Charged black points in general relativity coupled to the logarithmic U(1) gauge theory

Harald H. Soleng*
Theory Division, CERN, CH-1211 Geneva 23, Switzerland
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The exact solution for a static spherically symmetric field outside a charged point particle is found in a nonlinear U(1) gauge theory with a logarithmic Lagrangian. The electromagnetic self-mass is finite, and for a particular relation between mass, charge, and the value of the nonlinearity coupling constant λ, the electromagnetic contribution to the Schwarzschild mass is equal to the total mass. If we also require that the singularity at the origin be hidden behind a horizon, the mass is fixed to be slightly less than the charge. This object is a *black point.*

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The singularity problem of Einstein's general theory of relativity has sometimes been regarded as a "crisis in physics" [1]. It is hard to accept a theory in which space-time itself breaks down and where the Riemann tensor is predicted to diverge on a singularity which can be reached along a timelike curve. In general, physicists seem to have had less trouble with analogous singularities in gauge theories. One reason might be that in gauge theories the diverging curvature tensors are curvatures not of space-time but of *internal* spaces. Yet, in a sense, these internal dimensions are just as real as the external dimensions of everyday space-time. Therefore we should feel just as embarrassed by these singularities as by the singularities of Einstein's theory. In addition, near a point charge not only does the Faraday tensor diverge, but also the electromagnetic energy-momentum tensor blows up. Thus, through the gravitational field equations, such electromagnetic singularities are also producing singularities in space-time.

Within the framework of electromagnetism, an action for a bounded field strength was proposed long ago by Born and Infeld [2]. Altshuler [3] considered nonlinear electrodynamics as a possible mechanism for inflation, and devised a Lagrange multiplier scheme for constructing nonsingular field theories. This method was later invoked to realize the *limiting curvature hypothesis* in cosmological theories [4,5]. In two-dimensional space-times it has been applied both to black holes [6] and cosmological models [7]. But nonlinear electrodynamics is not only inspired by the desire to find nonsingular field theories; Heisenberg and Euler [8] discovered that vacuum polarization effects can be simulated classically by a nonlinear theory. Also in string theory one has found effective actions describing nonlinear electromagnetism [9].

In this Brief Report I investigate the logarithmic U(1) gauge theory which is contained in the class of theories constructed by Altshuler [3]. This particular case was omitted in the analysis of nonlinear charged black holes carried out by de Oliveira [10]. While this particular theory appears to have no direct relation to superstring theory, it serves as a toy model illustrating that certain nonlinear field theories can produce particlelike solutions which can realize the *limiting curvature hypothesis* also for gauge fields.

I shall find the classical (nonlinear) electromagnetic and gravitational fields for a static charged point particle. For the electromagnetic field there are in general two invariants which need to be bounded: $I_1 \equiv F_{\alpha\beta}F^{\alpha\beta}$ and $I_2 \equiv {}^*F_{\alpha\beta}F^{\alpha\beta}$. For a static, charged point particle, the latter invariant vanishes identically. Therefore I shall only consider I_1.

The action $S = \int \mathcal{L}\sqrt{-g}d^4x$ is specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{16\pi\lambda}\left[\lambda R - \ln\left(1 + \lambda F_{\alpha\beta}F^{\alpha\beta}\right)\right], \tag{1}$$

where geometrized units [1] with $G = c = 1$ have been employed. In these units the constant λ has dimension $(\text{length})^2$. The lowest-order terms of the Lagrangian are

$$\mathcal{L} = \frac{1}{16\pi}\left[R - F_{\alpha\beta}F^{\alpha\beta} + \frac{\lambda}{2}(F_{\alpha\beta}F^{\alpha\beta})^2 + O(\lambda^2)\right]. \tag{2}$$

To second order, and when $I_2 = 0$, both the Born-Infeld [2] and the Euler-Heisenberg [8] actions can be represented by the logarithmic Lagrangian. With the action (1), the energy-momentum tensor is

$$8\pi T_{\mu\nu} = 2\left(1 + \lambda F_{\alpha\beta}F^{\alpha\beta}\right)^{-1} F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{2}g_{\mu\nu}\frac{1}{\lambda}\ln(1 + \lambda F_{\alpha\beta}F^{\alpha\beta}). \tag{3}$$

The inhomogeneous electromagnetic field equations are

$$\left[\left(1 + \lambda F_{\alpha\beta}F^{\alpha\beta}\right)^{-1} F^{\mu\nu}\right]_{;\nu} = 4\pi J^{\mu}. \tag{4}$$

The homogeneous (cyclic) equations are identities which remain unchanged.

Let us now consider a charged point particle at rest. Thus the space-time metric is given by the spherically symmetric static metric. In Schwarzschild coordinates the line element is [11]

*Electronic address: Soleng@surya11.cern.ch

$$ds^2 = C(r)^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 - C(r)^2 dt^2. \quad (5)$$

The electromagnetic vector potential is given by

$$A^\mu = \frac{V(r)}{C(r)} \delta^\mu{}_4 \quad (6)$$

relative to the natural orthonormal frame $\boldsymbol{\omega}^1 = C(r)^{-1}\mathbf{d}r$, $\boldsymbol{\omega}^2 = r\mathbf{d}\theta$, $\boldsymbol{\omega}^3 = r\sin\theta\mathbf{d}\phi$, and $\boldsymbol{\omega}^4 = C(r)\mathbf{d}t$. The only nonvanishing component of the Faraday tensor is the radial component of the electric field,

$$F_{14} = -V'(r). \quad (7)$$

With these assumptions, the electromagnetic field equation (4) takes the form

$$2\,V'(r) - 4\,\lambda\,V'(r)^3 + r\,V''(r) + 2\,\lambda\,r\,V'(r)^2\,V''(r) = 0. \quad (8)$$

The first integral is

$$V'(r) = \frac{r^2 - \sqrt{8\lambda Q^2 + r^4}}{4\lambda Q}. \quad (9)$$

By comparison with the case $\lambda = 0$, we have identified the integration constant with the charge Q. Integrating once more, we find

$$V(r) = \frac{r^3 - r\sqrt{8\lambda Q^2 + r^4}}{12\lambda Q} - \frac{\sqrt{2}Qr\ {}_2F_1(\frac{1}{4},\frac{1}{2};\frac{5}{4};\frac{-r^4}{8\lambda Q^2})}{3\sqrt{\lambda}|Q|} + C_1, \quad (10)$$

where ${}_2F_1$ is the generalized hypergeometric function and C_1 is a constant.

For small r we find a linear potential

$$V(r) = -\frac{Q}{\sqrt{2\lambda}|Q|}r + O(r^3). \quad (11)$$

A linear ultraviolet potential has also been found [12] in a Kaluza-Klein model based on a five-dimensional Lovelock theory. Apart from an irrelevant constant, which has been neglected, the Coulomb potential is the leading term for large r:

$$V(r) = \frac{Q}{r} - \frac{2}{5}\frac{\lambda Q^3}{r^5} + O(1/r^8). \quad (12)$$

With this exact solution for the electromagnetic field, we can integrate Einstein's field equations. Note that the 44-component of the Einstein tensor for the metric (5) can be written

$$G_{44} = \frac{1}{r^2}\frac{d}{dr}\{r[1 - C(r)^2]\}. \quad (13)$$

Consequently, Einstein's field equations reduce to

$$C(r)^2 = 1 - \frac{8\pi}{r}\int_0^r r'^2 T_{44}(r')dr' - \frac{2M_0}{r}. \quad (14)$$

The last term is a contribution to the Schwarzschild mass coming from a source at the origin. From now on we shall set $M_0 = 0$. We note however that a nonzero M_0 generates a more Schwarzschild-like space-time structure. The explicit form of T_{44} can now be computed from Eqs. (3) and (7). Despite the complexity of the resulting integrand, the integral can be evaluated exactly. The result (with $M_0 = 0$) is

$$C(r)^2 = 1 + \frac{5\,r^2}{18\,\lambda} - \frac{5\sqrt{8\,\lambda\,Q^2 + r^4}}{18\,\lambda} - \frac{4\sqrt{2}|Q|\ {}_2F_1(\frac{1}{4},\frac{1}{2};\frac{5}{4};-\frac{r^4}{8\lambda Q^2})}{9\sqrt{\lambda}} - \frac{r^2}{6\,\lambda}\ln\left(\frac{-r^4 + \sqrt{8\lambda Q^2 r^4 + r^8}}{4\lambda Q^2}\right). \quad (15)$$

For small r the metric coefficient is

$$C(r)^2 = 1 - \frac{\sqrt{2}\,|Q|}{\sqrt{\lambda}} + \frac{5r^2}{18\,\lambda} - \frac{r^2}{6\lambda}\ln\left(\frac{r^2}{\sqrt{2\lambda}|Q|}\right) + O(r^3). \quad (16)$$

Even though this metric seems to be well behaved at the origin, there is still a curvature singularity there; at small radii the leading order of the Kretschmann invariant is

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{8Q^2}{\lambda r^4} + O(1/r^2). \quad (17)$$

This singularity should, however, not come as a surprise; we have not attempted to limit the space-time curvature. On the other hand, this singularity is much weaker than the singularities of the conventional Reissner-Nordström and Schwarzschild space-times.

From Eq. (16) one finds that $C(r)$ changes sign near the origin if $\lambda < 2Q^2$. This means that there is a horizon at a small radius and that the model is a black hole if λ is small. If $\lambda = 2Q^2$, then the horizon is at the origin. Such an object is a pointlike black hole and we shall call these objects *black points*. This is not the first occurrence of a pointlike black hole; charged dilatonic black holes [13,14] with a dilaton coupling constant $a > 0$ also reduce to black holes with a vanishing horizon radius in the extremal case. For $a > 1$ the dilatonic black points behave physically as elementary particles [15]. In the limit $\lambda \to 0$, we expect to recover properties of the Reissner-Nordström solution and the appearance of a horizon agrees with this expectation. There is however no inner horizon for any nonzero λ.

At large radii we get the asymptotic form

$$C(r)^2 = 1 - \frac{2^{5/4}\,\Gamma(\frac{1}{4})^2\,|Q|^{3/2}}{9\sqrt{\pi}\,\lambda^{1/4}\,r} + \frac{Q^2}{r^2} + O(1/r^3), \quad (18)$$

where $\Gamma(x)$ is the *gamma function.* We note that a Schwarzschild mass has been generated by the field (another contribution to the Schwarzschild mass term can be added by assuming a point mass $M_0 \neq 0$ at the origin). It is possible that effects of this type can explain a charged particle's mass in terms of electromagnetic field energy. It is therefore of interest to see what the size of the λ coupling must be in order that this is the case. For a particle with charge Q and rest mass $M = m_0$, the electromagnetic contribution to the Schwarzschild mass is equal to the rest mass if and only if $\lambda = \lambda_0$, where

$$\lambda_0 = \frac{2\,\mu_0^4\,Q^6}{m_0^4} \tag{19}$$

and

$$\mu_0 \equiv \frac{\Gamma(\frac{1}{4})^2}{9\sqrt{\pi}} \approx 0.824033. \tag{20}$$

For $\lambda > \lambda_0$ there is a positive point mass at the center of symmetry, and if $\lambda < \lambda_0$, the central mass must be negative. A negative central mass is also found in the pure Reissner-Nordström case; here the well-known Reissner-Nordström repulsion must be caused by a genuinely negative gravitational mass.

The presence of a nonzero λ implies that the Coulomb interaction changes character at a radius $r_{\rm cr}$ where

$$V'(r)^2\big|_{r=r_{\rm cr}} = 1/\lambda. \tag{21}$$

Using the solution (9), we find that the critical radius is given by

$$r_{\rm cr} = \lambda^{1/4}|Q|^{1/2}. \tag{22}$$

At smaller scales the electromagnetic field becomes effectively r independent.

It has long been conjectured that all or nearly all of the mass of the lightest charged particle is of electromagnetic origin. If we insert the value $\lambda = \lambda_0$ with $Q = e$ and $m_0 = m_e$ (the electron mass), we find $r_{\rm cr} \approx 3\times10^{-13}$ cm. This is of the same size as the classical electron radius or around 100 MeV in energy units. This might look appealing, but the model fails because in this case the critical value of λ is about an order of magnitude larger than the size of the corresponding coupling in the Euler-Heisenberg [8] action. High-precision experiments in QED rule out such a large value of λ.

It is perhaps more natural to look for these effects at the Planck scale [16]. Indeed, there is a cosmic censorship argument that leads to λ at such a large scale. In addition to the requirement that the whole mass be generated by the field, one can also demand that the space-time singularity at the origin should not be naked. From the small-distance behavior of the metric (16), one finds that $r = 0$ is a horizon if

$$|Q| = \sqrt{\lambda/2}. \tag{23}$$

This extremal (in the sense that it is on the verge of becoming a naked singularity) solution describes a black point. Since the "point gravity" (the analogue of the "surface gravity" of a black hole) and the horizon area vanish, both the Hawking temperature and the entropy formally vanish, but for these objects the statistical description is probably inappropriate [17]. If we combine the constraint (23) with Eq. (19), we get a unique value for the mass:

$$m_0 = \mu_0 Q. \tag{24}$$

Using $Q = e/3$ by analogy with quarks, as suggested by Rosen [18], gives $m_0 \approx 5.1\times10^{-7}$ g $= 2.9\times10^{17}$ GeV.

The Planck scale plays a role in low-energy physics; in geometrized units the elementary unit of charge is $e = \sqrt{\alpha}\ell_P$. Since charge implies an electromagnetic field, and since this field must have an energy that is equivalent to a rest mass, one should naturally expect any charged particle to have a mass not much smaller than the Planck mass. Nature is different. The great puzzle it presents to us is not why the electron has a mass but why its mass is so small. The solution to this problem must be sought at the Planck scale.

Note added in proof. After completion of this work, M. Cvetič informed me about two papers describing black points in effective heterotic string theory [19].

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[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[2] M. Born and L. Infeld, Proc. R. Soc. London **A144**, 425 (1934).
[3] B. L. Altshuler, Class. Quantum Grav. **7**, 189 (1990).
[4] V. Mukhanov and R. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992).
[5] R. Brandenberger, V. Mukhanov, and A. Sornborger, Phys. Rev. D **48**, 1629 (1993).
[6] M. Trodden, V. F. Mukhanov, and R. H. Brandenberger, Phys. Lett. B **316**, 483 (1993).
[7] R. Moessner and M. Trodden, Phys. Rev. D **51**, 2801 (1995).
[8] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
[9] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **163B**, 123 (1985).
[10] H. P. de Oliveira, Class. Quantum Grav. **11**, 1469 (1994).
[11] Since the energy-momentum tensor is radially boost invariant, we need only one independent function in the metric.
[12] H. H. Soleng and Ø. Grøn, Ann. Phys. (N.Y.) **240**, 432 (1995).
[13] G. W. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).
[14] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).
[15] C. F. E. Holzhey and F. Wilczek, Nucl. Phys. **B380**, 447 (1992).
[16] R. H. Brandenberger, in *Proceedings of the International Workshop on Planck Scale Physics*, Puri, India, 1994, edited by J. Maharana (World Scientific, Singapore, 1995).
[17] J. Preskill, P. Schwarz, A. Shapere, S. Trivedi, and F. Wilczek, Mod. Phys. Lett. A **6**, 2353 (1991).

[18] N. Rosen, Found. Phys. **24**, 1563 (1994).
[19] M. Cvetič and D. Youm, "Dyonic BPS saturated black holes of heterotic string on a six-torus," University of Pennsylvania Report No. UPR-672-T, hep-th/9507090 (unpublished); "BPS saturated and non-extreme states in Abelian Kaluza-Klein theory and effective $N = 1$ supersymmetric string vacua," in Proceedings of Strings 95: Future Perspectives in String Theory, Los Angeles, 1995, University of Pennsylvania Report No. UPR-675-T, hep-th/9508058 (unpublished).
[20] H. H. Soleng, CARTAN: a MATHEMATICA package for tensor computations, gr-qc/9502035; CARTAN: User's Guide and Reference Manual, NORDITA Report No. 94/63 (unpublished); World Wide Web URL address: `http://suryall.cern.ch/users/soleng/CARTAN/`.