(1+1)-dimensional QCD with fundamental bosons and fermions

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(Received 12 June 1995)

We analyze the properties of mesons in (1+1)-dimensional QCD with bosonic and fermionic "quarks" in the large N_c limit. We study the spectrum in detail and show that it is impossible to obtain massless mesons including boson constituents in this model. We quantitatively show how the QCD mass inequality is realized in two-dimensional QCD. We find that the mass inequality is close to being an equality even when the quarks are light. Methods for obtaining the properties of "mesons" formed from boson and/or fermion constituents are formulated in an explicit manner convenient for further study. We also analyze how the physical properties of the mesons such as confinement and asymptotic freedom are realized.

PACS number(s): 12.38.Aw, 11.10.Kk, 11.15.Pg, 12.40.Yx

I. INTRODUCTION

In confining gauge theories, physically observable particles at low energies have no gauge charge and are bound states of the charged matter that appear in the gauge theory Lagrangian. A prototype of such a theory is QCD wherein the gauge group is SU(3) and the charged matter fields are vector fermions in the fundamental representation. Several natural generalizations of QCD exist; we may use gauge groups other than SU(3), we may use chiral fermions, or we may choose representations more complicated than the fundamental representation for the matter fields. In this case, we cannot put in too many or too large representations if we want to preserve asymptotic freedom. Another generalization is to consider boson constituents as well as fermion constituents.

It is this last, as well as the first, generalization in two dimensions in the large-N_c limit that we shall investigate in this work. The possibility of boson constituents arise necessarily in several contexts, such as technicolor with multiple stages of symmetry breaking [1], QCD or technicolor with supersymmetry [2], so-called bosonic technicolor [3], as well as the standard model [4]. When both boson and fermion constituents exist in the theory, "meson" states of both Bose and Fermi statistics may arise. In general, it is difficult to derive the properties of the bound states from first principles. By using the large-N_c limit in two dimensions, we may analyze the properties of these meson states concretely.

Two-dimensional QCD in the large-N_c limit has greatly contributed to our current understanding of the gauge theory dynamics by providing us with a model where the properties are explicitly calculable analytically. Also, two-dimensional QCD has been used to test the validity of various approaches and approximation schemes applied to QCD. The model was first solved by 't Hooft [5] and some further physical properties such as some current matrix elements, the asymptotic freedom of mesons were studied in some subsequent works [6,7]. The formulation was extended to include boson-boson bound states [8,9] and boson-fermion bound states [10]. Mesons made only from fermionic quarks or the bosonic quarks obey Bose statistics but the boson-fermion bound state obeys Fermi statistics. (Hereafter, often referred to as ff, bb, and bf cases.)

In this work, we shall extend the investigation of the physical properties of the mesons made from fermions and generalize the results to the mesons made from bosons only and bosons and fermions. The results will enable us to compare the three cases and see the differences and similarities that arise between mesons made from constituents of various statistics. The spectrum of mesons is investigated both analytically and numerically. We will study the case when the quarks are heavy analytically. For mesons involving light quarks, we establish a number of results analytically and further analyze the problem numerically. The numerical approach will be formulated explicitly in all the three ff, bf, and bb cases which should be useful for further study.

We use these results to see how the QCD mass inequality [11,12] applies to two-dimensional QCD in the large-N_c limit. Given two different types of quarks, the QCD mass inequality states that the meson made from the same quarks is on average lighter than the meson made from different quarks. This nontrivial inequality, however, does not tell us by how much these meson masses differ, a question which we are able to answer analytically in some cases and numerically for all quark masses. Also, while there is no reason to doubt this important inequality, as was pointed out in the original articles themselves,

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the inequality is not completely rigorous. We find it satisfying to be able to study how the inequality is realized in a simplified version of QCD. To our knowledge, the QCD mass inequality has not been previously shown to be satisfied in such a concrete manner. Here, the inequality is applied to ff, bf, and bb cases. Except in the bf case, the inequality is necessarily an equality when the constituent quark masses are the same. In the bf case, such needs not be the case and indeed we find that it is always a strict inequality in the bf case.

The paper is organized as follows. In Sec. II, we briefly review the integral equation satisfied by the wave function of the meson in the large-N_c limit, partly to fix the notation. In Sec. III, we analytically study the static properties of mesons when the quark masses are large. We further formulate the bound-state equation as a linear eigenvalue problem for general values of the quark masses and analyze the problem numerically. In Sec. IV, we analyze some matrix elements and see how confinement and asymptotic freedom is realized in the bf case. We conclude with a brief discussion.

II. WAVE FUNCTION OF MESONS

In this section, we briefly summarize the equations meson wave functions satisfy and some basic properties of the solutions in the ff [5], bf [10], and bb [9] cases. The classical Lagrangian of QCD coupled to fermions and bosons is

$$-\mathcal{L} = \tfrac{1}{4}\mathrm{tr}(F_{\mu\nu}^2) + \sum_f \bar{\psi}_f(\not{D} + m_f)\psi_f + \sum_b (|D\phi_b|^2 + m_b^2|\phi_b|^2). \tag{2.1}$$

Both fermions and bosons are in the fundamental representation. We shall refer to the fields ψ_f, ϕ_b as "quarks." We fix the gauge to the light-cone gauge $A_- = A^+ = 0$, where $a^\pm = a_\mp \equiv (a^1 \pm a^0)/\sqrt{2}$. Light-cone gauge has the advantage that there are no gluon self-interactions in 1+1 dimensions.

We take the large-N_c limit by letting N_c go to infinity keeping g^2N_c and the quark masses to be of $O[(g^2N_c)^0]$. There are contributions of $O[(g^2N_c)^0]$ to the quark propagators which compete with the classical contribution. These contributions may be incorporated by solving the Schwinger-Dyson equations recursively to obtain the full propagators as

$$\begin{aligned} S(p;m) &= \left[i\not{p} + i\frac{g^2N_c}{2\pi}\left(\frac{\mathrm{sgn}(p_-)}{\lambda_-} - \frac{1}{p_-}\right)\gamma^+ + m\right]^{-1} \\ &= \left[-i\not{p} - i\frac{g^2N_c}{2\pi}\left(\frac{\mathrm{sgn}(p_-)}{\lambda_-} - \frac{1}{p_-}\right)\gamma^+ + m\right] \\ &\quad \times D(p;m), \\ D(p;m) &= \left[p^2 + m^2 + \frac{g^2N_c}{\pi}\left(\frac{|p_-|}{\lambda_-} - 1\right)\right]^{-1}. \end{aligned} \tag{2.2}$$

We note that the quantum correction to the mass is identical both for the fermionic and the bosonic quarks. In this work, we will use an infrared cutoff λ_- which seems to be more convenient for deriving the physical properties of the mesons. Other infrared regularizations may be used, but of course do not affect physical results [6,7].

A meson is formed as a quark-antiquark bound state and its wave function satisfies

$$\mu^2\tilde{\varphi}(x) = \left(\frac{\beta_a - 1}{x} + \frac{\beta_b - 1}{1-x}\right)\tilde{\varphi}(x) - P\int_0^1 \frac{dy}{(y-x)^2}\tilde{U}(x,y)\tilde{\varphi}(y) \equiv \tilde{H}\tilde{\varphi}(x), \tag{2.3}$$

where

$$\tilde{U}(x,y) \equiv \begin{cases} 1 & ff, \\ \frac{x+y}{2x} & bf, \\ \frac{(x+y)(2-x-y)}{4x(1-x)} & bb. \end{cases} \tag{2.4}$$

Here, $\beta_{f,b} \equiv m_{f,b}^2\pi/(g^2N_c)$, namely the mass squared of the quarks counted in the units of the QCD scale. These equations may be obtained by summing the graphs of the ladder type. Here and below, we refer to the solution of the Bethe-Salpeter equation, $\tilde{\varphi}(x)$, as the wave function of the meson, in analogy with the nonrelativistic case and in accordance with the previous literature. $x \equiv p_-/r_-$ is the momentum fraction carried by the quark and $1-x$ by the antiquark in the infinite momentum frame. In the $1/N_c$ expansion, sea quarks are suppressed by $1/N_c$ and gluons have no physical degrees of freedom so that all the momentum of the meson is carried by the quark and the antiquark. $P\int$ denotes the principal part integral defined by

$$P\int dx f(x) \equiv \frac{1}{2}\int dx\left[f(x+i\epsilon) + f(x-i\epsilon)\right]_{\epsilon\to 0}. \tag{2.5}$$

In the bf and the bb cases, the above meson equations are not Hermitian with respect to the standard measure $\int_0^1 dx$ and we shall often use the conjugated equation

$$\mu^2\varphi(x) = \left(\frac{\beta_a - 1}{x} + \frac{\beta_b - 1}{1-x}\right)\varphi(x) - P\int_0^1 \frac{dy}{(y-x)^2}U(x,y)\varphi(y) \equiv H\varphi(x), \tag{2.6}$$

$$U(x,y) = \begin{cases} 1 & ff, \\ \frac{x+y}{2\sqrt{xy}} & bf, \\ \frac{(x+y)(2-x-y)}{4\sqrt{x(1-x)}\sqrt{y(1-y)}} & bb. \end{cases} \tag{2.7}$$

The appropriate conjugation is

$$\varphi(x) = \tilde{\varphi}(x) \times \begin{cases} 1 & ff, \\ \sqrt{x} & bf, \\ \sqrt{x(1-x)} & bb. \end{cases} \tag{2.8}$$

A formula we find useful is

$$(\phi, H\psi) = \int_0^1 dx \left(\frac{\beta_a}{x} + \frac{\beta_b}{1-x} + J(x)\right) \overline{\phi(x)}\psi(x) + \frac{1}{2}\int_0^1 dx \int_0^1 \frac{dy}{(y-x)^2} U(x,y)[\overline{\phi(x)} - \overline{\phi(y)}] \times [\psi(x) - \psi(y)], \tag{2.9}$$

where

$$J(x) \equiv \begin{cases} 0 & ff, \\ -\frac{1}{1-x}\left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right) & bf, \\ -\frac{1}{x(1-x)} + \frac{\pi}{4\sqrt{x(1-x)}} & bb. \end{cases} \tag{2.10}$$

Note that the expression no longer involves the principal part integral. From this equation, it immediately follows that in the ff case when the bare quark masses are positive ($\beta_a, \beta_b \geq 0$), the meson mass is positive [5]. In the bf, bb cases, it is clear that the meson mass is positive when $\beta_a, \beta_b > 1$. Another property that may be obtained from this formula is that for the same quark-antiquark masses, the mass of the meson is the lightest for the bb case, heaviest in the ff case. Here and below, we often refer to the bare masses of the quarks $m_{f,b}$ as quark masses. It should perhaps be commented here that the quark masses $m_{f,b}$, or equivalently $\beta_{f,b}$, are not directly physically observable and that they receive quantum corrections. However, we note that the quantum corrections to both the fermionic and the bosonic quark are identical so that it is consistent to compare the quark masses on the same footing.

The wave function vanishes at the boundary as $\varphi(x) \sim x^{\gamma_{f,b}}$ where x denotes the momentum fraction of the fermionic or the bosonic constituent. $\gamma_{f,b}$ is determined by the equations

$$\begin{array}{ll} \beta_f - 1 + \pi\gamma_f \cot(\pi\gamma_f) = 0 & f, \\ \beta_b - 1 - \pi\gamma_b \tan(\pi\gamma_b) = 0 & b. \end{array} \tag{2.11}$$

Since the meson equations are Hermitian, the spectrum is real. The widths of the mesons are of $O(N_{\text{flavors}}/N_c)$ and are suppressed in the large-N_c limit. For higher mass states, the approximate wave functions and the respective masses are

$$\varphi(x) \simeq \sqrt{2}\sin \pi k x, \quad (\text{meson mass})^2 \simeq g^2 N_c \pi k, \quad k = 1, 2, 3, \ldots, \tag{2.12}$$

resulting in a linear Regge-type trajectory for large k. Here, k labels the kth lightest meson bound state formed from the quark-antiquark pair. We note that the relativistic effects are important in this problem, even qualitatively; in the nonrelativistic case, the bound-state mass squared behaves as $k^{2/3}$ rather than linearly, with respect to k.

III. STATIC PROPERTIES OF MESONS

In this section, we obtain and analyze the static properties of mesons using both analytical and numerical methods.

A. Heavy quarks

When the quark masses are heavy, we may use the variational method to obtain the meson wave function analytically. This method was previously applied to the ff case when the quarks and antiquarks have equal mass [6]. Here, we extend the analysis to include the case when the quark and the antiquark have different masses and further generalize this to the bf and bb cases. We refer to the "heavy quark" when the quark mass is large compared to the QCD scale ($m \gg g\sqrt{N_c} \Leftrightarrow \beta \gg 1$).

Using the trial function

$$\varphi_0^{\text{HQ}}(x) = \left(\frac{c}{\pi}\right)^{1/4} e^{-c(x-x_0)^2/2} \tag{3.1}$$

we obtain

$$(\varphi_0^{\text{HQ}}, H\varphi_0^{\text{HQ}}) = \frac{\beta_a}{x_0} + \frac{\beta_b}{1-x_0} + \left(\frac{\beta_a}{x_0^3} + \frac{\beta_b}{(1-x_0)^3}\right)\frac{1}{2c} + \sqrt{\pi c} + J(x_0) + O(\beta c^{-2}). \tag{3.2}$$

Varying with respect to x_0 and c we obtain

$$\begin{aligned} x_0 &= \frac{\sqrt{\beta_a}}{\sqrt{\beta_a} + \sqrt{\beta_b}}[1 + O(c^{-1})] \\ &= \frac{m_a}{m_a + m_b}[1 + O(c^{-1})], \\ c &= (\pi\beta_a\beta_b)^{-1/3}(\sqrt{\beta_a} + \sqrt{\beta_b})^{8/3}[1 + O(c^{-1})]. \end{aligned} \tag{3.3}$$

Consequently,

$$\mu = (\sqrt{\beta_a} + \sqrt{\beta_b})\left[1 + \frac{3}{4}\pi^{1/3}(\beta_a\beta_b)^{-1/6}(\sqrt{\beta_a} + \sqrt{\beta_b})^{-2/3} + w_s\right], \tag{3.4}$$

where

$$w_s = \begin{cases} 0 & ff, \\ -\frac{1}{2\sqrt{\beta_b}}\left(\frac{1}{\sqrt{\beta_a}} - \frac{1}{\beta_a^{1/4}(\sqrt{\beta_a} + \sqrt{\beta_b})^{1/2}}\right) & bf, \\ -\frac{1}{2\sqrt{\beta_a\beta_b}} + \frac{\pi}{4(\beta_a\beta_b)^{1/4}(\sqrt{\beta_a} + \sqrt{\beta_b})} & bb. \end{cases} \tag{3.5}$$

Here, we have obtained a clear physics picture of the meson as a bound state of the quark and antiquark with the momentum distributed in proportion to their masses. As the $q, \bar{q}$ masses become larger, the momentum distribution becomes narrower as $\beta^{-1/3}$ and this simple constituent quark picture becomes more accurate.

Since we are considering different masses for the quark and the antiquark, it may seem we should analyze the effect of including a variational function that is asymmetric with respect to the center of the wave function, such as $(x-x_0)\exp[-c(x-x_0)^2/2]$. It can be immediately shown that to the order we considered above, including this function in the variational problem does not change the results at all.

To systematically obtain the wave function when the quark masses are heavy we may use an orthonormal basis for the wave functions:

$$\varphi_n^{\rm HQ} \equiv \left(\frac{c}{\pi}\right)^{1/4} \frac{1}{\sqrt{n!}} H_n(\sqrt{2c}(x-x_0))e^{-c(x-x_0)^2/2}, \quad (3.6)$$

where H_n is the nth Hermite polynomial defined by

$$e^{tz-t^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(z). \quad (3.7)$$

To determine x_0, c to leading order, as in (3.3), we need the matrix elements of the Bethe-Salpeter equation to $O(\beta^{1/3})$. To this order, the statistics is unimportant. In all the three ff, bf, and bb cases, the matrix elements of H are

$$(\varphi_m^{\rm HQ}, H\varphi_n^{\rm HQ}) = \begin{cases} \frac{\beta_a}{x_0} + \frac{\beta_b}{1-x_0} + \frac{(2n+1)}{2c}\left(\frac{\beta_a}{x_0^3} + \frac{\beta_b}{(1-x_0)^3}\right) + \sqrt{\pi c}\left[1 - \sum_{k=1}^{n}(-1)^k \frac{n!(2k-3)!!}{(k!)^2(n-k)!}\right] + O(1), & m = n, \\ \frac{1}{(2c)^{(m-n)/2}}\sqrt{\frac{m!}{n!}}\left[\frac{\beta_a}{x_0^{m-n+1}} + \frac{\beta_b}{(1-x_0)^{m-n+1}}\right] & \\ \quad -\sqrt{\pi c}\sum_{k=0}^{n}(-1)^{n-k}\frac{\sqrt{m!n!}(m+n-2k-3)!!}{k!(n-k)!(m-k)!} + O(1), & m > n, m-n \equiv 0 \bmod 2, \\ \frac{1}{(2c)^{(m-n)/2}}\sqrt{\frac{m!}{n!}}\left[-\frac{\beta_a}{x_0^{m-n+1}} + \frac{\beta_b}{(1-x_0)^{m-n+1}}\right] + O(1), & m > n, m-n \equiv 1 \bmod 2. \end{cases} \quad (3.8)$$

The one-dimensional variational problem just using $\varphi_0^{\rm HQ}(x)$ is not quite enough to determine the lowest meson mass to the order in $\beta^{-2/3}$ given above in (3.4). It turns out, however, that only when we consider trial functions $\varphi_n^{\rm HQ}(x)$, with $n \geq 4$ that the meson mass is affected. When we include the variational functions with $n \geq 4$, we find that the physical effect is to decrease the numerical coefficient of the second term in the square brackets of (3.4) by less than 1%. The last term w_s is not affected at all. The smallness of the corrections is not surprising; while the contribution of more complicated variational functions are not suppressed by powers of $\beta^{-2/3}$, they are suppressed numerically just as the contribution from the higher excited states to the ground-state energy are suppressed in perturbation theory in quantum mechanics. This suppression is strong since there is a contribution to the meson mass only when we include variational functions with $n \geq 4$. A couple of comments are in order. First, the boundary conditions at $x = 0,1$ are not exactly satisfied in this approach, so that the expansion is not completely systematic. However, since the value of the wave function at the boundaries is exponentially suppressed as $\exp(-{\rm const}\times\beta^{2/3})$, the approximation should be good when the quarks are heavy. Second, the growth of the matrix elements with the increase in the dimension of the variational space as seen in (3.8) indicates that the expansion in $\beta^{-2/3}$ is an asymptotic series. This situation is quite common in quantum theories [13]. Further analyzing the problem through this variational approach does not change the physics picture obtained above and we shall not pursue this further here.

B. Variational methods using powers of the momentum fraction

An effective method, both analytically and numerically, for obtaining approximate solutions to the integral equation for the meson wave function, (2.3) and (2.6) is to reduce the problem to a finite dimensional diagonalization problem. An example of a variational approach particularly effective in the case of heavy quarks was given above. When the quark masses are light, a variational scheme that is effective is to use a basis

$$v_{2k} = [x(1-x)]^{\gamma+k},$$
$$v_{2k+1} = [x(1-x)]^{\gamma+k}(1-2x), \quad k = 0,1,2,\ldots . \quad (3.9)$$

This scheme was previously used in the ff case [14] and more recently in the investigations of the Schwinger model [15]. Below, we treat the ff, bb cases when the quark and the antiquark masses are identical ($\beta_a = \beta_b = \beta$). In this case, the integral equation (2.6) is reduced to finding the solution of the generalized eigenvalue problem

$$\det(H_{ij} - \mu^2 N_{ij}) = 0, \quad i,j = 0,1,2,\ldots, \quad (3.10)$$

where

$$H_{ij} \equiv (v_i, Hv_j) \equiv K_{ij} + U_{ij}, \quad N_{ij} \equiv (v_i, v_j), \quad (3.11)$$

$$N_{2k,2l} = B(2\gamma + k + l + 1, 2\gamma + k + l + 1), \quad N_{2k+1,2l+1} = \frac{B(2\gamma + k + l + 1, 2\gamma + k + l + 1)}{4\gamma + 2k + 2l + 3}, \tag{3.12}$$

$$K_{2k,2l} = (\beta - 1)B(2\gamma + k + l, 2\gamma + k + l), \quad K_{2k+1,2l+1} = \frac{(\beta - 1)}{4\gamma + 2k + 2l + 1}B(2\gamma + k + l, 2\gamma + k + l), \tag{3.13}$$

$$U_{2k,2l} = \begin{cases} \frac{(\gamma+k)(\gamma+l)}{2(2\gamma+k+l)} B(\gamma + k, \gamma + k)B(\gamma + l, \gamma + l) & ff, \\ \frac{8kl+(8\gamma+1)(k+l)+8\gamma^2+2\gamma}{4(2\gamma+k+l)} B(\gamma + k + 1/2, \gamma + k + 1/2)B(\gamma + l + 1/2, \gamma + l + 1/2) & bb, \end{cases} \tag{3.14}$$

$$U_{2k+1,2l+1} = \begin{cases} \frac{(\gamma+k)(\gamma+l)}{2(2\gamma+k+l)(2\gamma+k+l+1)} B(\gamma + k, \gamma + k)B(\gamma + l, \gamma + l) & ff, \\ \frac{4kl+(4\gamma+1)(k+l)+4\gamma^2+2\gamma}{2(2\gamma+k+l)(2\gamma+k+l+1)} B(\gamma + k + 1/2, \gamma + k + 1/2)B(\gamma + l + 1/2, \gamma + l + 1/2) & bb, \end{cases} \tag{3.15}$$

H_{ij},$N_{ij} = 0$ when $i + j \equiv 1 \bmod 2$. The boundary conditions mentioned in the previous sections require that $\gamma > 0$ which guarantees that the meson masses and the matrix elements given above are finite.

It is illuminating to study the simplest one-dimensional case analytically. In this case, the meson mass squared is

$$\langle H \rangle_\gamma \equiv \frac{H_{00}}{N_{00}} = (\beta - 1)\frac{4\gamma + 1}{\gamma} + \begin{cases} \frac{\gamma}{4} \frac{B^2(\gamma,\gamma)}{B(2\gamma+1,2\gamma+1)} & ff, \\ (\gamma + \frac{1}{4}) \frac{B^2(\gamma+1/2,\gamma+1/2)}{B(2\gamma+1,2\gamma+1)} & bb. \end{cases} \tag{3.16}$$

We may vary $\gamma(> 0)$ to obtain an upper bound on the meson mass. In the ff case, when the quark mass is positive ($\beta > 0$) the problem is well behaved and it was already established that the meson mass is positive using (2.9). When $\beta = 0$ the meson mass is zero. When $\beta < 0$, $\langle H \rangle_\gamma$ is not only negative but is unbounded from below. The situation parallels that of the $1/r^2$ potential in quantum mechanics; when the coupling is smaller than the critical value, the problem becomes ill behaved. In the bb case, this critical coupling is at $\beta = 1$; when $\beta < 1$, the meson mass is unbounded from below and is not acceptable physically. As a result, the bosonic quark mass needs to satisfy the condition $\beta \geq 1$, which we shall adopt from now on. When $\beta \geq 1$ the meson mass is positive and the problem is well behaved. Using this variational approach, we may establish that $0 < \mu^2 < \pi^2/4$ for $\beta = 1$. Consequently, it is not possible to obtain a massless bb meson in this model. Combined with the QCD mass inequality to be explained below, this excludes the possibility of obtaining massless mesons with boson constituents.

We may use this formulation for the numerical computation for the spectrum. In the bf case, it is natural to choose γ to satisfy the boundary behavior (2.11). This formulation is most convenient for the ff case with light quark masses $\beta \lesssim 1$.

The basis we have chosen is not orthonormal. An orthonormal basis is

$$v_i' = x^{\gamma_0}(1 - x)^{\gamma_1} G_n(2\gamma_0 + 2\gamma_1 + 1, 2\gamma_0 + 1; x), \tag{3.17}$$

where $G_n(a, b; x)$ denotes the nth Jacobi polynomial. While we may compute the matrix elements of the diagonalization problem analytically using the orthogonal basis, we have not found a compact general formula for the matrix elements as in (3.10) and we shall not use this approach here.

C. Multhopp's method

Another numerical method we shall employ, which is sometimes called Multhopp's wing dynamics, is valid for all quark masses in the three ff, bf, and bb cases. This method has previously been applied to the ff case with mesons formed from quarks with identical masses in [16]. Here, we apply this method to a meson made from a quark and an antiquark of different mass in the ff, bf, and bb cases. We approximate the meson wave function by

$$\tilde{\varphi}(x) = \sum_{n=1}^{K} a_n \sin n\theta, \quad x \equiv \frac{1 + \cos\theta}{2}. \tag{3.18}$$

Using integration by parts, the integral equation (2.3) for the meson wave function reduces to

$$\mu^2 \tilde{\varphi} = \sum_{n=1}^{K} M_n(\theta) a_n, \tag{3.19}$$

$$M_n(\theta) \equiv 2\left(\frac{\beta_a - 1}{1 + \cos\theta} + \frac{\beta_b - 1}{1 - \cos\theta}\right)\sin n\theta + 2\pi\left(\frac{n \sin n\theta}{\sin\theta} + B_n(\theta)\right), \tag{3.20}$$

where

$$B_n(\theta) \equiv \begin{cases} 0 & ff, \\ \frac{\cos\theta\,\cos n\theta}{2(1+\cos\theta)} & bf, \\ \frac{-\cos\theta\,\cos n\theta+\delta_{n,1}/8}{\sin^2\theta} & bb. \end{cases} \quad (3.21)$$

We solve the equation by evaluating it at the points $\theta = \theta_n$, $n = 1, 2, \ldots, K$ where we used the notation $\theta_j \equiv \pi j/(K+1)$. Using the relation

$$\sum_{k=1}^{K} \sin\theta_{nk} \sin\theta_{mk} = \frac{K+1}{2}\delta_{nm} \quad (3.22)$$

the problem reduces to a finite-dimensional matrix eigenvalue problem

$$\sum_k A_{nk} a_k = \mu^2 a_n,$$

$$A_{mn} \equiv \frac{2}{K+1} \sum_{k=1}^{K} \sin\theta_{mk} M_n(\theta_k). \quad (3.23)$$

We now comment on the relative merits of the numerical schemes we used. First, we note that when solving the problem numerically, it is beneficial both in terms efficiency and in terms of accuracy to be able to compute the matrix elements of the diagonalization problem analytically. If one does not insist on this property, we may use (2.9) to compute the matrix elements by numerically performing the double integration for any basis satisfying the boundary conditions. Multhopp's method is robust in that it tends to produce matrix elements of order 1 even when K is large and does not give rise to almost singular matrices numerically. Also, the diagonalization problem is well behaved for all values of the quark masses. Except in the region of light quark mass ($B \lesssim 1$) the convergence is sufficiently fast; $K = 200$ is more than enough to obtain the meson mass to eight digits. $K = 800$ is a reasonable computations task requiring the order of an hour of CPU time on a current workstation. For the ff case when the quarks are light and are of equal mass, the basis involving powers of the momentum fraction is useful. A ten-dimensional diagonalization suffices to obtain the spectrum to five significant digits. However this approach tends to lead to almost singular matrices numerically as we increase the dimension of the variational space. It may be possible to overcome this problem by a clever choice of basis. The convergence is slower than the Multhopp's method except in the ff case with light quark masses of equal mass.

D. The meson spectrum and the QCD mass inequality

In all the three ff, bf, and bb cases, the spectrum leads to a linear Regge trajectory for the higher mass states as we saw in (2.12). Here, we plot the examples for three sets of parameters: (i) $\beta_a = \beta_b = 1$; (ii) $\beta_a = 1$, $\beta_b = 10$; and (iii) $\beta_a = \beta_b = 10$ for ff, bf, and bb cases in Fig. 1. Here and below, unless otherwise stated, the numerical errors are too small to be visible on the plots.

Let us now analyze how the QCD inequality is realized in QCD in 1+1 dimensions in the large-N_c limit. The QCD mass inequality was originally shown for the four-dimensional case [11,12]. At a formal level, the inequality may be extended to the ff, bb, and bf cases in two dimensions. Given the analytical form of the masses when the quark masses are heavy, we may analyze how the QCD mass inequality is satisfied in this case. Comparing the masses, we obtain

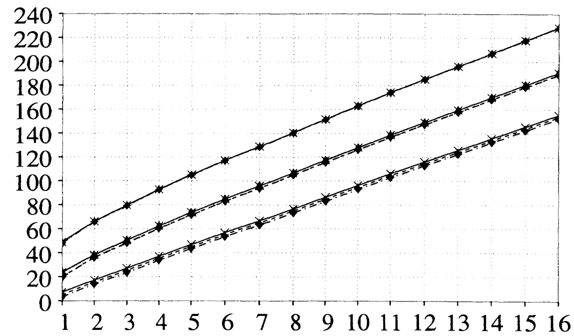

FIG. 1. The spectrum of mesons for the cases (i) $\beta_a = \beta_b = 1$; (ii) $\beta_a = 1$, $\beta_b = 10$; and (iii) $\beta_a = \beta_b = 10$. The horizontal axis is labeled by the number when we count from the lightest meson ("principal quantum number") and the vertical axis is labeled by the meson mass squared counted in the units of the QCD scale ($g^2 N_c/\pi$). The spectrum for the ff, bf, and bb cases are joined by solid, dashed, and dot-dashed lines, respectively. The ff, bf, and bb cases are barely distinguishable for (i) and indistinguishable in the other cases.

$$2\mu_{ab} - (\mu_{aa} + \mu_{bb}) = \frac{3\pi^{1/3}}{2(\beta_a\beta_b)^{1/6}} \left[\left(\sqrt{\beta_a} + \sqrt{\beta_b}\right)^{1/3} - 2^{2/3}\left(\beta_a^{1/6} + \beta_b^{1/6}\right) \right] + O(\beta^{-1/2}) \geq 0 \quad (3.24)$$

to this order in all three ff, bf, and bb cases. We see that the mass difference arises not at leading order but at next order since the leading order only contains the rest masses of the quarks. Furthermore, we see that the mass inequality can be an equality to this order only if the masses of the quarks are the same. In the ff and bb cases, this inequality is an exact equality when the masses of the bound quark and antiquark are the same ($\beta_a = \beta_b$) since the mesons being compared can be the same meson. When applying the mass inequality to the bf case, this is not the case so that it can be and is a proper inequality. In the bf case μ_{ab} in the inequality is the mass of bf fermionic meson and the μ_{aa}, μ_{bb} are the masses of the ff, bb mesons. When $\beta_a = \beta_b (\equiv \beta)$, we may compute to next order and obtain

$$2\mu_{ab} - (\mu_{aa} + \mu_{bb}) = \frac{1}{\sqrt{\beta}} \left(\sqrt{2} - 1 - \frac{\pi}{8}\right) \sim \frac{0.022}{\sqrt{\beta}} > 0. \quad (3.25)$$

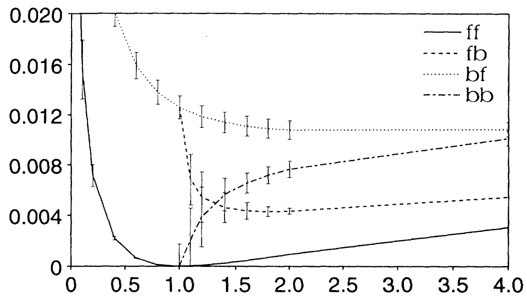

FIG. 2. The relative QCD mass inequality for $\beta_a = 1$. fb (bf) means that β_a is the mass of the fermionic (bosonic) quark.

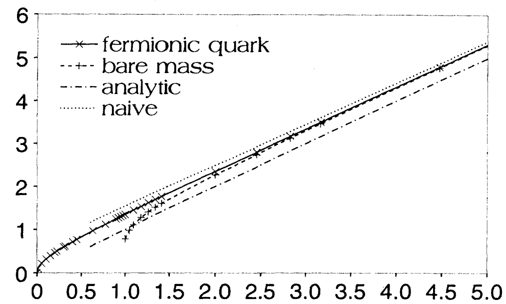

FIG. 4. The constituent quark masses of the fermion and the boson quarks computed from the meson masses compared against the naive quark mass, $\sqrt{\beta_{f,b}}$. "Analytic" refers to the formula derived analytically from (3.4).

As mentioned above, since the expansion in $\beta^{-2/3}$ is an asymptotic one, this analytic derivation of the inequality is not rigorous and perhaps should be considered illustrative. The numerical results below clearly show that the inequality in the bf case is a proper inequality.

The QCD mass inequality may be analyzed numerically for arbitrary values of the quark mass. We plot the normalized mass inequality $[\mu_{ab} - (\mu_{aa} + \mu_{bb})/2]/\mu_{ab}$, which is dimensionless, against β_b in Figs. 2 and 3 for $\beta_a = 1$ and 10, respectively. We immediately see that the QCD mass inequality is a proper inequality in the bf case. The values for the normalized mass inequality at $\beta = 0$, while not visible on the plots, are finite and can go up to 0.3 and 0.1 for the $\beta = 1$ and 10 cases, respectively. Also, even though the relative mass inequality is increasing as we increase β_b, at some point, the relative inequality starts to decrease and is never more than 0.012 and 0.002 for $\beta_b > \beta_a$ in the $\beta = 1$ and 10 cases, respectively. For $\beta = 1$ the numerical data can have appreciable errors as shown on the plots. The errors were crudely estimated as follows. Since the numerical data have not completely converged, we linearly extrapolated the finite-dimensional results by using the analytically known $\beta_a = \beta_b = 0$ ff case as a guide. The error in the extrapolation was estimated by the statistical error in the extrapolation when various sets of data were taken.

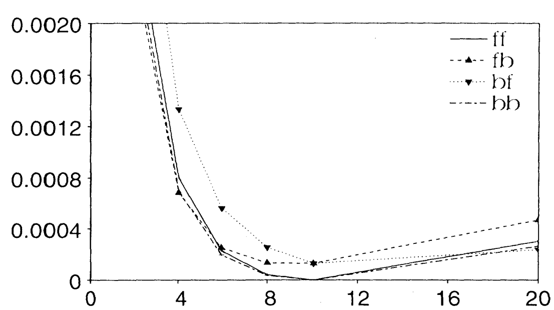

FIG. 3. The relative QCD mass inequality for $\beta_a = 10$. fb (bf) means that β_a is the mass of the fermionic (bosonic) quark.

One property that strikes the eye is that the QCD mass inequality is surprisingly small, even when the quark masses are of order 1 in QCD scale. Here, we remind the reader that the QCD scale is taken to be $g^2 N_c/\pi$, which is a conservative choice when we consider that the Regge slope parameter is larger by a factor of π^2 as in (2.12). This near saturation of the inequality is an indication of how well the constituent quark picture works in this model. This is in agreement with the general argument concerning the validity of the constituent quark picture in the large-N_c limit [17]. It is naturally of interest, then, to compare the constituent quark mass against the naive or the bare quark mass, which is done in Fig. 4. In this figure, $\mu_{aa}/2$ is plotted against the bare quark mass. The constituent quark mass is close neither to the bare quark mass m_q nor the quantum corrected mass $\sqrt{m_q^2 - g^2 N_c/\pi}$, but is larger than m_q by an amount of $O(1)$ when the mass is of the order of the QCD scale. This shows that even though the constituent quark picture seems valid, the constituent quark is not the naive quark but a "dressed" quark. When the quark mass is heavy, the difference between the constituent quark mass and the naive quark mass may be understood from analyzing the nonrelativistic linear potential.[1]

IV. $q\bar{q}$ HADRON VERTEX, CONFINEMENT, AND ASYMPTOTIC FREEDOM

In this section, we analyze some other physical properties of the "bf meson." The results are somewhat similar to the ff case [6] and the bb case [9]. In the bf case, we should note that unlike the ff and the bf cases, the "meson" is a fermion.

[1]We would like to thank H. Kawai for pointing this out.

First, let us analyze the $q\bar{q}$ scattering matrix. The equation satisfied by the $q\bar{q}$ scattering matrix is essentially that of the equation satisfied by the meson wave function (2.3). First we make use of the Lorentz structure of the scattering matrix and reduce it to its essential scalar component by defining $T_{\alpha,\beta}(p,p';r) = (\gamma_-)_{\alpha\beta}T(p,p';r)$. The reduced matrix element satisfies the equation

$$T(p,p';r) = ig^2L(p_-,p'_-) + 2ig^2N_c\int\frac{d^2k}{(2\pi)^2}D(k)\hat{S}(k-r)L(k_-,p_-)T(k,p';r). \tag{4.1}$$

Here we defined $L(x,y) \equiv (x+y)/(x-y)^2$ and we denoted by $\hat{S}(p)$ the component of the fermion propagator that contributes to the scattering matrix $\gamma_-S(p)\gamma_- \equiv 2\gamma_-\hat{S}(p)$. This equation may be solved in a manner closely related to that of the meson equation (2.3). Define $\phi(p_-,p;r)$ as

$$\phi(p_-,p;r) \equiv \int dp_+D(p)\hat{S}(p-r)T(p,p';r) \tag{4.2}$$

so that

$$T(p,p';r) = ig^2L(p_-,p'_-) + ig^2N_c\int\frac{dk_-}{2\pi}L(k_-,p_-)\phi(k_-,p;r). \tag{4.3}$$

p_+ in (4.2) may be integrated out explicitly to obtain the following equation for ϕ:

$$\tilde{H}\phi(x,y;r) = \mu^2\phi(x,y;r) + \frac{\pi^2}{N_cr_-x}L(x,y), \tag{4.4}$$

where $\tilde{H}$ was defined in (2.3). We may obtain the solution to this equation using the meson wave functions as

$$\phi(x,x';r) = \frac{\pi g^2}{r_-}\sum_k\frac{1}{r^2-r_k^2} \times\int_0^1 dy\tilde{\varphi}_k(x)\overline{\tilde{\varphi}_k(y)}L(x',y). \tag{4.5}$$

Provided that $\{\tilde{\varphi}_k\}$ satisfies the following properties.

(1) The "Schrödinger equation" for the meson:

$$\tilde{H}\tilde{\varphi}_k(x) = \mu_k^2\tilde{\varphi}_k(x). \tag{4.6}$$

(2) Completeness:

$$x\sum_k\tilde{\varphi}_k(x)\overline{\tilde{\varphi}_k(y)} = \delta(x-y). \tag{4.7}$$

(3) Orthogonality:

$$\int_0^1 dxx\overline{\tilde{\varphi}_k(x)}\tilde{\varphi}_l(x) = \delta_{kl}. \tag{4.8}$$

Corresponding properties in the ff case have been proven rigorously in [18].

Using the expression for the reduced scattering matrix (4.5) we may reconstruct the scattering matrix as

$$T(x,x';r) = \frac{ig^2}{r_-}L(x,x') + \sum_k\frac{2ir_-}{r^2-r_k^2}\Phi_k(x)\overline{\Phi_k(x')}, \tag{4.9}$$

where the q-$\bar{q}$-meson vertex $\Phi_k(x)$ is defined as

$$\Phi_k(x) = \frac{g^2}{2r_-}\left(\frac{N_c}{\pi}\right)^{1/2}\int_0^1 dyL(x,y)\tilde{\varphi}_k(y). \tag{4.10}$$

We see that the $q\bar{q}$ scattering matrix may be broken up into the q-$\bar{q}$-meson vertex and the single-particle poles corresponding to the mesons which are solutions to the bound-state equation. All the intermediate physical states to this order in $1/N_c$ are mesons and we see that the colored particles are indeed confined. At higher orders in $1/N_c$, more complicated intermediate states such as two-particle cuts will appear.

To investigate the properties of the meson, we compute some matrix elements. Define a fermionic operator

$$F_\mu^{ab} \equiv \sum_i\phi_a^{i\dagger}\gamma_\mu\psi_b^i, \tag{4.11}$$

where i is the color index. Denoting the kth meson state as h_k, we may obtain some matrix elements such as

$$\begin{aligned}\langle 0|F_-^{ab}|h_k\rangle &= -2i\left(\frac{N_c}{\pi}\right)^{1/2}\int_0^1 dx\tilde{\varphi}_k(x),\\ \langle 0|F_+^{ab}|h_k\rangle &= 2\left(\frac{N_c}{\pi}\right)^{1/2}\frac{m_f}{r_-}\int_0^1 dx\frac{\tilde{\varphi}_k(x)}{x}.\end{aligned} \tag{4.12}$$

Some correlation functions of these fermionic "currents" may also be obtained. Define the correlation function

$$M_{\mu\nu}(q) \equiv \int d^2xe^{iqx}\left\langle 0\middle|TF_\mu^{ab\dagger}(x)F_\nu^{ab}(0)\middle|0\right\rangle, \tag{4.13}$$

$$M_{--}(q) = \frac{q_-}{q^2}\sum_k\left(\int_0^1 dx\tilde{\varphi}_k(x)\right)^2. \tag{4.14}$$

The behavior of this element in the deep-inelastic region is

$$M_{--}(q) \sim \frac{q_-}{q^2}\ln\frac{q^2}{m^2} \quad\text{for } q^2 \gg m^2, g^2N_c. \tag{4.15}$$

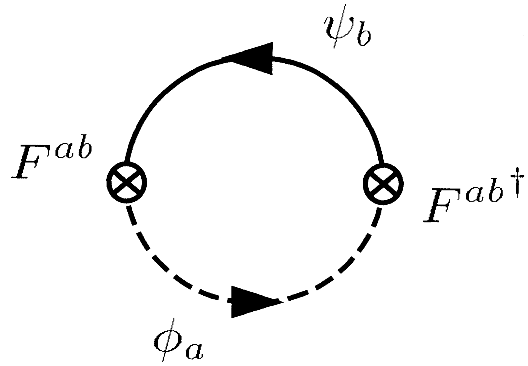

FIG. 5. Correlation function of the fermionic currents in the free theory.

This agrees with the correlation function obtained in the free theory by computing the Feynman graph in Fig. 5. This is, of course, none other than the statement of asymptotic freedom in this case. For comparison, the current matrix elements for the ff and the bb case are

$$M_{--}(q) \sim \begin{cases} \frac{q_-}{q^2} & ff \\ \frac{q_-^2}{q^2} \ln \frac{q^2}{m^2} & bb \end{cases} \quad \text{for } q^2 \gg m^2, g^2 N_c. \quad (4.16)$$

V. DISCUSSION

In particle physics and in other areas of physics, such as condensed matter, we often need to consider particles bound together by gauge interactions involving boson constituents. It is important to have a concrete example where such phenomenon is analyzed from first principles. In particular, the precise correspondence between QCD without quarks and string theory has been recently established in two dimensions [19]. It is of import to elucidate this correspondence when dynamical matter is coupled to QCD, putting the previous work on this subject [5,20] on a more rigorous footing. In this regard, it is crucial to understand in detail the properties of free strings, which are none other than the mesons in QCD. In this paper, we have analyzed the properties of bound states involving boson constituents both analytically and numerically. We believe that this concrete model should contribute to the physical understanding of bound states involving boson constituents.

There are a few intriguing aspects of our results which were not anticipated prior to computation: In mesons involving bosons, we found that it was not possible to construct massless bound states without additional interaction other than the gauge interaction. This is consistent with the result of [21] where it is found that the phase transition between the broken symmetry phase and the symmetric phase in this model is of first order. This behavior differs from the ff case where zero mass quarks produced a massless meson. It is possible that interactions may change some of these properties. Interactions that may be naturally added are the $|\phi|^4$ interaction and the Yukawa interaction. There are, in addition, interaction terms that are renormalizable in 1+1 dimensions but not in 3+1 dimensions, such as $(\bar{\psi}\psi)^2$ and any gauge-invariant scalar self-interaction terms.

In this work, we analyzed the QCD mass inequalities in 1+1 dimensions for mesons made from bosons and/or fermions. It is important to analyze this nonperturbative inequality in a model when possible. We have shown how the inequality works quantitatively in 1+1 dimensions. An interesting aspect is that the QCD mass inequality is close to being saturated; as we have seen, to at most few percent relatively. Naively, there is no reason to expect that it should be so small and this is a sign that the constituent model works well even when the quark masses are light. We need to ask if these properties are artifacts of the model we used, namely the large-N_c limit and the fact that the model is formulated in 1+1 dimensions. In the large-N_c limit, quark loops are suppressed for group-theoretical reasons and one may wonder if this is the cause for the saturation. First, even in two-dimensional QCD in the large-N_c limit, the inequality is not exactly saturated and we know of no solid argument short of a concrete calculation that shows that the inequality is close to being saturated. Furthermore, in recent studies in 1+1 dimensions, it has been shown that even when the quark loops are not suppressed for group-theoretical reasons, the lighter hadrons are very well approximated by a small definite number of partons, so that the quark loops are suppressed dynamically [22]. While it is unknown whether this feature is preserved in higher dimensions, we should keep in mind the successes of the constituent quark model [17,23].

(1+1)-dimensional QCD has served particle physics well as a testing ground for various properties of QCD. In this paper, we have made an effort to present the method for computing the wave function and the spectrum of ff, bb, and bf bound states in (1+1)-dimensional QCD in a concrete manner. We believe that this will be useful for further investigations in this field.

ACKNOWLEDGMENTS

We would like to thank Eric D'Hoker, Norisuke Sakai, and Hidenori Sonoda for numerous discussions and encouragement. We would like to thank K. Harada and M. Taniguchi for illuminating discussion and for pointing out the references in [15]. This work was supported in part by the Grant in Aid for Scientific Research from the Ministry of Education, Science and Culture and the Japan Society for the Promotion of Science for Young Scientists.

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