelevant for the calculation of entropy. I thank D. Fursaev for this remark.

and omitting higher terms we have

$$Z=\bar{Z}\left(1-2\pi\xi(1-\alpha)\left\langle\int_\Sigma\phi^2\right\rangle_{\bar{Z}}\right), \tag{4.5}$$

where the average $\langle\ \rangle_{\bar{Z}}$ is taken with respect to the measure defined by the functional integral

$$\bar{Z}=\int[\mathcal{D}\phi]\exp\left(-\frac{1}{2}\int_{E_\alpha}\phi(-\Box_\xi)\phi\right). \tag{4.6}$$

Equivalently, this can be written as

$$\ln Z=\ln\bar{Z}-2\pi\xi(1-\alpha)\left\langle\int_\Sigma\phi^2\right\rangle_{\bar{Z}}. \tag{4.7}$$

For $\ln\bar{Z}$ the following heat kernel expansion is known [20]:

$$\ln\bar{Z}=-\frac{1}{2}\ln\det(-\Box_\xi)=\frac{1}{2}\int_{\epsilon^2}^\infty\frac{ds}{s}\mathrm{Tr}\bar{K}_{E_\alpha}(s),$$
$$\bar{K}_{E_\alpha}(s)=e^{-s\Box_\xi}=\frac{1}{(4\pi s)^{\frac{d}{2}}}\sum_n\bar{a}_n s^n,\quad s\to 0, \tag{4.8}$$

where the coefficients $\bar{a}_n(x,x)$ generally take the form

$$\bar{a}_n(x,x)=\bar{a}_n^{\rm st}(x,x)+\bar{a}_{n,\alpha}(x,x)\delta_\Sigma. \tag{4.9}$$

The $\bar{a}_n^{\rm st}(x,x)$ are standard [26] heat kernel coefficients given by the local functions of curvature tensors (see Appendix B). The second term in (4.9) has support only on the singular surface $\Sigma$; $\bar{a}_{n,\alpha}(x,x)$ is a local function of projections of curvature tensors on the subspace normal to $\Sigma$. The exact form of the coefficients $\bar{a}_{n,\alpha}(x,x)$ is given in Appendix B.

On the other hand, by standard arguments we have

$$\langle\phi(x)\phi(x')\rangle=\Box_\xi^{-1}=\int_{\epsilon^2}^\infty ds\,e^{-s\Box_\xi}. \tag{4.10}$$

Inserting (4.8)–(4.10) into (4.7) we finally obtain

$$\ln Z=\frac{1}{2}\int_{\epsilon^2}^\infty\frac{ds}{s}\mathrm{Tr}K_{E_\alpha}(s),$$
$$\mathrm{Tr}K_{E_\alpha}(s)=\mathrm{Tr}\bar{K}_{E_\alpha}(s)-4\pi\xi(1-\alpha)s\mathrm{Tr}_\Sigma\bar{K}_{E_\alpha}(s), \tag{4.11}$$

where the $x$ integration in $\mathrm{Tr}_\Sigma$ is taken only over the surface $\Sigma$. Identity (4.11) allows us to write the following expansion for the heat kernel $K_{E_\alpha}(s)$:

$$\mathrm{Tr}K_{E_\alpha}(s)=\frac{1}{(4\pi s)^{d/2}}\sum_n a_n s^n,$$
$$a_n=\int_{E_\alpha}\bar{a}_n(x,s)-4\pi\xi(1-\alpha)\int_\Sigma\bar{a}_{n-1}(x,s). \tag{4.12}$$

Since we are interested only in the first order of $(1-\alpha)$ we may take $\bar{a}_{n-1}=\bar{a}_{n-1}^{\rm st}$ in the right-hand side (RHS) of (4.12) neglecting the corresponding $\bar{a}_{n-1,\alpha}$ term. One can see that $a_n$ has the same volume part $\bar{a}_n^{\rm st}$ as (4.9) [see (B3)]:

$$a_0^{\rm st}(x)=1\ ,\quad a_1^{\rm st}=(\tfrac{1}{6}-\xi)\bar{R},$$
$$a_2^{\rm st}(x)=\tfrac{1}{180}\bar{R}^2_{\mu\nu\alpha\beta}-\tfrac{1}{180}\bar{R}^2_{\mu\nu}-\tfrac{1}{6}(\tfrac{1}{5}-\xi)\Box\bar{R}+\tfrac{1}{2}(\tfrac{1}{6}-\xi)^2\bar{R}^2. \tag{4.13}$$

The difference appears in the surface term. For the few first coefficients we obtain [cf. (B4)]

$$a_{0,\alpha}=0,\quad a_{1,\alpha}=4\pi(1-\alpha)\left[\frac{1}{6}\left(\frac{1+\alpha}{2\alpha}\right)-\xi\right],$$

$$a_{2,\alpha}=4\pi(1-\alpha)\left(\frac{1}{6}-\xi\right)\left[\frac{1}{6}\left(\frac{1+\alpha}{2\alpha}\right)-\xi\right]\bar{R}-\frac{\pi}{180}\left(\frac{1-\alpha^4}{\alpha^3}\right)(\bar{R}_{ij}-2\bar{R}_{ijij}), \tag{4.14}$$

where $\bar{R}_{ii}=\bar{R}_{\mu\nu}n_i^\mu n_i^\nu$ and $\bar{R}_{ijij}=\bar{R}_{\mu\nu\lambda\rho}n_i^\mu n_i^\lambda n_j^\nu n_j^\rho$.

Now we are ready to calculate the divergences of the effective action $W_{\rm eff}=-\ln Z$. In four dimensions the infinite part of the effective action is

$$W_{\rm div}=-\frac{1}{32\pi^2}\left(\frac{1}{2}\frac{a_0}{\epsilon^4}+\frac{a_1}{\epsilon^2}+2a_2\ln\frac{L}{\epsilon}\right), \tag{4.15}$$

where $L$ is the infrared cutoff. Because of the same property (4.9) of the coefficients $a_n$, (4.12), the $W_{\rm div}$ is a sum of volume and surface parts (3.1). Combining volume part of the one-loop action (4.15) with the tree-level one (3.2) we can see that divergences (under $\epsilon\to 0$) are absorbed in the standard renormalization of the coupling constants [26]:

$$\frac{1}{G_{\rm ren}}=\frac{1}{G_B}+\frac{1}{2\pi\epsilon^2}\left(\frac{1}{6}-\xi\right),$$

$$c_{1,\rm ren}=c_{1,B}-\frac{1}{32\pi^2}\left(\frac{1}{6}-\xi\right)^2\ln\frac{L}{\epsilon},$$

$$c_{2,\rm ren}=c_{2,B}+\frac{1}{32\pi^2}\frac{1}{90}\ln\frac{L}{\epsilon},$$

$$c_{3,\rm ren}=c_{3,B}-\frac{1}{32\pi^2}\frac{1}{90}\ln\frac{L}{\epsilon}. \tag{4.16}$$

On the other hand, applying the formula

$$S_{\rm div}=(\alpha\partial_\alpha-1)W_{\rm div}|_{\alpha=1}\ ,$$

we obtain the divergence of the entropy:

$$S_{\rm div}=\frac{1}{8\pi\epsilon^2}\left(\frac{1}{6}-\xi\right)A_\Sigma+\left[\frac{1}{4\pi}\left(\frac{1}{6}-\xi\right)^2\int_\Sigma\bar{R}-\frac{1}{16\pi}\frac{1}{45}\int_\Sigma(\bar{R}_{ii}-2\bar{R}_{ijij})\right]\ln\frac{L}{\epsilon}. \tag{4.17}$$

We see that the complete entropy which is the sum of the tree level $S(G_B, c_{i,B})$, (3.4), and $S_{\rm div}(\epsilon)$, (4.17), becomes finite by the same renormalization of the constants (4.16) which renormalizes the effective action. So the identity (3.5) indeed holds.

For the minimal coupling ($\xi = 0$) the expression (4.17) has been obtained in [13]. In the conformal invariant case ($\xi = \frac{1}{6}$) the Newton constant $G$ and the coupling $c_1$ are not renormalized. Correspondingly, there are no area $A_\Sigma$ and $\int_\Sigma R$ contributions to the entropy (4.17) which is remarkably determined by only the conformally invariant expression $\int_\Sigma (R_{ii} - 2R_{ijij})$.

It should be noted that our proof of the main statement is based on the nice property of the heat kernel coefficients $a_n$. Namely, up to $(1-\alpha)^2$ terms the exact $a_n$ on the manifold $E_\alpha$ is equal to the standard volume coefficient $\bar{a}_n^{\rm st}$ considered on the manifold $E_\alpha$:

$$a_n(E_\alpha) = \int_{E_\alpha} \bar{a}_n^{\rm st}(x,x) + O((1-\alpha)^2) \qquad (4.18)$$

if one applies the formulas of Appendix A for curvatures on $E_\alpha$. Then up to $(1-\alpha)^2$ the renormalization of the entropy, (3.5), directly follows from the renormalization of the effective action, (3.6).

The curvature terms enter the matter action of the fields of different spins that gives rise to difficulties in operating with the entanglement entropy [27]. We believe that our result can be certainly generalized also for these cases.

## V. REMARKS

One can look at the entanglement entropy given by expression (2.6) from quite a different point of view. Consider the whole system (gravity plus matter) at arbitrary temperature $T = (2\pi\beta)^{-1}$. Define its partition function as given by the Euclidean functional integral over all fields (including the metric) which are periodical with respect to the imaginary time coordinate $\tau$ with period $2\pi\beta$. For a static gravitational field this means that the Euclidean manifold possesses Abelian isometry along the Killing vector $\partial_\tau$ with period $2\pi\beta$.

Standardly [28], the temperature is specified by giving the period in the time direction at infinity or at a box of finite radius, $T^{-1} = \int g_{\tau\tau}^{1/2} d\tau$. At infinity, for a static asymptotically flat metric ($g_{\tau\tau} \to 1$), which we only consider here, this definition of temperature is the same as above.

The assumption that the system includes a black hole means that there exists a surface $\Sigma$ (horizon) which is a fixed point of the isometry. Semiclassically, we take a metric satisfying these conditions and evaluate the quantum contribution of matter fields on this background. Then (2.5) and (2.7) are exactly such a partition function with the effective action $W(\beta, g_{\mu\nu}) = -\ln Z$ the functional of the temperature $\beta^{-1}$ and the metric $g_{\mu\nu}$. Taking its variation with respect to the periodicity $\beta$ (when $g_{\mu\nu}$ fixed) gives us the statistical (entanglement) entropy $S_{\rm ent} = (\beta\partial_\beta - 1)W(\beta, g_{\mu\nu})$ considered above. It should be noted that this is an off-shell approach [21] when the Euclidean time coordinate periodicity $\beta$ and metric $g_{\mu\nu}$ are nonrelated quantities. There exists a conical singularity on the horizon $\Sigma$ for a general metric $g_{\mu\nu}$ which contributes to the (off-shell) entropy $S_{\rm ent}$.

On the other hand, taking the temperature to be fixed we can find the corresponding equilibrium configuration which is the extremum of the effective action $W(\beta, g_{\mu\nu})$. The entanglement entropy then is worth comparing with the thermodynamical entropy[4] of a black hole. The latter is determined by the total response of the one-loop free energy $F$ ($\beta F = W$) of the system being in thermal equilibrium on variation of temperature. So we must compare the free energies of the two configurations being in equilibrium at slightly different temperatures. The equilibrium configuration corresponding to the fixed temperature $\beta$ is found from the extreme equation $\frac{\delta W(\beta, g_{\mu\nu})}{\delta g_{\mu\nu}}|_\beta = 0$. This extremum of the effective action is reached on regular manifolds without conical singularities and the equilibrium metric is a function of the temperature $\beta$ and parameters fixing the macrostate of the system (mass $M$, charge $Q$, etc.).[5] For an equilibrium state the parameter $\beta$ enters the free energy $\beta F = W(\beta, g_{\mu\nu}(\beta))$ both as an argument characterizing the Euclidean time coordinate periodicity [it is the same as in the off-shell expression $W(\beta, g_{\mu\nu})$] and through the equilibrium metric $g_{\mu\nu}(\beta)$. The equilibrium free energy $\beta F = W(\beta, g_{\mu\nu}(\beta))$ gives us the thermodynamical entropy $S^{\rm td} = (\beta d_\beta - 1)W(\beta, g_{\mu\nu}(\beta))$. Note that for equilibrium states the total derivative $d_\beta W = \partial_\beta W + \frac{\delta W(\beta, g_{\mu\nu})}{\delta g_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\delta \beta}$ coincides with the partial one defined as differentiating with respect to periodicity only under a metric fixed $g_{\mu\nu}$. Then we obtain that two entropies indeed coincide, $S^{\rm td} = S_{\rm ent}$.

However, in order to calculate $S^{\rm td}$ we must know exactly the form of the quantum-corrected configuration $g_{\mu\nu}(\beta)$, which is normally beyond our knowledge. On the other hand the calculation of $S_{\rm ent}$ does not require such information and we can obtain exactly the entropy (off shell) as a function of metric and its derivatives on the horizon $\Sigma$. It should be emphasized that there is no contribution to $W(\beta, g_{\mu\nu}(\beta))$ due to the conical singularity and we deal with the standard ultraviolet divergences coming from the bulk terms in the effective action. They result in the corresponding divergences of the entropy which are obviously regularized by the standard renormalization of the gravitational couplings. So in terms of the thermodynamical entropy our main statement holds automatically.

*Note added.* After this paper had been submitted for publication there appeared a paper by F. Larsen and F. Wilczek [Report No. hep-th/9506066 (unpublished)] devoted to a related subject. In particular, they prove the

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[4]I wish to thank V.P. Frolov for discussing this point.

[5]Really the minimization of the functional $W(\beta, g_{\mu\nu})$ under fixed $\beta$ includes also variations in the space of macroparameters. Therefore, $M$ is a function of $\beta$ and $Q$ in the equilibrium state.

formulated statement for the Newton constant and different types of matter: fermions and gauge fields. It is also noted that the entanglement entropy, defined as $S = -\mathrm{Tr}\rho \ln \rho$, is positive definite for any finite matrix $\rho$. However, the formal field theoretical expression obtained is not obviously a positive quantity. This can be also seen from our expression (4.17). In principle, it is unlikely that we should expect that conclusions made for usual finite systems are also valid for the field system with an infinite number of degrees of freedom. An indication of this is the ultraviolet divergence of observable quantities in the field theory which incorporates the renormalization procedure absent in usual quantum mechanics. Nevertheless, the positivity of the entropy seems to be an important requirement which probably bounds the renormalized values of the gravitational couplings $(G, c_1, c_2, c_3)$ (which are really ambiguous in the theory) and/or the multiplet of the matter fields contributing to the black hole entropy. This problem needs further investigation.

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## APPENDIX A: CURVATURE TENSORS ON $E_\alpha$ [22]

Consider space $E_\alpha$ which is an $\alpha$-fold covering of a smooth manifold $E$ along the Killing vector $\partial_\varphi$ generating Abelian isometry. Let the surface $\Sigma$ be a stationary point of this isometry and near $\Sigma$ space $E_\alpha$ looks as a direct product $\Sigma \times \mathcal{C}_\alpha$ of the surface $\Sigma$ and two-dimensional cone $\mathcal{C}_\alpha$ with angle deficit $\delta = 2\pi(1-\alpha)$. Outside the singular surface $\Sigma$ the space $E_\alpha$ has the same geometry as the smooth manifold $E$. In particular, their curvature tensors coincide. However, at the surface $\Sigma$ there exists a conical singularity which results in a singular ($\delta$-function-like) contribution to the curvatures. To extract this contribution exactly one can use some regularization procedure, replacing the singular space $E_\alpha$ by a sequence of regular manifolds $\tilde{E}_\alpha$. In the limit $\tilde{E}_\alpha \to E_\alpha$ we obtain the result [22]

$$\begin{aligned} R^{\mu\nu}{}_{\alpha\beta} &= \bar{R}^{\mu\nu}{}_{\alpha\beta} + 2\pi(1-\alpha)[(n^\mu n_\alpha)(n^\nu n_\beta) \\ &\quad -(n^\mu n_\beta)(n^\nu n_\alpha)]\delta_\Sigma, \\ R^\mu{}_\nu &= \bar{R}^\mu{}_\nu + 2\pi(1-\alpha)(n^\mu n_\nu)\delta_\Sigma, \\ R &= \bar{R} + 4\pi(1-\alpha)\delta_\Sigma, \end{aligned} \tag{A1}$$

where $\delta_\Sigma$ is the delta function: $\int_{\mathcal{M}} f\delta_\Sigma = \int_\Sigma f$; $n^k = n^k_\mu dx^\mu$ are two orthonormal vectors orthogonal to $\Sigma$, $(n_\mu n_\nu) = \sum_{k=1}^2 n^k_\mu n^k_\nu$ and the quantities $\bar{R}^{\mu\nu}{}_{\alpha\beta}$, $\bar{R}^\mu{}_\nu$, and $\bar{R}$ are computed in the regular points $E_\alpha/\Sigma$ by the standard method.

These formulas can be applied to define the integral expressions

$$\int_{E_\alpha} R = \alpha \int_E \bar{R} + 4\pi(1-\alpha)\int_\Sigma , \tag{A2}$$

$$\int_{E_\alpha} R^2 = \alpha \int_E \bar{R}^2 + 8\pi(1-\alpha)\int_\Sigma \bar{R} + O((1-\alpha)^2) , \tag{A3}$$

$$\int_{E_\alpha} R^{\mu\nu}R_{\mu\nu} = \alpha \int_E \bar{R}^{\mu\nu}\bar{R}_{\mu\nu} + 4\pi(1-\alpha)\int_\Sigma \bar{R}_{ii} + O((1-\alpha)^2) , \tag{A4}$$

$$\int_{E_\alpha} R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} = \alpha \int_E \bar{R}^{\mu\nu\lambda\rho}\bar{R}_{\mu\nu\lambda\rho} + 8\pi(1-\alpha) \times \int_\Sigma \bar{R}_{ijij} + O((1-\alpha)^2) , \tag{A5}$$

where $\bar{R}_{ii} = \bar{R}_{\mu\nu}n^\mu_i n^\nu_i$ and $\bar{R}_{ijij} = \bar{R}_{\mu\nu\lambda\rho}n^\mu_i n^\lambda_i n^\nu_j n^\rho_j$.

The first integrals in the right-hand part of (A2)–(A5) are defined on the smooth space $E$; they are proportional to $\alpha$. The terms $O((1-\alpha)^2)$ in (A3)–(A5) are really something like $\delta^2_\Sigma$ . They are ill defined and turn out to be dependent on the regularization prescription and singular in the limit $\tilde{E}_\beta \to \mathcal{M}_\beta$. But these terms are not important, for example, in the calculation of the entropy.

## APPENDIX B: THE HEAT KERNEL EXPANSION OF THE OPERATOR $(-\Box + \xi\bar{R})$ ON $E_\alpha$ [20]

Consider on space $E_\alpha$, possessing an Abelian isometry, the operator $-\Box_\xi = -\Box + \xi\bar{R}$ , where $\bar{R}$ is the regular part of the scalar curvature $R$ on $E_\alpha$ [see (A1)]. Then we have the heat kernel expansion

$$\ln\det(-\Box_\xi) = \int_{\epsilon^2}^\infty \frac{ds}{s}\mathrm{Tr}\bar{K}_{E_\alpha}(s),$$

$$\bar{K}_{E_\alpha}(s) = e^{-s\Box_\xi} = \frac{1}{(4\pi s)^{\frac{d}{2}}}\sum_n \bar{a}_n s^n, \quad s \to 0, \tag{B1}$$

where

$$\bar{a}_n(x,x) = \bar{a}^{\mathrm{st}}_n(x,x) + \bar{a}_{n,\alpha}(x,x)\delta_\Sigma \tag{B2}$$

is the sum of the standard coefficient $\bar{a}_n^{\rm st}(x,x)$ for a smooth manifold $E$ [26],

$$a_0^{\rm st}(x)=1\ ,\quad a_1^{\rm st}=(\tfrac{1}{6}-\xi)\bar{R},$$
$$a_2^{\rm st}(x)=\tfrac{1}{180}\bar{R}^2_{\mu\nu\alpha\beta}-\tfrac{1}{180}\bar{R}^2_{\mu\nu}-\tfrac{1}{6}(\tfrac{1}{5}-\xi)\Box\bar{R}$$
$$+\tfrac{1}{2}(\tfrac{1}{6}-\xi)^2\bar{R}^2, \qquad (B3)$$

and a part coming from the singular surface $\Sigma$ (stationary point of the isometry):

$$a_{0,\alpha}=0,\quad a_{1,\alpha}=\frac{\pi}{3}\frac{(1-\alpha)(1+\alpha)}{\alpha}\int_\Sigma\sqrt{\gamma}d^2\theta\ ,$$

$$a_{2,\alpha}=\frac{\pi}{3}\frac{(1-\alpha)(1+\alpha)}{\alpha}\int_\Sigma(\tfrac{1}{6}-\xi)\bar{R}\sqrt{\gamma}d^2\theta$$
$$-\frac{\pi}{180}\frac{(1-\alpha)(1+\alpha)(1+\alpha^2)}{\alpha^3}$$
$$\times\int_\Sigma(\bar{R}_{\mu\nu}n^\mu_i n^\nu_i-2\bar{R}_{\mu\nu\alpha\beta}n^\mu_i n^\alpha_i n^\nu_j n^\beta_j)\sqrt{\gamma}d^2\theta, \quad (B4)$$

where $n^i$ are two vectors orthogonal to the surface $\Sigma$ ($n^\mu_i n^\nu_j g_{\mu\nu}=\delta_{ij}$) and $\gamma$ is a metric on the surface $\Sigma$.

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