Black hole entropy and the Hamiltonian formulation of diffeomorphism invariant theories

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(Received 5 July 1995)

Path integral methods are used to derive a general expression for the entropy of a black hole in a diffeomorphism invariant theory. The result, which depends on the variational derivative of the Lagrangian with respect to the Riemann tensor, agrees with the result obtained from Noether charge methods by Iyer and Wald. The method used here is based on the direct expression of the density of states as a path integral (the microcanonical functional integral). The analysis makes crucial use of the Hamiltonian form of the action. An algorithm for placing the action of a diffeomorphism invariant theory in Hamiltonian form is presented. Other path integral approaches to the derivation of black hole entropy include the Hilbert action surface term method and the conical deficit angle method. The relationships between these path integral methods are presented.

PACS number(s): 04.70.Dy, 04.20.Fy, 04.60.Gw

I. INTRODUCTION

Noether charge methods have led Iyer and Wald [1,2] to the discovery of two elegant expressions for the entropy of a stationary black hole in a diffeomorphism invariant theory in D spacetime dimensions. The first expression is

$$\mathcal{S}_{\rm BH} = 2\pi \int_{\mathcal{H}} \boldsymbol{Q}[t] \ , \tag{1.1}$$

where $\mathcal{H}$ denotes the black hole bifurcation surface and $\boldsymbol{Q}[t]$ is the Noether charge $(D-2)$-form associated with the horizon Killing field t^a. The second expression is

$$\mathcal{S}_{\rm BH} = -2\pi \int_{\mathcal{H}} d^{D-2}x \sqrt{\sigma} \epsilon_{ab} \epsilon_{cd} U_0^{abcd} \ , \tag{1.2}$$

where σ is the determinant of the metric on $\mathcal{H}$, ϵ_{ab} is the binormal of $\mathcal{H}$, and U_0^{abcd} is the variational derivative [3] of the Lagrangian with respect to the Riemann tensor $\mathcal{R}_{abcd}$. Equation (1.2) is a generalization of the result obtained in Ref. [4] for black hole entropy in a theory described by a Lagrangian that depends on at most first derivatives of the Riemann tensor. The equivalence of expressions (1.1) and (1.2) is demonstrated in Ref. [2].

More recently, Iyer and Wald [5] and Nelson [6] compared the Noether charge approach with various path integral derivations of black hole entropy. The path integral methods all originate, ultimately, with the observation made by Gibbons and Hawking [7] that the partition function for the gravitational field can be expressed as a path integral.[1] These path integral methods were developed within the context of specific theories, such as Einstein gravity or Lovelock gravity. They include (i) the direct expression of $\exp(\mathcal{S}_{\rm BH})$ in terms of a path integral (the microcanonical functional integral) [9,10], (ii) the expression of $\mathcal{S}_{\rm BH}$ in terms of the Hilbert action surface term [11,12], and (iii) the derivation of $\mathcal{S}_{\rm BH}$ in terms of a nonclassical spacetime with a conical singularity [12,13]. Using the language and techniques of the Noether charge formalism, Iyer and Wald showed that the path integral methods (i) and (ii) yield the result (1.1), the black hole entropy expressed as the integral of the Noether charge $\boldsymbol{Q}[t]$, when applied to an arbitrary diffeomorphism invariant theory. Nelson has analyzed the relationship between the path integral method (iii) and the Noether charge result (1.1).

Section IV of this paper contains a direct derivation of the result (1.2), the black hole entropy expressed in terms of the variational derivative of the Lagrangian. This derivation is based on the microcanonical functional integral method (i) and bypasses the Noether charge for-

[1]The original calculation of Gibbons and Hawking was inconsistent. Their result for the partition function implies a negative value for the heat capacity. On the other hand, general arguments show that the heat capacity is necessarily positive for any system that can be characterized by a partition function. This problem was overcome by York [8] who showed that the partition function yields a positive value for the heat capacity if the boundary conditions in the path integral are imposed at a finite spatial location.

0556-2821/95/52(12)/7011(16)/$06.00

malism altogether. There may be some advantage to this. The Noether charge formalism is a useful tool for deriving the first law of black hole mechanics but, by itself, it does not provide a logically complete derivation of black hole entropy. In order to extract the black hole entropy from the first law of black hole mechanics one must, in principle, supplement the Noether charge analysis with the quantum field theory scattering calculation [14] that leads to the identification of surface gravity (divided by 2π) with black hole temperature. On the other hand, the path integral approach, although formal, does provide a logically complete framework in which black hole entropy can be derived and analyzed. For this reason, insights into the mysteries of black hole entropy, such as its statistical origin, should be obtained more easily from within the path integral formalism.

The microcanonical functional integral method (i) is reviewed in Sec. II. The derivation of Eq. (1.2) in Sec. IV makes crucial use of the Hamiltonian form of the action, which is derived in Sec. III. The three path integral methods mentioned above, having a common origin, are closely related to one another. The logical connections between the microcanonical functional integral method (i), the Hilbert action surface term method (ii), and the conical deficit angle method (iii) are discussed in Sec. V.

An important part of the present analysis is contained in Sec. III, where an algorithm is developed that allows one to place the action for a diffeomorphism invariant theory in Hamiltonian form. Specifically, it is shown that any action can be placed in "almost Hamiltonian" form, which differs from a true Hamiltonian form by the presence of extra undifferentiated variables (referred to as the χ's) in the Hamiltonian constraint $\mathcal{C}_\perp$. A true Hamiltonian form of the action is obtained when the χ's are eliminated through the solution of their algebraic equations of motion. If the rank of the matrix formed from the second derivatives of $\mathcal{C}_\perp$ with respect to the χ's is not maximal, then there are constraints on the canonical variables. These constraints must be added to the action via Lagrange multipliers. In practice, it might not be possible to solve analytically the algebraic equations of motion for the χ's, for example, if the equations of motion include high-order polynomials. (It might also occur that the solution of the equations of motion for the χ's is not unique. In that case, the action and the system it describes splits into separate self-consistent Hamiltonian theories.) In Appendix A the familiar Hamiltonian form of the action for Einstein gravity coupled to Maxwell electrodynamics is derived using the algorithm developed in Sec. III.

A fourth path integral method considered by Iyer and Wald is (iv) the calculation of $\exp(\mathcal{S}_{\rm BH})$ as the enhancement factor for the rate of black hole pair creation relative to the pair creation rate for matter distributions. They show [5] that method (iv) yields the entropy expression (1.1) when applied to an arbitrary diffeomorphism invariant theory. In Ref. [15], it was shown that the enhancement in the black hole creation rate, method (iv), must agree with the entropy as calculated from the microcanonical functional integral, method (i). Therefore, the results of Ref. [15] along with those obtained here constitute a derivation of Eq. (1.2) as the enhancement factor for black hole pair creation in a diffeomorphism invariant theory.

The analysis presented here applies to any stationary spacetime with bifurcate Killing horizon. This includes not only certain black hole spacetimes but also, for example, Rindler spacetime. With periodic identifications in the extra dimensions, the bifurcation surface $\mathcal{H}$ of the Rindler horizon has the topology of a $(D-2)$-torus. Equation (1.2) gives the associated entropy. For definiteness, I will typically use the terminology appropriate for black hole spacetimes. Some key results concerning the surface gravity of a bifurcate Killing horizon, which are used in the analysis of Sec. IV, are derived in Appendix B.

II. MICROCANONICAL FUNCTIONAL INTEGRAL

In the microcanonical functional integral formalism, the density of states ν is expressed directly as a path integral [9,10]:

$$\nu = \sum_{\mathcal{M}} \int Dg\, D\psi\, \exp\Big(\mathcal{S}[g,\psi]\Big)\ . \tag{2.1}$$

Here, $\mathcal{S}$ is the action, which is a functional of the metric g_{ab} and a collection of matter fields denoted by ψ. Also, $\sum_{\mathcal{M}}$ denotes a sum over manifolds $\mathcal{M}$ of different topologies, subject to the requirement that the boundary $\partial\mathcal{M}$ should have topology $\mathcal{B}\times S^1$. For the purpose of describing the thermodynamics associated with a horizon, it is most convenient to choose $\mathcal{B}$ to have the same topology as the bifurcation surface. Thus, for the case of black hole spacetimes, $\mathcal{B}$ is a $(D-2)$-sphere. For the case of Rindler spacetime $\mathcal{B}$ is a $(D-2)$-torus.

The boundary conditions on the metric and matter fields in the path integral for the density of states ν involve fixation of those quantities on $\partial\mathcal{M}$ that characterize the states of the system. In the terminology of traditional thermodynamics, these are the extensive variables including, for example, internal energy and electric charge. These quantities appear at the classical level as functions of the canonical variables q^α and p_α. Thus, consider the action $\mathcal{S}$ written in Hamiltonian form:

$$\mathcal{S}[\lambda,q,p] = i\int_{S^1} dt \int_\Sigma d^dx \Big(p_\alpha \dot{q}^\alpha - \lambda^A \mathcal{C}_A(q,p)\Big) + (\text{boundary terms})\ , \tag{2.2}$$

where $d = D-1$ is the dimensionality of space Σ. The Lagrange multipliers are denoted by λ^A and $\mathcal{C}_A(q,p)$ are the constraints. The particular form (2.2) for the action follows from spacetime diffeomorphism invariance and the assumption that under reparametrizations in t the canonical variables transform as scalars and the Lagrange multipliers transform as scalar densities [16].

There are two types of boundary terms that appear in the Hamiltonian form of the action for a diffeomorphism invariant theory on a manifold $\mathcal{M}$. The first is a term on the boundary $\partial\mathcal{M} = \mathcal{B} \times S^1$ of the spacetime manifold. By the argument given in Ref. [15], the action (2.2) appropriate for the density of states ν contains no such boundary terms at $\partial\mathcal{M}$. Otherwise, if boundary terms at $\partial\mathcal{M}$ were present, the boundary conditions would include fixation of quantities that depend on the Lagrange multipliers (and, hence, do not depend solely on the canonical variables q^α and p_α).

The second type of boundary term that appears in the Hamiltonian form of the action arises only if the boundary of space Σ includes an element $\mathcal{H}$ in addition to the generic leaf $\mathcal{B}$ of the foliation of $\partial\mathcal{M} = \mathcal{B} \times S^1$; that is, if $\partial\Sigma = \mathcal{H} \cup \mathcal{B}$. This situation occurs in particular when the spacetime manifold has topology $\mathcal{M} = \mathcal{B} \times \boldsymbol{R}^2$ and the leaves of the foliation terminate at a common surface $\mathcal{H}$, considered to be the "origin" of the $\boldsymbol{R}^2$ plane. Such boundary terms are derived as follows. Start with the action in Lagrangian form, expressed as an integral over $\mathcal{M}$ (plus possible boundary terms). Now excise a region from $\mathcal{M}$ surrounding $\mathcal{H}$, so that $\mathcal{M}$ has the product topology $\mathcal{B} \times (\text{annulus}) = \Sigma \times S^1$ (where $\Sigma = \mathcal{B} \times I$, with I a real line interval). The boundary $\partial\mathcal{M}$ then consists of two copies of $\mathcal{B} \times S^1$, where one copy coincides with the original boundary of $\mathcal{M}$ and the other copy coincides with the boundary of the excised region. The passage from the Lagrangian form of the action to the Hamiltonian form of the action proceeds as usual, with various boundary terms appearing at the boundary of the excised region. One then takes the limit in which the excised region shrinks to zero, being careful to ensure that the geometry is smooth at $\mathcal{H}$. The second type of boundary term is a term on the boundary of the excised region that survives this limit.

The entropy of a stationary black hole is computed as follows [9,10]. First, express the Lorentzian solution in stationary coordinates, $ds^2 = \tilde{g}_{ab}\, dx^a dx^b$, $\psi = \tilde{\psi}$, where $\tilde{g}_{ab}$ and $\tilde{\psi}$ are t independent. Next, choose boundary conditions for the path integral (2.1) that coincide with the boundary values (as constructed from the canonical data of a $t =$ const slice with boundary element $\mathcal{B}$) of the black hole spacetime. The path integral for ν will have an extremum in the topological sector $\mathcal{M} = \mathcal{B}\times\boldsymbol{R}^2$ that consists of the complex black hole solution $ds^2 = \bar{g}_{ab}\, dx^a dx^b$, $\psi = \bar{\psi}$. The complex black hole is obtained from the Lorentzian black hole by the substitution $t \to -it$. The $t =$ const slices of the Lorentzian and complex black hole solutions coincide in the sense that their canonical data agree [17,9,15]. In particular, the data on the boundary element $\mathcal{H}$ of the spatial slices of the complex black hole coincide with the data on the bifurcation surface of the Lorentzian black hole. In the zero-loop approximation the density of states is given by $\nu \approx \exp(\mathcal{S}[\bar{g},\bar{\psi}])$, where $\mathcal{S}[\bar{g},\bar{\psi}]$ is the action evaluated at the complex black hole solution. The entropy of the black hole is then

$$\mathcal{S}_{\rm BH} \approx \mathcal{S}[\bar{g},\bar{\psi}] \ , \tag{2.3}$$

the logarithm of the density of states.

When the action is expressed in Hamiltonian form, the evaluation of the entropy in Eq. (2.3) is simple. Because the complex spacetime $\bar{g}$, $\bar{\psi}$ is a stationary solution of the classical equations of motion, both the $p_\alpha \dot{q}^\alpha$ terms and the constraint terms in Eq. (2.2) vanish. The only contribution to the entropy comes from the boundary terms. As discussed above (see Ref. [15]), there are no boundary terms at $\partial\mathcal{M}$. There are, however, boundary terms at $\mathcal{H}$. The entropy arises entirely from the evaluation of these terms at the complex black hole solution.

III. ACTION FOR DIFFEOMORPHISM INVARIANT THEORIES

The action for an arbitrary diffeomorphism invariant theory of the metric g_{ab} and tensor matter fields ψ can be expressed in the manifestly covariant form [2]

$$\mathcal{S}[g,\psi] = i\int_{\mathcal{M}} d^D x \sqrt{-g}\,\mathcal{L} \ , \tag{3.1a}$$

$$\mathcal{L} = \mathcal{L}\Big(g_{ab}, \mathcal{R}_{bcde}, \nabla_{a_1}\mathcal{R}_{bcde}, \ldots, \nabla_{(a_1}\cdots\nabla_{a_m)}\mathcal{R}_{bcde}, \psi, \nabla_{a_1}\psi, \ldots, \nabla_{(a_1}\cdots\nabla_{a_\ell)}\psi\Big) \ , \tag{3.1b}$$

where ∇_a is the spacetime covariant derivative.

I will use the action (3.1) as the starting point. Note, however, that the derivation of the entropy (1.2) in this paper is not restricted just to the case of tensor matter fields. For example, ψ can include the components A^α_a of the Yang-Mills connection (α is the internal index), where the covariant derivative ∇_a of Eq. (3.1b) acts on A^α_a as a collection of covariant vectors. Likewise, ψ can include the tetrad field $(e^\mu)_a$. In this case, the Lagrangian should include a term $\Lambda^{ab}[g_{ab} - (e^\mu)_a\eta_{\mu\nu}(e^\nu)_b]$ that links the tetrad to the metric, where Λ^{ab} is an

[2] The work of Anderson and Torre [18] implies that the Lagrangian (3.1b) can be written in terms of covariant derivatives of the Riemann tensor in which the symmetrization over covariant derivatives is extended to the second and fourth slots of the Riemann tensor itself. The arguments of this section remain valid whether or not the Lagrangian is written in this way.

independently varied field (included among the ψ) and $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,\ldots,+1)$. With the tetrad appearing as a dynamical variable one can also include coupling to the Dirac field (see, for example, Ref. [19]).

The goal of this section is to place the action (3.1) in Hamiltonian form (2.2), and thereby derive the relevant boundary terms at $\mathcal{H}$. I will assume that the manifold topology is $\mathcal{M} = \Sigma \times I$. In Sec. IV, where the density of states (2.1) is evaluated, the factor I is periodically identified to form a circle S^1.

A. Elimination of derivatives of the Riemann tensor

The first step in the derivation of the Hamiltonian form of the action (3.1) is the elimination of derivatives of the Riemann tensor. The highest derivative, namely the mth derivative, can be eliminated as follows. Introduce a set of auxiliary fields $U_m^{a_1\cdots a_m bcde}$ and $V^m_{a_1\cdots a_m bcde}$, and write the action as

$$\mathcal{S}[g,\psi,U_m,V^m] = i\int_{\mathcal{M}} d^Dx\sqrt{-g}\Big\{\mathcal{L}\Big(g,\mathcal{R},\nabla_{a_1}\mathcal{R},\ldots,\nabla_{(a_1}\cdots\nabla_{a_{m-1})}\mathcal{R},V^m_{a_1\cdots a_m},\psi\text{'s}\Big) + U_m^{a_1\cdots a_m}\Big[\nabla_{(a_1}\cdots\nabla_{a_m)}\mathcal{R} - V^m_{a_1\cdots a_m}\Big]\Big\} . \tag{3.2}$$

Note that the indices on the Riemann tensor, and the corresponding indices on U_m and V^m, have been suppressed. Also, the notation ψ's is used for the matter fields and their derivatives. The actions (3.1) and (3.2) are equivalent. This is demonstrated by substituting the solution of the classical equations of motion for U_m and V^m, namely,

$$0 = \frac{1}{i\sqrt{-g}}\frac{\delta\mathcal{S}}{\delta U_m^{a_1\cdots a_m}} = \nabla_{(a_1}\cdots\nabla_{a_m)}\mathcal{R} - V^m_{a_1\cdots a_m} , \tag{3.3a}$$

$$0 = \frac{1}{i\sqrt{-g}}\frac{\delta\mathcal{S}}{\delta V^m_{a_1\cdots a_m}} = -U_m^{a_1\cdots a_m} + \frac{\partial\mathcal{L}}{\partial V^m_{a_1\cdots a_m}} , \tag{3.3b}$$

into the action (3.2). The result is the action (3.1). Now integrate by parts in Eq. (3.2) to remove one derivative from $\nabla_{(a_1}\cdots\nabla_{a_m)}\mathcal{R}$, and discard the boundary term. This leads to the action

$$\mathcal{S}[g,\psi,U_m,V^m] = i\int_{\mathcal{M}} d^Dx\sqrt{-g}\Big\{\mathcal{L}\Big(g,\mathcal{R},\nabla_{a_1}\mathcal{R},\ldots,\nabla_{(a_1}\cdots\nabla_{a_{m-1})}\mathcal{R},V^m_{a_1\cdots a_m},\psi\text{'s}\Big) - \Big[(\nabla_{a_m}U_m^{a_1\cdots a_m})\nabla_{(a_1}\cdots\nabla_{a_{m-1})}\mathcal{R} + U_m^{a_1\cdots a_m}V^m_{a_1\cdots a_m}\Big]\Big\} , \tag{3.4}$$

in which the highest derivative of the Riemann tensor is the $(m-1)$th derivative.

The action (3.4) yields the same equations of motion as the original action (3.1), but (3.4) is not entirely equivalent to (3.1) because boundary terms were discarded in its derivation. The change of boundary terms implies a change of boundary conditions for the variational problem. However, the boundary terms at $\partial\mathcal{M}$ that should be present in the final Hamiltonian form (2.2) of the action are known: there should be no boundary terms at $\partial\mathcal{M}$. Thus, we are free to discard any boundary terms that arise through integration by parts in spacetime. At the end of the analysis, any remaining boundary terms at $\partial\mathcal{M}$ must be eliminated from the Hamiltonian form of the action anyway.

The algorithm described above can be iterated until all derivatives of the Riemann tensor have been eliminated. In the process, sets of auxiliary variables U_{m-1}, V^{m-1}, $\ldots$, U_1, V^1 are introduced, which serve to eliminate the $(m-1)$th, $\ldots$, (1)th derivatives of $\mathcal{R}_{bcde}$. The resulting action is

$$\mathcal{S}[g,\psi,U_m,V^m,\ldots,U_1,V^1] = i\int_{\mathcal{M}} d^Dx\sqrt{-g}\Big\{\mathcal{L}(g,\mathcal{R},V^1,\ldots,V^m,\psi\text{'s}) - \Big[(\nabla_{a_1}U_1^{a_1})\mathcal{R} + U_1^{a_1}V^1_{a_1}\Big] - \Big[(\nabla_{a_2}U_2^{a_1a_2})V^1_{a_1} + U_2^{a_1a_2}V^2_{a_1a_2}\Big] - \cdots - \Big[(\nabla_{a_m}U_m^{a_1\cdots a_m})V^{m-1}_{a_1\cdots a_{m-1}} + U_m^{a_1\cdots a_m}V^m_{a_1\cdots a_m}\Big]\Big\} . \tag{3.5}$$

Now isolate the Riemann tensor by introducing one more set of auxiliary fields, U_0 and V^0. (This is one more iteration of the algorithm, but without the integration by parts.) The action becomes

$$\mathcal{S}[g,\psi,U_m,V^m,\ldots,U_0,V^0] = i\int_{\mathcal{M}} d^Dx\sqrt{-g}\Big\{\mathcal{L}(g,V^0,\ldots,V^m,\psi\text{'s}) + \Big[U_0\mathcal{R} - U_0V^0\Big] - \Big[(\nabla_{a_1}U_1^{a_1})V^0 + U_1^{a_1}V^1_{a_1}\Big] - \cdots - \Big[(\nabla_{a_m}U_m^{a_1\cdots a_m})V^{m-1}_{a_1\cdots a_{m-1}} + U_m^{a_1\cdots a_m}V^m_{a_1\cdots a_m}\Big]\Big\} . \quad (3.6)$$

Observe that when the V equations of motion hold,

$$0 = -U_0 - \nabla_{a_1}U_1^{a_1} + \frac{\partial\mathcal{L}}{\partial V^0} ,$$

$$\vdots$$

$$0 = -U_{m-1}^{a_1\cdots a_{m-1}} - \nabla_{a_m}U_m^{a_1\cdots a_m} + \frac{\partial\mathcal{L}}{\partial V^{m-1}_{a_1\cdots a_{m-1}}} ,$$

$$0 = -U_m^{a_1\cdots a_m} + \frac{\partial\mathcal{L}}{\partial V^m_{a_1\cdots a_m}} ,$$

and the U equations of motion hold,

$$0 = \mathcal{R} - V^0 ,$$

$$0 = \nabla_{a_1}V^0 - V^1_{a_1} ,$$

$$\vdots$$

$$0 = \nabla_{(a_m}V^{m-1}_{a_1\cdots a_{m-1})} - V^m_{a_1\cdots a_m} ,$$

the variable U_0 equals

$$U_0^{bcde} = \frac{\partial\mathcal{L}}{\partial(\mathcal{R}_{bcde})} - \nabla_{a_1}\left(\frac{\partial\mathcal{L}}{\partial(\nabla_{a_1}\mathcal{R}_{bcde})}\right) + \cdots + (-1)^m\nabla_{a_1}\cdots\nabla_{a_m}\left(\frac{\partial\mathcal{L}}{\partial(\nabla_{(a_1}\cdots\nabla_{a_m)}\mathcal{R}_{bcde})}\right) . \quad (3.7)$$

This is the variational derivative [3] of the Lagrangian $\mathcal{L}$ with respect to the Riemann tensor $\mathcal{R}_{bcde}$.[2]

B. Elimination of higher-order derivatives of the matter fields

The second step in the derivation of the Hamiltonian form of the action is the elimination of all but the first covariant derivatives of the matter fields. This can be achieved by applying the same algorithm that was used in the elimination of derivatives of the Riemann tensor. The resulting action depends on the first covariant derivative of the following fields: $U_1,\ldots,U_m$, some (or all) of the matter fields ψ, and some of the auxiliary fields that were introduced in the elimination of the higher-order derivatives of ψ. I will denote these fields collectively by Ψ'. The covariant derivatives $\nabla_a\Psi'$ can be isolated as linear terms in the Lagrangian through the introduction of yet another set of auxiliary fields, just as the Riemann tensor was isolated by the introduction of the fields U_0, V^0. The action now takes the form

$$\mathcal{S}[g,\Psi] = i\int_{\mathcal{M}} d^Dx\sqrt{-g}\Big\{U_0^{abcd}\mathcal{R}_{abcd} + f(g,\Psi,\nabla_a\Psi')\Big\} , \quad (3.8)$$

where $\nabla_a\Psi'$ appear linearly in f with coefficients that are independent variables. In Eq. (3.8), Ψ denotes the original matter fields ψ, the auxiliary fields U and V, and the auxiliary fields that were introduced in the elimination of the higher-order derivatives of ψ and the isolation of the first derivatives $\nabla_a\Psi'$. Thus, Ψ' is the subset of fields Ψ that appear differentiated in the action. The presentation below is simplified if we assume that the fields Ψ' are covariant in their tensor indices. (As discussed at the beginning of this section, some matter fields might also carry internal indices, such as a Yang-Mills index or a tetrad index.) This assumption entails no loss of generality—for each field with a contravariant tensor index that appears differentiated in the action, say ψ^a, a simple change of variables $\psi_a = g_{ab}\psi^b$, $g_{ab} = g_{ab}$ allows us to replace ψ^a with ψ_a as the fundamental variable.

C. Spacetime decomposition

The third step in the derivation of the Hamiltonian form of the action is the introduction of a spacetime

[3]It is possible to carry out the above analysis without the fields V. For example, in order to eliminate the mth derivative of the Riemann tensor from the action (3.1), one can perform a Legendre transformation in which $\nabla_{(a_m}(\nabla_{a_1}\cdots\nabla_{a_{m-1})}\mathcal{R})$ play the role of velocities and $U_m^{a_1\cdots a_m}$ play the role of momenta, and follow this with an integration by parts. [Equivalently, V^m can be eliminated from the action (3.4) by substitution of the solution of the V^m equation of motion.] One must allow for the possibility that the relationship between the velocities and momenta is not invertible, signaling the presence of constraints. The key equation (3.7) and the form (3.8) of the action (see below) can be deduced in this way, in spite of the fact that the constraints are not explicitly known.

split. Let spacetime have topology $\mathcal{M} = \Sigma \times I$ and let t label the hypersurfaces of the foliation Σ. The unit normal of the hypersurfaces is $u_a = -N\nabla_a t$, where $N = [-(\nabla_a t)g^{ab}(\nabla_b t)]^{-1/2}$ defines the lapse function. The hypersurface metric is defined by $h_{ab} = g_{ab} + u_a u_b$, so the spacetime metric g_{ab} becomes

$$g_{ab} = h_{ab} - u_a u_b \ . \tag{3.9}$$

The Gauss, Codazzi, and Ricci equations imply (see, for example, Ref. [20])

$$\begin{aligned}\mathcal{R}_{abcd} = R_{abcd} &+ 2K_{a[c}K_{d]b} + 4(D_{[a}K_{b][c})u_{d]} \\ &+4(D_{[c}K_{d][a})u_{b]} - 4u_{[a}\Big(\pounds_u K_{b][c} + K^e_{b]}K_{e[c} \\ &+(D_{b]}D_{[c}N)/N\Big)u_{d]} \ ,\end{aligned} \tag{3.10}$$

where $\pounds_u$ is the Lie derivative along u^a and R_{abcd}, $K_{ab} = -\frac{1}{2}\pounds_u h_{ab}$, and D_a denote the Riemann tensor, extrinsic curvature, and covariant derivative of the $t =$ const hypersurfaces, respectively.

Under the spacetime decomposition the tensor fields Ψ are projected normally and tangentially to the $t =$ const hypersurfaces. For example, ψ_a is split into its projections $u^a\psi_a$ and $h^b_a\psi_b$ and ψ^a is split into $u_a\psi^a$ and $h^a_b\psi^b$. The first derivative $\nabla_a\Psi'$ of a covariant tensor Ψ' is split into hypersurface covariant derivatives and normal Lie derivatives of the projections of Ψ'. For example, for a scalar field ψ we have $\nabla_a\psi = -u_a\pounds_u\psi + D_a\psi$ and for a covariant vector field ψ_a we have

$$\begin{aligned}\nabla_a\psi_b = u_a u_b\pounds_u(u^c\psi_c) &- u_a\pounds_u(h^c_b\psi_c) + D_a(h^c_b\psi_c) \\ &-u_b D_a(u^c\psi_c) + K_{ab}u^c\psi_c - 2u_{(a}K^c_{b)}\psi_c \\ &-u_a u_b\psi_c(D^c N)/N + u_a u^c\psi_c(D_b N)/N \ .\end{aligned} \tag{3.11}$$

Note that the Lie derivatives $\pounds_u$ and the spatial covariant derivative of the lapse function, in the combination $(D_a N)/N$, appear linearly in $\nabla_a\psi_b$. Also note that $\pounds_u(h^c_b\psi_c)$ is a spatial tensor; that is, $u^b\pounds_u(h^c_b\psi_c) = 0$. The decomposition for the derivatives of higher rank covariant tensors is similar to that in Eq. (3.11). I will use $\mathcal{P}\Psi$ as a shorthand notation for the normal and tangential projections of the fields Ψ.

With the spacetime split described above, the action (3.8) becomes

$$\begin{aligned}\mathcal{S} = i\int_{\mathcal{M}} d^D x\sqrt{-g}\Big\{ &U_0^{abcd}\Big[R_{abcd} + 2K_{ac}K_{bd}\Big] + 8U_0^{abcd}u_d(D_a K_{bc}) - 4U_0^{abcd}u_a u_d\Big[\pounds_u K_{bc} + K^e_b K_{ec} + (D_b D_c N)/N\Big] \\ &+f(h_{ab}, h^{ab}, \mathcal{P}\Psi, \pounds_u(\mathcal{P}\Psi'), D_a(\mathcal{P}\Psi'), K_{ab}, (D_a N)/N)\Big\} \ ,\end{aligned} \tag{3.12}$$

where $\pounds_u(\mathcal{P}\Psi')$ and $(D_a N)/N$ appear linearly in f. The action (3.12) contains a second time derivative in the term $\pounds_u K_{bc} = -\frac{1}{2}\pounds_u\pounds_u h_{ab}$. This can be removed by promoting the extrinsic curvature K_{bc} to an independent variable. Thus, introduce an auxiliary variable P^{ab} and write the action as

$$\begin{aligned}\mathcal{S} = i\int_{\mathcal{M}} d^D x\Big\{ &NP^{ab}\Big[\pounds_u h_{ab} + 2K_{ab}\Big] + \sqrt{-g}\,U_0^{abcd}\Big[R_{abcd} + 2K_{ac}K_{bd}\Big] + 8\sqrt{-g}\,U_0^{abcd}u_d(D_a K_{bc}) \\ &-4\sqrt{-g}\,U_0^{abcd}u_a u_d\Big[\pounds_u K_{bc} + K^e_b K_{ec} + (D_b D_c N)/N\Big] \\ &+\sqrt{-g}\,f(h_{ab}, h^{ab}, \mathcal{P}\Psi, \pounds_u(\mathcal{P}\Psi'), D_a(\mathcal{P}\Psi'), K_{ab}, (D_a N)/N)\Big\} \ .\end{aligned} \tag{3.13}$$

The action (3.12) is recovered when the solution of the P^{ab}, K_{ab} equations of motion is substituted into the action (3.13).

Now choose a time flow vector field t^a such that $t^a\nabla_a t = 1$, and define the shift vector by $V^a = h^a_b t^b$ (not to be confused with the auxiliary fields $V^0, \ldots, V^m$). The Lie derivative $\pounds_u$, acting on h_{ab}, K_{ab}, and the covariant tensors $\mathcal{P}\Psi'$, is expressed as $N\pounds_u = \pounds_t - \pounds_V$. With the fields mapped from $\mathcal{M}$ to $\Sigma \times I$, the Lie derivatives $\pounds_t$ along t^a become ordinary time derivatives (denoted by a dot) and $\sqrt{-g} = N\sqrt{h}$ where h is the determinant of the metric h_{ij} on Σ. The action (3.13) becomes

$$\begin{aligned}\mathcal{S}[N, V, h, K, P, \mathcal{P}\Psi] = i\int dt\int_\Sigma d^d x\Big\{ &P^{ij}\Big[\dot{h}_{ij} - D_{(i}V_{j)} + 2NK_{ij}\Big] + N\sqrt{h}\,U_0^{ijk\ell}\Big[R_{ijk\ell} + 2K_{ik}K_{j\ell}\Big] \\ &-8N\sqrt{h}\,U_0^{ijk\perp}(D_i K_{jk}) - 4\sqrt{h}\,U_0^{\perp ij\perp}\Big[\dot{K}_{ij} - \pounds_V K_{ij} + NK^k_i K_{kj} + (D_i D_j N)\Big] \\ &+N\sqrt{h}\,f(h_{ij}, h^{ij}, \mathcal{P}\Psi, \pounds_u(\mathcal{P}\Psi'), D_i(\mathcal{P}\Psi'), K_{ij}, (D_i N)/N)\Big\} \ ,\end{aligned} \tag{3.14}$$

where $N\pounds_u(\mathcal{P}\Psi') = (\mathcal{P}\Psi')^\cdot - \pounds_V(\mathcal{P}\Psi')$ and $i, j, \ldots$ are indices for tensors on Σ. Also, the notation $U_0^{abc\perp} = -U_0^{abcd}u_d$ and $U_0^{\perp bc\perp} = U_0^{abcd}u_a u_d$ has been used.

D. Hamiltonian form of the action

The action (3.14) is linear in time derivatives and Lagrange multipliers. That is, each term in the Lagrangian of Eq. (3.14) depends linearly on either the time derivative of a field, or the lapse function N, or the shift vector V^i. [Recall that $\pounds_u(\mathcal{P}\Psi')$ and $(D_aN)/N$ appear linearly in f.] Therefore each term in the Lagrangian transforms as a scalar density under reparametrizations in t, and no terms such as $V^i h_{ij} V^j/N$ appear. Moreover, the coefficients of $\dot{h}_{ij}$, $\dot{K}_{ij}$, and $(\mathcal{P}\Psi')^\cdot$ are independent variables. Specifically, the coefficient of $\dot{h}_{ij}$ is P^{ij}, the coefficient of $\dot{K}_{ij}$ is $-4\sqrt{h}U_0^{\perp ij\perp}$, and the coefficient of $(\mathcal{P}\Psi')^\cdot$ is given in terms of the auxiliary fields that were introduced in the process of isolating $\nabla_a\Psi'$ as a linear factor in the action. It follows that these coefficients (denoted p_α) are the canonical momenta conjugate to the coordinates h_{ij}, K_{ij}, and $\mathcal{P}\Psi'$ (denoted q^α). After all, consider what happens if one tries to identify both q^α and p_α as coordinates, and to define conjugate momenta Π^q_α and Π^α_p. The definition of these momentum variables leads to sets of second class constraints, namely, $\Pi^q_\alpha = p_\alpha$ and $\Pi^\alpha_p = 0$. Elimination of these constraints through the Dirac brackets effectively eliminates the new momenta, and reveals the interpretation of q^α and p_α as canonically conjugate variables.

Now remove the spatial derivatives from the lapse function N and the shift vector V^i through integrations by parts. The action (3.14) takes the "almost Hamiltonian" form

$$\mathcal{S}[N,V,q,p,\chi] = i\int dt\int_\Sigma d^dx\Big[p_\alpha\dot{q}^\alpha - N\mathcal{C}_\perp(q,p,\chi) - V^i\mathcal{C}_i(q,p)\Big] \\ +i\int dt\int_{\partial\Sigma} d^{d-1}x\sqrt{\sigma}\Big[-4n_iU_0^{\perp ij\perp}D_jN + (\text{terms} \sim N \text{ and } V^i)\Big] , \qquad (3.15)$$

where n^i is the outward pointing unit normal of the boundary of space, $\partial\Sigma$, and σ is the determinant of the metric on $\partial\Sigma$. The "terms $\sim N$ and V^i" that appear at the boundary $\partial\Sigma$ are proportional to the undifferentiated lapse function N and undifferentiated shift vector V^i. Observe that the action for any diffeomorphism invariant theory can be put into the form (3.15), since no special assumptions were made in its derivation.

The action (3.15) is not quite in Hamiltonian form because it contains certain extra undifferentiated variables χ in addition to the canonical variables q^α and p_α and Lagrange multipliers N and V^i. These variables include, for example, the normal and tangential projections of the auxiliary fields $V^0,\ldots,V^m$. Note that the χ's appear only in the Hamiltonian constraint $\mathcal{C}_\perp$. They do not appear in the momentum constraint $\mathcal{C}_i$, or in the boundary terms proportional to V^i, because $\mathcal{C}_i$ and the boundary terms proportional to V^i all originate from the Lie derivatives $\pounds_u$ in the combination $p_\alpha(\dot{q}^\alpha - \pounds_V q^\alpha)$. The χ's also do not appear in the boundary terms proportional to N. To see this, one should recall that these boundary terms arise through integration by parts that eliminate spatial derivatives of N. Inspection of Eq. (3.10) shows that the spatial derivatives of N appear in the combination $N\pounds_u K_{bc} + D_bD_cN$. Thus, the coefficient of D_iD_jN in the action equals the momentum conjugate to K_{ij}, and involves no χ's. Likewise, inspection of Eq. (3.11) shows that the spatial derivatives of N appear in combination with Lie derivatives in such a way that the coefficients of D_iN involve the momenta conjugate to $\mathcal{P}\Psi'$ and $\mathcal{P}\Psi'$ itself, but no χ's.

A true Hamiltonian form of the action is obtained from Eq. (3.15) by elimination of the variables χ through the solution of their algebraic equations of motion:

$$\frac{i}{N}\frac{\delta\mathcal{S}}{\delta\chi} = \frac{\partial\mathcal{C}_\perp}{\partial\chi} = 0 . \qquad (3.16)$$

That is, one solves the set of equations (3.16) and inserts the solution back into the action. If the matrix of second derivatives of $\mathcal{C}_\perp$ with respect to the χ's has vanishing determinant, then Eqs. (3.16) are not independent and cannot be solved for all of the χ's as functions of the canonical variables q^α and p_α. In that case Eqs. (3.16) include constraints on the canonical variables. In any case, when the conditions (3.16) are imposed the Hamiltonian constraint $\mathcal{C}_\perp$ is independent of χ.

The constraints that arise through the elimination of the χ's must be incorporated into the action principle via Lagrange multipliers. Let $\mathcal{C}_A$ denote the complete set of constraints for the system—the constraints that arise through Eqs. (3.16) as well as the Hamiltonian constraint $\mathcal{C}_\perp$ and momentum constraints $\mathcal{C}_i$. Likewise, let λ^A denote the complete set of Lagrange multipliers associated with the constraints $\mathcal{C}_A$, including the lapse function N and shift vector V^i. The action, which is now in Hamiltonian form, reads

$$\mathcal{S}[\lambda,q,p] = i\int dt \int_\Sigma d^dx\Big(p_\alpha \dot{q}^\alpha - \lambda^A \mathcal{C}_A(q,p)\Big) + i\int dt \int_{\partial\Sigma} d^{d-1}x\sqrt{\sigma}\Big[-4n_i U_0^{\perp ij\perp} D_j N + (\text{terms} \sim N \text{ and } V^i)\Big] \, . \tag{3.17}$$

This is Eq. (2.2) with the boundary terms displayed somewhat more explicitly.

The details of the elimination of the variables χ must be carried out on a case-by-case basis. The instructive example of Einstein gravity coupled to Maxwell electrodynamics is presented in Appendix A.

E. Comments

Several comments are in order. First, consider the situation in which one of the constraints, say, $\mathcal{C}_1$, is simply one of the momentum variables, say, p_1. Then the equation of motion for the Lagrange multiplier λ_1 is $p_1 = 0$, and the equation of motion for p_1 yields an expression for λ_1 in terms of the other Lagrange multipliers, q^α, and p_α. These equations can be used to eliminate p_1 and λ^1 from the variational principle—effectively one just sets p_1 equal to zero. In this way the pair q^1, p_1 is removed from the list of dynamical variables in the theory, although in general the action still depends on q^1 (undifferentiated in time). The resulting situation is similar to that encountered in the "almost Hamiltonian" form (3.15) of the action, in that the action depends on an extra variable. (The key difference is that the constraints might depend on spatial derivatives of q^1, and also that q^1 might appear in the boundary terms. The variables χ, on the other hand, appeared only undifferentiated in the Hamiltonian constraint.) Now one can attempt to eliminate q^1 through the solution of its equation of motion. If the q^1 equation of motion can be solved for q^1, then insertion of this solution into the action yields a new Hamiltonian form of the action for the system in which q^1 and p_1 are completely excluded. If the q^1 equation of motion depends only on the canonical variables (other than q^1 and p_1), then q^1 is a Lagrange multiplier and should be left alone. It might happen that the q^1 equation of motion cannot be solved for q^1 and also does not yield a constraint. In this case one can always stick to the Hamiltonian form of the action that includes q^1, p_1 and the constraint $\mathcal{C}_1 = p_1$.

Although the situation in which one of the constraints is equal to a momentum variable might appear to be of academic interest only, it in fact occurs in the examples of Maxwell electrodynamics and Einstein gravity. In electrodynamics the variable that plays the role of q^1 is the normal projection of the electromagnetic potential, which becomes a Lagrange multiplier for the Gauss's law constraint. In Einstein gravity one must first perform a canonical transformation on the variables q^α, p_α to bring the action to a form in which one of the momentum variables is constrained to vanish. The variables (q^1 and p_1) that are eliminated in this way are the extrinsic curvature K_{ij} and its conjugate. Details can be found in Appendix A.

As a final comment, observe that the Hamiltonian $H = \int_\Sigma d^dx\lambda^A\mathcal{C}_A$ + (boundary terms) obtained from the action functional (3.17) is not necessarily either the total Hamiltonian or the extended Hamiltonian [16]. If the extended Hamiltonian of the system is desired, one can start with the Hamiltonian H and treat the constraints $\mathcal{C}_A = 0$ as primary constraints. The preservation in time of the primary constraints can lead to secondary constraints. One then proceeds to the classification of constraints as first or second class, and to the construction of the extended Hamiltonian [16]. For the purpose of this paper, it is not necessary that the extended Hamiltonian appear in the Hamiltonian form (3.17) of the action. What really matters is that the variational principles based on the action functionals (3.1) and (3.17) are equivalent.

IV. BLACK HOLE ENTROPY

We are now in a position to compute the entropy of a stationary spacetime with bifurcate Killing horizon using the microcanonical functional integral method. It is assumed that the spacetime metric and matter fields $\{\tilde{g}_{ab}, \tilde{\psi}\}$ satisfy the classical equations of motion that follow from the action (3.1). Let t^a denote the Killing vector field that vanishes on the bifurcation surface $\mathcal{H}$. Consider a spacelike hypersurface Σ_0 whose boundaries consist of an outer boundary $\mathcal{B}$ and the bifurcation surface $\mathcal{H}$. Note that Σ_0 lies within a single "wedge" of the spacetime where t^a is timelike. Now extend Σ_0 into a foliation of the wedge by stationary hypersurfaces t = const, where $t^a\nabla_a t = 1$. Also choose t^a as the time flow vector field.[4] Then the solution $\{\tilde{g}_{ab}, \tilde{\psi}\}$ can be written in Hamiltonian

[4]One can choose a time flow vector field $t^a - \Omega\phi^a$ where Ω is constant and ϕ^a is a spatial Killing vector field, if such a Killing vector field exists. This changes the shift vector by $-\Omega\phi^a$, so the argument given in Appendix B that V^a vanishes at $\mathcal{H}$ no longer holds. However, the overall results of the analysis are unchanged because in the action (4.1) below the extra nonvanishing boundary terms at $\mathcal{H}$ just cancel corresponding boundary terms at $\mathcal{B}$. This can be seen by reversing the steps that generate the boundary terms proportional to V^a: Sum the identity $p_\alpha \pounds_V q^\alpha = 0$ over canonical pairs, integrate over Σ, then integrate by parts and use the momentum constraint. Experience with black hole thermodynamics in the context of Einstein gravity [17,21] shows that the shift vector defined as the spatial projection of t^a, not $t^a - \Omega\phi^a$, is the physically correct definition for the product of inverse temperature and chemical potential.

form $\{\tilde{\lambda}, \tilde{q}, \tilde{p}\}$, where $\tilde{\lambda}$, $\tilde{q}$, and $\tilde{p}$ are t independent.

As discussed in Sec. II, the entropy is obtained by evaluating the action (2.2) at the complex solution $\{\bar{\lambda}, \bar{q}, \bar{p}\}$ with periodic identification in t. The complex solution $\{\bar{\lambda}, \bar{q}, \bar{p}\}$ is obtained from the real Lorentzian solution $\{\tilde{\lambda}, \tilde{q}, \tilde{p}\}$ by the substitution $t \to -it$. Under reparametrizations in t the canonical variables transform as scalars and the Lagrange multipliers transform as scalar densities, so it follows [15] that $\bar{\lambda} = -i\tilde{\lambda}$, $\bar{q} = \tilde{q}$, and $\bar{p} = \tilde{p}$. The orbits of the Killing vector field t^a in the complex spacetime form closed curves (circles) around the bifurcation surface $\mathcal{H}$.

According to the discussion of Sec. II, the action (2.2) contains no boundary terms at the spacetime boundary $\partial\mathcal{M} = \mathcal{B} \times S^1$ where the boundary data appropriate for the microcanonical functional integral are fixed [15]. The boundary terms at $\mathcal{H}$ are just the boundary terms displayed in Eq. (3.17). Thus, the action takes the form

$$\mathcal{S}[\lambda, q, p] = i \int_{S^1} dt \int_\Sigma d^dx \Big(p_\alpha \dot{q}^\alpha - \lambda^A \mathcal{C}_A(q,p)\Big) + i \int_{S^1} dt \int_{\mathcal{H}} d^{d-1}x \sqrt{\sigma} \Big[-4 n_i U_0^{\perp ij \perp} D_j N + (\text{terms} \sim N \text{ and } V^i)\Big] , \tag{4.1}$$

where $\partial\Sigma = \mathcal{H} \cup \mathcal{B}$. The boundary terms at $\mathcal{H}$ should be understood in terms of a limiting procedure, as discussed in Sec. II. That is, the boundary terms are defined by an integral over the boundary of an excised region that surrounds the bifurcation surface $\mathcal{H}$, and the limit is taken as the excised region shrinks to $\mathcal{H}$.

In Appendix B it is shown that the lapse function $\tilde{N}$ and shift vector $\tilde{V}^a$, and hence also $\bar{N}$ and $\bar{V}^a$, vanish in the limit as the bifurcation surface is approached. Thus, all boundary "terms $\sim N$ and V^i" in the action (4.1) vanish upon evaluation at the complex solution. This assumes that the coefficients of the "terms $\sim N$ and V^i", which depend solely on the canonical variables, are well behaved at the bifurcation surface $\mathcal{H}$. Now, because $\{\bar{\lambda}, \bar{q}, \bar{p}\}$ is a stationary solution of the classical equations of motion, the $p_\alpha \dot{q}^\alpha$ terms and the constraint terms vanish in the evaluation of the action. The result is that the entropy (2.3) becomes

$$\mathcal{S}_{\rm BH} \approx \mathcal{S}[\bar{\lambda}, \bar{q}, \bar{p}] = -4i \int_{S^1} dt \int_{\mathcal{H}} d^{d-1}x \sqrt{\sigma} n_i U_0^{\perp ij \perp} D_j \bar{N} . \tag{4.2}$$

The right-hand side of this expression is evaluated at the complex solution $\{\bar{\lambda}, \bar{q}, \bar{p}\}$. However, for notational simplicity, the bars have been omitted from the p's and q's in Eq. (4.2). No ambiguity arises since the canonical variables for the Lorentzian and complex solutions agree. The bar is retained on the lapse function since the Lagrange multipliers for the Lorentzian and complex solutions differ by a factor of $-i$. In the analysis below I will continue the practice of placing bars or tildes only over the Lagrange multipliers.

As shown in Appendix B, the gradient of the lapse function is related to the surface gravity $\tilde{\kappa}$ by $\lim D_i \tilde{N} = -\lim \tilde{\kappa} n_i$, where the limit is taken in which the bifurcation surface $\mathcal{H}$ is approached along a $t = \text{const}$ hypersurface Σ. (Note that n^i is the outward pointing normal of $\partial\Sigma$ at the bifurcation surface, so n^i points "radially inward" towards $\mathcal{H}$.) Thus, we obtain

$$\mathcal{S}_{\rm BH} \approx 4 \int_{S^1} dt \int_{\mathcal{H}} d^{d-1}x \sqrt{\sigma} n_i U_0^{\perp ij \perp} n_j \tilde{\kappa} . \tag{4.3}$$

Since the surface gravity of a spacetime with bifurcate Killing horizon is constant over the horizon [22], $\tilde{\kappa}$ can be removed from the integral over $\mathcal{H}$. Now, the proper circumference of the circular orbits of t^a is $\int_{S^1} dt \sqrt{-\bar{N}^2 + \bar{V}^i \bar{V}_i}$.[5] From Appendix B we have the result $\lim \tilde{V}^i / \tilde{N} = 0$, so the proper circumference, in the limit as the bifurcation surface is approached, equals $\int_{S^1} dt \tilde{N}$. The expression $\tilde{\kappa} = -\lim n^j D_j \tilde{N}$ then shows that $\int_{S^1} dt\, \tilde{\kappa}$ equals the rate of change of circumference with respect to radius for these orbits. The complex geometry will be smooth at $\mathcal{H}$, and satisfy the classical equations of motion there, only if the period in S^1 is chosen such that $\int_{S^1} dt\, \tilde{\kappa} = 2\pi$. The entropy is then

$$\mathcal{S}_{\rm BH} \approx 8\pi \int_{\mathcal{H}} d^{d-1}x \sqrt{\sigma} n_i U_0^{\perp ij \perp} n_j . \tag{4.4}$$

The right-hand side of this expression for $\mathcal{S}_{\rm BH}$ depends only on the canonical variables, so it can be evaluated either at the complex solution $\{\bar{\lambda}, \bar{q}, \bar{p}\}$ or at the Lorentzian solution $\{\tilde{\lambda}, \tilde{q}, \tilde{p}\}$.

Recalling the definition $U_0^{\perp bc \perp} = \tilde{U}_0^{abcd} \tilde{u}_a \tilde{u}_d$ and using the expression $\tilde{\epsilon}_{ab} = 2 \lim \tilde{u}_{[a} n_{b]}$ for the binormal of $\mathcal{H}$, we have

$$\mathcal{S}_{\rm BH} \approx -2\pi \int_{\mathcal{H}} d^{d-1}x \sqrt{\sigma} \tilde{\epsilon}_{ab} \tilde{\epsilon}_{cd} \tilde{U}_0^{abcd} . \tag{4.5}$$

This is the main result, Eq. (1.2), for the entropy of a spacetime with bifurcate Killing horizon. Here, U_0^{abcd} is the variational derivative (3.7) of the Lagrangian with respect to the Riemann tensor.

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[5]This is $\int ds$, where $ds^2 = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt)$ with $dx^i = 0$, evaluated at the complex solution.

V. OTHER PATH INTEGRAL METHODS

A. Hilbert action surface term

The relationship between the Hilbert action surface term method (ii) [11,12] and the microcanonical functional integral method (i) can be understood as follows. Consider first the logical outline of the microcanonical functional integral method. The action $\mathcal{S}$ is the integral of the Lagrangian $\mathcal{L}$ over the manifold $\mathcal{M} = \mathcal{B}\times \mathbb{R}^2$. The integral is split into two pieces, an integral over the excised region $\mathcal{B}\times$disk that contains the bifurcation surface $\mathcal{H}$, and an integral over the remainder of the manifold, $\mathcal{B}\times$ annulus. Schematically, we have

$$\mathcal{S} = \int_{\mathbb{R}^2} \mathcal{L} = \int_D \mathcal{L} + \int_A \mathcal{L} \ , \qquad (5.1)$$

where D and A refer to the disk and the annulus, respectively, and the factor $\mathcal{B}$ has been suppressed for notational simplicity. The first integral vanishes in the limit in which the excised region D shrinks to $\mathcal{H}$, under the assumption that $\mathcal{L}$ is smooth. The second integral is written in Hamiltonian form, which yields a volume integral $\mathcal{S}_C$ [the integral of $p_\alpha \dot{q}^\alpha - \lambda^A \mathcal{C}_A$ in Eq. (4.1)] and terms at the "inner" boundary ∂A_i and "outer" boundary ∂A_o of the annulus. The terms at the outer boundary ∂A_o are discarded. In the limit as the disk shrinks to $\mathcal{H}$, the terms at the inner boundary ∂A_i become boundary terms $BT_{\mathcal{H}}$ at the bifurcation surface [the boundary terms in Eq. (4.1)]. Thus, the action becomes

$$\mathcal{S} \to \mathcal{S}_C + BT_{\mathcal{H}} \ . \qquad (5.2)$$

When $\mathcal{S}$ is evaluated at the complex solution, the canonical action $\mathcal{S}_C$ vanishes so that

$$\mathcal{S} \to BT_{\mathcal{H}} \ . \qquad (5.3)$$

Only the term proportional to the gradient of the lapse function N remains in $BT_{\mathcal{H}}$ since, as shown in Appendix B, N and V^i both vanish at the bifurcation surface.

In the Hilbert action surface term method (ii), the action integral is split according to

$$\mathcal{S} = \int_{\mathbb{R}^2} \mathcal{L} = \int_D \mathcal{L} + \int_{\partial D} \ell + \int_A \mathcal{L} + \int_{\partial A_i} \ell \ , \qquad (5.4)$$

where $\int \ell$ is the "Hilbert action surface term"; that is, $\int \ell$ is the surface term that must be added to $\int \mathcal{L}$ such that the boundary conditions include fixation of the metric on the boundary. Equation (5.4) is, of course, equivalent to Eq. (5.1) since the Hilbert action surface terms at ∂D and ∂A_i cancel one another. As in the microcanonical functional integral method, the integral over D vanishes in the limit in which the excised region shrinks to $\mathcal{H}$. Also the integral over A can be written in Hamiltonian form. The resulting action is

$$\mathcal{S} \to \int_{\partial D} \ell + \mathcal{S}_C + BT'_{\mathcal{H}} \ , \qquad (5.5)$$

where $BT'_{\mathcal{H}} = BT_{\mathcal{H}} + \int_{\partial A_i} \ell$. (Again, the terms at the outer boundary are discarded.)

The integrand in $BT'_{\mathcal{H}}$ must be linear in the lapse and shift in order to transform properly under reparametrizations in t. But, in fact, the integrand in $BT'_{\mathcal{H}}$ cannot depend on spatial derivatives of N or V^i. (Actually, spatial derivatives in a direction tangent to the boundary are allowed, since these can be removed through integration by parts.) This can be understood as follows. The boundary conditions appropriate for $\mathcal{S}_C + BT'_{\mathcal{H}}$ (which equals $\int_A \mathcal{L} + \int_{\partial A_i} \ell$ plus terms at the outer boundary ∂A_0) include fixation of the induced metric on ∂A_i, by definition of the Hilbert action surface term. Therefore, with the induced metric on ∂A_i denoted by γ_{mn}, we have

$$\begin{aligned}\delta(\mathcal{S}_C + BT'_{\mathcal{H}}) = {} & (\text{EOM's}) + (\text{BT's at } \partial A_0) \\ & + \int_{\partial A_i} \pi^{mn} \delta\gamma_{mn} \\ & + (\text{other BT's at } \partial A_i) \end{aligned} \qquad (5.6)$$

for some π^{mn}. Here, "EOM's" are terms that yield the classical equations of motion and "BT's at ∂A_0" are boundary terms at ∂A_0. The "other BT's at ∂A_i" are boundary terms at ∂A_i that involve variations of various matter fields and auxiliary fields, but do not involve variations of γ_{mn} or variations of derivatives of γ_{mn}. The induced metric γ_{mn} on the $[(D-1)$-dimensional] surface ∂A_i can be split into a lapse, shift, and $[(D-2)$-dimensional] spatial metric using the slices $t =$ const and time flow vector field t^a induced on ∂A_i. If the slices $t =$ const are orthogonal to ∂A_i, so that the unit normal of the slices lies in ∂A_i, then the lapse and shift components of γ_{mn} are just the restrictions of N and V^i to ∂A_i. If the slices $t =$ const are not orthogonal to ∂A_i, then the lapse and shift components of γ_{mn} are constructed algebraically from N and V^i through simple kinematical boost relations [23]. Consequently, the boundary terms at ∂A_i in Eq. (5.6) depend on the variations of N and V^i, but not on the variations of their derivatives. Since $\mathcal{S}_C$ contains no derivatives of N or V^i, Eq. (5.6) shows that $BT'_{\mathcal{H}}$ cannot contain derivatives of N or V^i.

When the action of Eq. (5.5) is evaluated at the complex solution, the canonical action $\mathcal{S}_C$ vanishes. The boundary term $BT'_{\mathcal{H}}$ is zero since its integrand is linear in the undifferentiated lapse and shift, and the lapse and shift vanish at $\mathcal{H}$. Therefore

$$\mathcal{S} \to \int_{\partial D} \ell \ , \qquad (5.7)$$

which shows that the entropy $\mathcal{S}_{\text{BH}} \approx \mathcal{S}(\bar{\lambda}, \bar{q}, \bar{p})$ equals the Hilbert action surface term for a small disk surrounding the bifurcation surface $\mathcal{H}$, evaluated at the complex solution. In effect, what has been shown is that the Hilbert action surface term must include the negative of the particular boundary term displayed in Eq. (4.1) that is proportional to the gradient of the lapse function. The minus sign is compensated by the fact that the normal n^i of ∂D points away from $\mathcal{H}$.

B. Conical deficit angle

The starting point for the canonical deficit angle method (iii) [12,13,6] for computing black hole entropy is the (grand canonical) partition function

$$Z(\beta) = \sum_{\mathcal{M}} \int Dg\, D\psi\, \exp\Big(\mathcal{S}_\beta[g,\psi]\Big) . \tag{5.8}$$

Here, $\mathcal{S}_\beta$ is the Hilbert action $\mathcal{S}_\beta = \int_{\mathcal{M}} \mathcal{L} + \int_{\partial\mathcal{M}} \ell$ and the inverse temperature is defined by the lapse component of the boundary metric γ_{mn} according to $\beta = i\int dt(\text{lapse})$. When evaluated at the complex black hole solution, β is the proper length in the S^1 direction as measured orthogonally to the stationary time slices in $\partial\mathcal{M}$. The relationship between $Z[\beta]$ and the density of states (2.1) is spelled out in detail in Ref. [9]. For our present purposes it is sufficient to note that the density of states is a function of the internal energy E, where E is defined by the variation of the Hilbert action with respect to the lapse component of γ_{mn}. Thus, the microcanonical action $\mathcal{S}$ of Eq. (2.1) and the Hilbert action $\mathcal{S}_\beta$ differ by boundary terms that include a term of the form $-\beta E$. The partition function is then given by the Laplace transform of the density of states:[6]

$$Z(\beta) = \int dE\, \nu(E) e^{-\beta E} . \tag{5.9}$$

With the relationship $\nu(E) \approx \exp[\mathcal{S}_{\rm BH}(E)]$, the leading order approximation to the integral in Eq. (5.9) is

$$\ln Z(\beta) \approx \mathcal{S}_{\rm BH}(E^*) - \beta E^* , \tag{5.10}$$

where E^* is the function of β such that $E = E^*(\beta)$ extremizes the exponential:

$$\left.\frac{\partial \mathcal{S}_{\rm BH}(E)}{\partial E}\right|_{E^*} = \beta . \tag{5.11}$$

Equations (5.10) and (5.11) just express $\ln Z(\beta)$ (which is $-\beta$ times the free energy) as a Legendre transform of the entropy $\mathcal{S}_{\rm BH}(E)$.

From the relationships above it is easy to show that the entropy is given by

$$\mathcal{S}_{\rm BH}(E^*) \approx \ln Z(\beta) - \beta \frac{\partial \ln Z(\beta)}{\partial\beta} . \tag{5.12}$$

The zero-loop approximation to the path integral (5.8) yields $\ln Z(\beta) \approx \mathcal{S}_\beta[\bar g, \bar\psi]$, where $\bar g$, $\bar\psi$ is the complex black hole solution that extremizes $\mathcal{S}_\beta[g,\psi]$ among configurations whose proper length (period) in the S^1 direction equals β at the boundary $\partial\mathcal{M}$. Then the entropy can be written as

$$\mathcal{S}_{\rm BH}(E^*) \approx \mathcal{S}_\beta[\bar g, \bar\psi] - \beta \lim \frac{\mathcal{S}_{\beta'}[\bar g', \bar\psi'] - \mathcal{S}_\beta[\bar g, \bar\psi]}{\beta' - \beta} , \tag{5.13}$$

where the limit is taken in which $\beta' \to \beta$. It is important to recognize that in taking the derivative with respect to β, one does *not* introduce a conical singularity in the metric $\bar g$ (contrary to claims made in the literature). Rather, when β is varied, the parameters of the solution $\bar g$, $\bar\psi$ (notably the black hole mass parameter) vary in such a way that the configuration remains a smooth solution of the classical equations of motion. Thus, in Eq. (5.13), the extremal configuration with period β' is the smooth solution denoted $\bar g'$, $\bar\psi'$.

Equation (5.13) can be evaluated with the following trick. Since the action $\mathcal{S}_{\beta'}$ is stationary at $\bar g'$, $\bar\psi'$, one can distort this configuration without affecting the value of the action to first order. Thus, replace $\bar g'$ with the metric $\bar g^\vee$ and leave $\bar\psi'$ alone. In principle, $\bar g^\vee$ could be any metric obtained from an infinitesimal variation of $\bar g'$. Consider as a particular choice for $\bar g^\vee$ the metric whose components are identical to the components of $\bar g$ in the stationary coordinate system, but with the period in coordinate time t adjusted so that the proper length in the S^1 direction at $\partial\mathcal{M}$ is β' rather than β. Thus, $\bar g^\vee$ is a smooth, regular solution of the classical equations of motion everywhere except at the bifurcation surface $\mathcal{H}$. At $\mathcal{H}$ the regularity condition $\int_{S^1} dt\, \tilde\kappa = 2\pi$ does not hold for $\bar g^\vee$, indicating the presence of a conical singularity.

With the replacement of $\bar g'$ by $\bar g^\vee$, the entropy becomes

$$\mathcal{S}_{\rm BH}(E^*) \approx \mathcal{S}_\beta[\bar g] - \beta \lim \frac{\mathcal{S}_{\beta'}[\bar g^\vee] - \mathcal{S}_\beta[\bar g]}{\beta' - \beta} . \tag{5.14}$$

(The dependence on matter fields ψ has been dropped for notational simplicity.) Now split the action $\mathcal{S}_\beta[\bar g]$ into an integral $(\int_D \mathcal{L})[\bar g]$ over the disk plus "other terms" that consist of an integral over the annulus A and boundary integrals at $\partial\mathcal{M}$. The integral over D vanishes in the limit as the disk shrinks to $\mathcal{H}$, since $\mathcal{L}$ is smooth. The other terms contain integrals over t and are proportional to β. Likewise, split $\mathcal{S}_{\beta'}[\bar g^\vee]$ into $(\int_D \mathcal{L})[\bar g^\vee]$ plus "other terms." The integral over D does not vanish in this case due to the conical singularity in $\bar g^\vee$. The other terms are identical to the other terms from $\mathcal{S}_\beta[\bar g]$ with the exception that they are proportional to β'. Therefore all "other terms" in expression (5.14) cancel, and the entropy becomes

$$\mathcal{S}_{\rm BH}(E^*) \approx -\beta \lim \frac{1}{\beta' - \beta} \left(\int_D \mathcal{L}\right)\Big|_{\bar g^\vee} . \tag{5.15}$$

The entropy is thus expressed in terms of a spacetime $\bar g^\vee$ with a conical singularity. The limit in Eq. (5.15) is

[6] To be precise, the inverse temperature is a function on the system boundary $\mathcal{B}$ and, correspondingly, the energy is a surface density [17,9]. Thus, the notation used here should be viewed as schematic. On the other hand, β and E can be interpreted as the "zero mode" parts of the inverse temperature and energy surface density. The final result (5.15) for the entropy can be understood in this way.

taken in which D shrinks to $\mathcal{H}$ and β' approaches β.

The correctness of the result (5.15) can be verified as follows. Let $\mathcal{L}$ be the integrand of Eq. (3.8).[7] The term proportional to f vanishes in the limit as the disk shrinks to $\mathcal{H}$. Likewise, most of the terms in $U_0^{abcd}\mathcal{R}_{abcd}$ do not contribute to the entropy—only the term that contains the curvature in the surface orthogonal to $\mathcal{H}$ survives as D shrinks to $\mathcal{H}$. This term captures the curvature of the conical singularity. Inserting a projection onto the binormal of $\mathcal{H}$, we have

$$\left(\int_D \mathcal{L}\right)\bigg|_{\bar{g}^\vee} = \frac{i}{4}\int_{D\times\mathcal{B}} d^Dx\left(\sqrt{-g}U_0^{efgh}\epsilon_{ef}\epsilon_{gh}\epsilon^{ab}\epsilon^{cd}\mathcal{R}_{abcd}\right)\bigg|_{\bar{g}^\vee} . \tag{5.16}$$

The binormal can be written as $\epsilon_{ab} = 2u_{[a}n_{b]}$ where n_a and u_a are orthogonal to $\mathcal{H}$ and to each other, and n^a lies in a t = const surface. Note that $u_a = -N\nabla_a t$ is imaginary when evaluated at a complex spacetime, so $-i\tilde{u}_a$ is the real unit vector with square $+1$ for the metric $\bar{g}^\vee$. The basic interpretation of the Riemann tensor gives

$$n^a(-iu^b)n^c\mathcal{R}_{abc}{}^d = \delta n^d/A_D , \tag{5.17}$$

where δn^d is the change in the vector n^c as it is parallel transported around the perimeter of the disk D (first in the n^a direction, then in the $-iu^b$ direction) and A_D is the area of the disk. If the disk has deficit angle α, then $\delta n^d = \alpha(-iu^d)$. Thus, we find $\epsilon^{ab}\epsilon^{cd}\mathcal{R}_{abcd} = -4\alpha/A_D$ and

$$\left(\int_D \mathcal{L}\right)\bigg|_{\bar{g}^\vee} = -\int_{D\times\mathcal{B}} d^Dx\left(\sqrt{g}\epsilon_{ab}\epsilon_{cd}U_0^{abcd}\alpha/A_D\right)\bigg|_{\bar{g}^\vee} . \tag{5.18}$$

Now, the deficit angle is given by $\alpha/(2\pi) = (\beta-\beta')/\beta$, so the entropy from Eq. (5.15) becomes

$$\mathcal{S}_{\rm BH}(E^*) \approx -2\pi\int_{\mathcal{H}} d^{D-2}x\left(\sqrt{\sigma}\epsilon_{ab}\epsilon_{cd}U_0^{abcd}\right)\bigg|_{\bar{g}} . \tag{5.19}$$

In the integrand above, $\epsilon_{ab}\epsilon_{cd}U_0^{abcd}$ is a spacetime scalar and in particular it is invariant under reparametrizations in t. Consequently, the integrand can be evaluated at the Lorentzian black hole solution $\tilde{g}$ rather than $\bar{g}$. The result agrees with Eq. (4.5).

ACKNOWLEDGMENTS

I would like to thank T. Jacobson for helpful communications and J.W. York for helpful discussions and comments on the manuscript.

APPENDIX A: MAXWELL ELECTRODYNAMICS AND EINSTEIN GRAVITY

Consider Maxwell electrodynamics in $D = 4$ spacetime dimensions coupled to the gravitational field. The electromagnetic field contribution to the action is

$$\mathcal{S}_M = -\frac{i}{4\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\,\nabla_{[a}A_{b]}g^{ac}g^{bd}\nabla_{[c}A_{d]} . \tag{A1}$$

In the form of Eq. (3.8) the action becomes

$$\mathcal{S}_M = i\int_{\mathcal{M}} d^4x\sqrt{-g}\Big\{\Pi^{ab}(\nabla_a A_b - \Lambda_{ab}) - (\Lambda_{[ab]}g^{ac}g^{bd}\Lambda_{[cd]})/(4\pi)\Big\} , \tag{A2}$$

where $\nabla_a A_b$ appears linearly. The auxiliary fields are Π^{ab} and Λ_{ab}. The "almost Hamiltonian" form [cf. Eq. (3.15)] of the action is

$$\mathcal{S}_M = i\int dt\int_\Sigma d^3x\sqrt{h}\Big\{-\Pi^{\perp\perp}\dot{A}_\perp + \Pi^{\perp i}\dot{A}_i - N\mathcal{C}^M_\perp - V^i\mathcal{C}^M_i\Big\} + i\int dt\int_{\partial\Sigma} d^2x\sqrt{\sigma}n_i\left\{NA_\perp\Pi^{\perp i} - NA^i\Pi^{\perp\perp} - V^jA_j\Pi^{\perp i}\right\} , \tag{A3}$$

where the electromagnetic field contribution to the Hamiltonian and momentum constraints is

[7] Iyer and Wald [5] conjectured that the canonical deficit angle method (iii) is limited to theories in which the Lagrangian is a linear function of the Riemann tensor. The results here show that any theory of the form (3.1) can be treated by the conical deficit angle method if the action is first put into the form (3.8) in which the Lagrangian is linear in $\mathcal{R}_{abcd}$.

$$\begin{aligned}\mathcal{C}_\perp^M &= -D^i(\Pi^{\perp\perp}A_i) + \Pi^{\perp\perp}\Lambda_{\perp\perp} + D_i(\Pi^{\perp i}A_\perp) - \Pi^{\perp i}\Big(K_i^j A_j + \Lambda_{\perp i}\Big)\\ &\quad +\Pi^{i\perp}\Big(D_iA_\perp - K_i^jA_j - \Lambda_{i\perp}\Big) - \Pi^{ij}\Big(D_iA_j - K_{ij}A_\perp - \Lambda_{ij}\Big)\\ &\quad +(\Lambda_{[ij]}h^{ik}h^{j\ell}\Lambda_{[k\ell]})/(4\pi) - (\Lambda_{[i\perp]}h^{ij}\Lambda_{[j\perp]})/(2\pi)\ , \end{aligned} \tag{A4a}$$

$$\mathcal{C}_i^M = -\Pi^{\perp\perp}D_iA_\perp - A_iD_j\Pi^{\perp j} - 2\Pi^{\perp j}D_{[j}A_{i]}\ . \tag{A4b}$$

The action is a functional of the coordinates (the q^α's) $A_\perp$ and A_i, the momenta (the p_α's) $-\sqrt{h}\Pi^{\perp\perp}$ and $\sqrt{h}\Pi^{\perp i}$, and the extra variables (the χ's) $\Pi^{i\perp}$, Π^{ij}, $\Lambda_{i\perp}$, $\Lambda_{\perp i}$, Λ_{ij}, and $\Lambda_{\perp\perp}$.

The χ's appear in the action undifferentiated and only in the function $\mathcal{C}_\perp^M$. In particular $\Pi^{i\perp}$, Π^{ij}, $\Lambda_{i\perp}$, $\Lambda_{\perp i}$, and Λ_{ij} appear quadratically in $\mathcal{C}_\perp^M$ and can be eliminated by the solution of their algebraic equations of motion. Those equations are straightforward to derive, and the solution is

$$\Pi^{i\perp} = -\Pi^{\perp i}\ , \tag{A5a}$$

$$\Pi^{ij} = -h^{ik}h^{j\ell}D_{[k}A_{\ell]}/(2\pi)\ , \tag{A5b}$$

$$\Lambda_{i\perp} = D_iA_\perp - K_i^jA_j\ , \tag{A5c}$$

$$\Lambda_{\perp i} = -4\pi h_{ij}\Pi^{\perp j} + D_iA_\perp - K_i^jA_j\ , \tag{A5d}$$

$$\Lambda_{ij} = D_iA_j - K_{ij}A_\perp\ . \tag{A5e}$$

Inserting this result into $\mathcal{C}_\perp^M$, one obtains

$$\begin{aligned}\mathcal{C}_\perp^M &= -D^i(\Pi^{\perp\perp}A_i) + \Pi^{\perp\perp}\Lambda_{\perp\perp} + A_\perp D_i\Pi^{\perp i}\\ &\quad +2\pi\Pi^{\perp i}h_{ij}\Pi^{\perp j} + D_{[i}A_{j]}D^{[i}A^{j]}/(4\pi)\end{aligned} \tag{A6}$$

for the electromagnetic field contribution to the Hamiltonian constraint.

The variable $\Lambda_{\perp\perp}$ appears in the action (A3), (A4b), and (A6) as a Lagrange multiplier associated with the constraint $\Pi^{\perp\perp} = 0$. As discussed in Sec. III, the variables $\Lambda_{\perp\perp}$, $\Pi^{\perp\perp}$ can be eliminated through their equations of motion, which amounts to setting $\Pi^{\perp\perp}$ equal to zero. The coordinate $A_\perp$ conjugate to $-\sqrt{h}\Pi^{\perp\perp}$ remains as an extra variable in the action, which now reads

$$\begin{aligned}\mathcal{S}_M = i\int dt\int_\Sigma d^3x\sqrt{h}\Big\{\Pi^{\perp i}\dot{A}_i - V^i\Big[-A_iD_j\Pi^{\perp j} + 2\Pi^{\perp j}D_{[i}A_{j]}\Big] - N\Big[A_\perp D_i\Pi^{\perp i} + 2\pi\Pi^{\perp i}h_{ij}\Pi^{\perp j} + D_{[i}A_{j]}D^{[i}A^{j]}/(4\pi)\Big]\Big\}\\ +i\int dt\int_{\partial\Sigma} d^2x\sqrt{\sigma}n_i\Big\{N\Pi^{\perp i}A_\perp - V^jA_j\Pi^{\perp i}\Big\}\ .\end{aligned} \tag{A7}$$

The equation of motion for $A_\perp$ yields the Gauss's law constraint $D_i\Pi^{\perp i} = 0$. Thus, $A_\perp$ is a Lagrange multiplier. Now make the changes of variables $\mathcal{E}^i = \sqrt{h}\Pi^{\perp i}$ for the momentum conjugate to A_i and $A_t = A_at^a = -NA_\perp + V^iA_i$ for the Lagrange multiplier. The result is

$$\begin{aligned}\mathcal{S}_M = i\int dt\int_\Sigma d^3x\Big\{\mathcal{E}^i\dot{A}_i - V^i\Big[2\mathcal{E}^jD_{[i}A_{j]}\Big] + A_t\Big[D_i\mathcal{E}^i\Big] - N\Big[2\pi\mathcal{E}^i\mathcal{E}_i/\sqrt{h} + \sqrt{h}D_{[i}A_{j]}D^{[i}A^{j]}/(4\pi)\Big]\Big\}\\ -i\int dt\int_{\partial\Sigma} d^2x\sqrt{\sigma}\,A_tn_i\mathcal{E}^i/\sqrt{h}\ ,\end{aligned} \tag{A8}$$

the Hamiltonian form of the action for the electromagnetic field coupled to gravity.

Now consider Einstein gravity in $D = 4$ spacetime dimensions:

$$\mathcal{S}_E = i\int_{\mathcal{M}} d^4x\sqrt{-g}g^{ac}g^{bd}\mathcal{R}_{abcd}\ , \tag{A9}$$

where Newton's constant equals $1/(16\pi)$. In the form of Eq. (3.6), or equivalently (3.8), the action becomes

$$\mathcal{S}_E = i\int_{\mathcal{M}} d^4x\sqrt{-g}\Big\{U_0^{abcd}(\mathcal{R}_{abcd} - V^0_{abcd}) + g^{ac}g^{bd}V^0_{abcd}\Big\}\ . \tag{A10}$$

It is assumed that U_0^{abcd} and V^0_{abcd} have the same symmetries as $\mathcal{R}_{abcd}$. The spacetime split leads to the action of Eq. (3.13), where the function f is given by

$$f = h^{ac}h^{bd}V^0_{abcd} + 2h^{ab}V^0_{\perp ab\perp} - U_0^{abcd}h_a^eh_b^fh_c^gh_d^hV^0_{efgh} + 4U_0^{abc\perp}h_a^dh_b^eh_c^fV^0_{def\perp} - 4U_0^{\perp ab\perp}h_a^ch_b^dV^0_{\perp cd\perp}\ . \tag{A11}$$

By mapping the fields from $\mathcal{M}$ to $\Sigma\times I$ and integrating by parts to remove spatial derivatives from the lapse and shift, one obtains the action in "almost Hamiltonian" form [cf. Eq. (3.15)]:

$$\mathcal{S}_E = i\int dt \int_\Sigma d^3x\Big\{P^{ij}\dot{h}_{ij} + Q^{ij}\dot{K}_{ij} - N\mathcal{C}^E_\perp - V^i\mathcal{C}^E_i\Big\} \\ +i\int dt\int_{\partial\Sigma} d^2x(\sqrt{\sigma}/\sqrt{h})n_i\Big\{-2P^{ij}V_j - 2Q^{ij}K_{jk}V^k + Q^{ij}D_jN - ND_jQ^{ij}\Big\} . \tag{A12}$$

Here, the notation $Q^{ij} = -4\sqrt{h}U_0^{\perp ij\perp}$ has been introduced, and the gravitational contribution to the Hamiltonian and momentum constraints is

$$\mathcal{C}^E_\perp = -D_iD_jQ^{ij} - 2P^{ij}K_{ij} - Q^{ij}K_i^kK_{kj} - \sqrt{h}U_0^{ijk\ell}(R_{ijk\ell} + 2K_{ik}K_{j\ell} - V^0_{ijk\ell}) \\ +4\sqrt{h}U_0^{ijk\perp}(2D_iK_{jk} - V^0_{ijk\perp}) - \sqrt{h}h^{ik}h^{j\ell}V^0_{ijk\ell} - (Q^{ij} + 2\sqrt{h}h^{ij})V^0_{\perp ij\perp} , \tag{A13a}$$

$$\mathcal{C}^E_i = -2D_jP^j_i + Q^{jk}D_iK_{jk} - 2D_j(Q^{jk}K_{ki}) . \tag{A13b}$$

The action is a functional of the coordinates (the q^α's) h_{ij} and K_{ij}, the momenta (the p_α's) P^{ij} and Q^{ij}, and the extra variables (the χ's) $U_0^{ijk\ell}$, $U_0^{ijk\perp}$, $V^0_{ijk\ell}$, $V^0_{ijk\perp}$, and $V^0_{\perp ij\perp}$.

The χ variables $U_0^{ijk\ell}$, $U_0^{ijk\perp}$, $V^0_{ijk\ell}$, and $V^0_{ijk\perp}$ can be eliminated by the solution of their algebraic equations of motion. That solution is

$$U_0^{ijk\ell} = h^{i[k}h^{\ell]j} , \tag{A14a}$$

$$U_0^{ijk\perp} = 0 , \tag{A14b}$$

$$V^0_{ijk\ell} = R_{ijk\ell} + 2K_{i[k}K_{\ell]j} , \tag{A14c}$$

$$V^0_{ijk\perp} = 2D_{[i}K_{j]k} . \tag{A14d}$$

Inserting this result into $\mathcal{C}^E_\perp$, one obtains

$$\mathcal{C}^E_\perp = -D_iD_jQ^{ij} - 2P^{ij}K_{ij} - Q^{ij}K_i^kK_{kj} - \sqrt{h}(R + K^2 - K_{ij}K^{ij}) - (Q^{ij} + 2\sqrt{h}h^{ij})V^0_{\perp ij\perp} . \tag{A15}$$

Clearly, the variable $V^0_{\perp ij\perp}$ plays the role of a Lagrange multiplier for the constraint $Q^{ij} + 2\sqrt{h}h^{ij} = 0$. The situation here is close to that discussed in Sec. III E in which a constraint (denoted $\mathcal{C}_1$) is given by a momentum variable (denoted p_1). In fact, the present theory can be placed in this form by a canonical transformation in which Q^{ij} is replaced by $Q^{ij} + 2\sqrt{h}h^{ij}$ as the momentum conjugate to K_{ij}. The form of the canonical transformation can be deduced from the relationship

$$P^{ij}\dot{h}_{ij} + Q^{ij}\dot{K}_{ij} = (P^{ij} + \sqrt{h}Kh^{ij} - 2\sqrt{h}K^{ij})\dot{h}_{ij} + (Q^{ij} + 2\sqrt{h}h^{ij})\dot{K}_{ij} - 2(\sqrt{h}K)^{\cdot} . \tag{A16}$$

Thus, define the new momenta

$$\bar{P}^{ij} = P^{ij} + \sqrt{h}Kh^{ij} - 2\sqrt{h}K^{ij} , \tag{A17a}$$

$$\bar{Q}^{ij} = Q^{ij} + 2\sqrt{h}h^{ij} , \tag{A17b}$$

and the action becomes

$$\mathcal{S}_E = i\int dt \int_\Sigma d^3x\Big\{\bar{P}^{ij}\dot{h}_{ij} + \bar{Q}^{ij}\dot{K}_{ij} - 2(\sqrt{h}K)^{\cdot} - N\mathcal{C}^E_\perp - V^i\mathcal{C}^E_i\Big\} \\ +i\int dt\int_{\partial\Sigma} d^2x(\sqrt{\sigma}/\sqrt{h})n_i\Big\{-2\bar{P}^{ij}V_j - 2\bar{Q}^{ij}K_{jk}V^k + \bar{Q}^{ij}D_jN - ND_j\bar{Q}^{ij} - 2\sqrt{h}D^iN + 2\sqrt{h}KV^i\Big\} , \tag{A18}$$

where

$$\mathcal{C}^E_\perp = -D_iD_j\bar{Q}^{ij} - 2\bar{P}^{ij}K_{ij} - \bar{Q}^{ij}K_i^kK_{kj} - \sqrt{h}(R - K^2 + K_{ij}K^{ij}) - \bar{Q}^{ij}V^0_{\perp ij\perp} . \tag{A19a}$$

$$\mathcal{C}^E_i = -2D_j\bar{P}^j_i + \bar{Q}^{jk}D_iK_{jk} - 2D_j(\bar{Q}^{jk}K_{ki}) . \tag{A19b}$$

The variables $\bar{Q}^{ij}$ and $V^0_{\perp ij\perp}$ (which play the role of p_1 and λ^1 in the discussion of Sec. III E) can be eliminated by setting $\bar{Q}^{ij}$ equal to zero. The action then reduces to

$$\mathcal{S}_E = i\int dt \int_\Sigma d^3x\Big\{\bar{P}^{ij}\dot{h}_{ij} - 2(\sqrt{h}K)^{\cdot} - V^i[-2D_j\bar{P}^j_i] - N\Big[-2\bar{P}^{ij}K_{ij} - \sqrt{h}(R - K^2 + K_{ij}K^{ij})\Big]\Big\}$$
$$+i\int dt \int_{\partial\Sigma} d^2x\sqrt{\sigma}n_i\left\{-2\bar{P}^{ij}V_j/\sqrt{h} - 2D^iN + 2KV^i\right\}, \quad \text{(A20)}$$

where K_{ij} (which plays the role of q^1) appears as an extra independent variable in addition to the canonical pair h_{ij}, $\bar{P}^{ij}$, the lapse function N, and the shift vector V^i.

The equation of motion for K_{ij} which follows from the action (A20) is

$$0 = -2\bar{P}^{ij} + 2\sqrt{h}Kh^{ij} - 2\sqrt{h}K^{ij}, \quad \text{(A21)}$$

and the solution of this equation is

$$\sqrt{h}K_{ij} = -\bar{P}_{ij} + \bar{P}h_{ij}/2. \quad \text{(A22)}$$

By substituting this result into Eq. (A20), we find the action for Einstein gravity in the Hamiltonian form

$$\mathcal{S}_E = i\int dt \int_\Sigma d^3x\Big\{-\dot{\bar{P}}^{ij}h_{ij} - V^i[-2D_j\bar{P}^j_i] - N\Big[(2\bar{P}^{ij}\bar{P}_{ij} - \bar{P}^2)/(2\sqrt{h}) - \sqrt{h}R\Big]\Big\}$$
$$+i\int dt \int_{\partial\Sigma} d^2x\sqrt{\sigma}n_i\left\{V_j(\bar{P}h^{ij} - 2\bar{P}^{ij})/\sqrt{h} - 2D^iN\right\}. \quad \text{(A23)}$$

Note that the "kinetic" term of Eq. (A23) is $-\dot{\bar{P}}^{ij}h_{ij}$, rather than the more usual $\bar{P}^{ij}\dot{h}_{ij}$. The difference is just a boundary term, $i\int_\Sigma d^3x\,\bar{P}$ at the initial and final times. If we had originally chosen the action (A9) to include a boundary term $2i\int_\Sigma d^3x\,\sqrt{h}K$ at the initial and final times, then we would have obtained $\bar{P}^{ij}\dot{h}_{ij}$ for the kinetic term in the Hamiltonian form of the action.

APPENDIX B: SURFACE GRAVITY

As described in Sec. IV, one wedge of the spacetime $\tilde{g}_{ab}$ is foliated into stationary hypersurfaces t = const with time flow vector field t^a, where t^a is the Killing vector field that vanishes at the bifurcation surface $\mathcal{H}$. In this appendix I will drop the tildes with the understanding that all relationships hold for the Lorentzian metric $\tilde{g}_{ab}$.

The lapse function N and shift vector V^a satisfy

$$t^a = Nu^a + V^a. \quad \text{(B1)}$$

This expression is well defined on the interior of the wedge. Consider the limit in which $\mathcal{H}$ is approached from within a t = const hypersurface, say, Σ_0. Since t^a vanishes at $\mathcal{H}$, it follows by contraction of Eq. (B1) successively with u_a and h^b_a that

$$\lim N = 0, \quad \lim V^a = 0. \quad \text{(B2)}$$

This result is used in Sec. IV to show that the "terms $\sim N$ and V^i" vanish at the boundary $\mathcal{H}$ of Σ.

The surface gravity of a Killing horizon equals [22]

$$\kappa = \lim(|t||a|), \quad \text{(B3)}$$

where $|t| = \sqrt{-t^at_a}$ is the magnitude of t^a and $|a| = \sqrt{a^ca_c}$ is the magnitude of the acceleration $a^c = (t^a\nabla_at^c)/|t|^2$ of the orbits of t^a. I will now show that, in effect, the shift vector V^a vanishes sufficiently rapidly in the limit as the bifurcation surface is approached from within Σ_0 so that the orbits of t^a become orthogonal to the t = const surfaces. Then the surface gravity can be expressed in terms of the acceleration of the unit normal u^a of the t = const surfaces.

Using the Killing vector field property $\nabla_{(a}t_{b)} = 0$, one can easily show that $|t|a_c = \nabla_c|t|$. Then the surface gravity (B3) becomes

$$\kappa = \lim\sqrt{(\nabla^a|t|)(\nabla_a|t|)}. \quad \text{(B4)}$$

Observe that the limit of $|t| = \sqrt{-t^at_a} = \sqrt{N^2 - V^aV_a}$ is a constant (namely, zero) as $\mathcal{H}$ is approached from within Σ_0. Thus we have $\lim\sigma^{ab}\nabla_b|t| = 0$ where $\sigma_{ab} = h_{ab} - n_an_b$ is the induced metric on $\mathcal{H}$ and n_a is the unit normal of $\mathcal{H}$ in Σ_0. Since also $t^a\nabla_a|t| = 0$, the gradient of $|t|$ lies entirely in the v^a direction,

$$\lim\nabla_a|t| = \lim v_av^b\nabla_b|t|, \quad \text{(B5)}$$

where v^a is the unit vector orthogonal to both σ_{ab} and t_a:

$$v^a \sim \lim(Nn^a + n^bV_bu^a). \quad \text{(B6)}$$

Because the surface gravity of a bifurcate Killing horizon is nonzero [22], it follows from Eq. (B4) that $\lim\nabla_a|t| \neq 0$. Also note that $\nabla_a|t|$ is spacelike, since it is orthogonal to the timelike vector t^a. Therefore, $\lim h^b_a\nabla_b|t| \neq 0$ and, since $\lim\sigma^{ab}\nabla_b|t| = 0$, we conclude that $\lim n^b\nabla_b|t| \neq 0$.

Now consider the limit of $V^a/|t|$ as $\mathcal{H}$ is approached from within Σ_0. This is an indeterminate form, 0/0.

We can apply l'Hôpital's rule and differentiate both numerator and denominator along the normal n^a direction within Σ_0. The derivative of the denominator, $n^b\nabla_b|t|$, has a nonzero limit by the argument above. The derivative of the numerator is

$$\begin{aligned} n^b\nabla_b V^a &= n^b\nabla_b(h^a_c t^c) \\ &= n^b t^c \nabla_b h^a_c + n^b h^a_c \nabla_b t^c \ . \end{aligned} \tag{B7}$$

Using $h^a_c = \delta^a_c + u^a u_c$, one can write the first term as

$$\begin{aligned} n^b t^c \nabla_b h^a_c &= n^b t^c \nabla_b (u^a u_c) \\ &= N K^{ab} n_b - u^a n^b K_{bc} V^c \ , \end{aligned} \tag{B8}$$

where $K_{ab} = -h^c_a \nabla_c u_b$ is the extrinsic curvature of Σ_0. The two terms in Eq. (B8) vanish in the limit because the lapse N and shift V^c both vanish in the limit. Using the relation $h^{ac} = \sigma^{ac} + n^a n^c$ and the Killing vector field property $\nabla_{(b}t_{c)} = 0$, one can write the second term of Eq. (B7) as

$$\begin{aligned} n^b h^{ac} \nabla_b t_c &= n^b \sigma^{ac} \nabla_b t_c \\ &= -n^b \sigma^{ac} \nabla_c t_b \\ &= -n^b \sigma^{ac} \nabla_c (N u_b + V_b) \\ &= N \sigma^{ac} K_{cb} n^b - n^b \sigma^{ac} \nabla_c V_b \ . \end{aligned} \tag{B9}$$

The first term in Eq. (B9) vanishes in the limit due to the factor of N, and the second term in Eq. (B.9) is zero in the limit since the derivative acts along the bifurcation surface where V_b vanishes. The result is that $\lim n^b\nabla_b V^a = 0$, so by l'Hôpital's rule we have $\lim V^a/|t| = 0$. Since $|t| = \sqrt{N^2 - V^b V_b}$, we also find $\lim(V^a/N)(1 - V^b V_b/N^2)^{-1/2} = 0$ which implies $\lim V^a/N = 0$.

The gradient of $|t|$ is given by

$$\nabla_a |t| = \frac{N\nabla_a N}{|t|} + \frac{V^b \nabla_a V_b}{|t|} \ . \tag{B10}$$

The results above show that the second term vanishes in the limit as $\mathcal{H}$ is approached from within Σ_0, and that $\lim N/|t| = 1$. Therefore Eq. (B10) yields $\lim \nabla_a |t| = \lim \nabla_a N$. We also find from Eq. (B6) that $v^a = n^a$. Thus, Eq. (B5) becomes

$$\lim \nabla_a N = \lim n_a n^b \nabla_b N \ . \tag{B11}$$

It follows that the surface gravity (B4) can be written as $\kappa = \lim |n^a \nabla_a N|$. If we choose, as in the main body of the paper, the unit normal n^a to point "radially inward" towards $\mathcal{H}$, the surface gravity becomes $\kappa = -\lim n^a \nabla_a N$. From the relationship (B11) we find the key result

$$\lim \nabla_a N = -\lim \kappa n_a \ . \tag{B12}$$

Finally, note that the surface gravity also can be expressed as $\kappa = -\lim N n^b A_b$, where (see Ref. [20]) $A_b = u^a \nabla_a u_b = h^c_b(\nabla_c N)/N$ is the acceleration of the unit normal of the $t = \text{const}$ hypersurfaces.

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