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Quantum mechanics of the dynamical zero mode in (1+1)-dimensional QCD on the light cone

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Motivated by the work of Kalloniatis, Pauli, and Pinsky, we consider the theory of light-cone quantized (1+1)-dimensional QCD on a spatial circle with periodic and antiperiodic boundary conditions on the gluon and quark fields, respectively. This approach is based on discretized light-cone quantization. We investigate the canonical structures of the theory. We show that the traditional light-cone gauge $A_- = 0$ is not available and the zero mode (ZM) is a dynamical field, which might contribute to the vacuum structure nontrivially. We construct the full ground state of the system and obtain the Schrödinger equation for the ZM in a certain approximation. Finally, the relationship between our results and those of Kalloniatis, Pauli, and Pinsky are discussed.

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I. INTRODUCTION

Quantum field theory on the light cone has been recently studied as a new powerful tool for understanding nonperturbative phenomena [1,2], especially in the theory of strong interaction (QCD) [3–7]. One of the most remarkable advantages in light-cone formalism is that vacuum is simple or trivial; i.e., the Fock vacuum is an eigenstate of the light-cone Hamiltonian [8]. On the other hand, in the usual equal-time formalism the vacuum contains an infinitely large number of particles. However, one simple and naive question arises here: how can we understand phenomena such as chiral symmetry breaking and confinement in such a simple vacuum?

As has already been indicated by many authors [9–11], zero modes of the fields might play an essential and important role there. Recently Kalloniatis, Pauli, and Pinsky have investigated about pure glue (1+1)-dimensional QCD (QCD_{1+1}) [an SU(2) non-Abelian gauge theory in 1+1 dimensions with classical sources coupled to the gluons] and have discussed the physical effects of the dynamical zero mode [12]. Note here that there are two kinds of zero modes of the fields. One is called a constrained zero mode, which is not an independent degree of freedom. Rather, it is dependent on the dynamical modes through the constraint equation. There have been many works on such a constrained zero mode related to the phenomena of phase transition in scalar field theory [13–16]. The other, which we treat here, is called a dynamical zero mode, it is a true dynamical independent field. Also Kalloniatis *et al.* have used the specific approach of discretized light-cone quantization (DLCQ) [2] in their analysis because this approach gives us an infrared regulated theory and the discretization of momenta facilitates putting the many-body problem on the computer. We shall follow their approach.

Our aim in this paper is to study the light-cone quantized QCD_{1+1} with fundamental fermions (quarks) coupled to the gauge fields (gluons) in more detail and give insight into the nontrivial QCD vacuum structure. The contents of this paper are as follows. In Sec. II, we study the canonical structures of QCD_{1+1} on the light-cone (Hamiltonian formalism) based on Dirac's treatment of the constraint system. We explicitly obtain a canonical light-cone Hamiltonian and Dirac brackets between physical quantities there. We also comment on the dynamical zero mode of the gluon fields in this section. In Sec. III, we quantize the theory developed in the previous section and construct a full ground state of the Hamiltonian. Furthermore we derive the Schrödinger equation for the zero mode in a specific coupling region. Section IV is devoted to a summary and discussion. The appendix explains our notation and conventions.

II. CLASSICAL THEORY: HAMILTONIAN FORMALISM

In this section, we study the canonical structures of light-cone QCD_{1+1}, where the space is a circle and the gauge group is SU(2). Let us start with Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \ . \tag{2.1}$$

Here $\Psi(x)$ is a quark field. Especially in two dimensions, the quark field (in a representation in which γ^5 is diagonal)

$$\Psi(x) = \begin{pmatrix} \Psi_R(x) \\ \Psi_L(x) \end{pmatrix} \tag{2.2}$$

is a two-component spinor in the fundamental representation [2]. R and L indicate chirality, which specifies only direction for massless fermions. While the filed $F^a_{\mu\nu}$ and the covariant derivative are defined as

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$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc} A^b_\mu A^c_\nu \ , \tag{2.3}$$

$$iD_\mu = i\partial_\mu - gA^a_\mu T^a \ , \tag{2.4}$$

where $A^a_\mu(x)$ is a "gluon" field and g is the coupling constant and T^a and ϵ^{abc} are the generators and the structure constant of the SU(2) gauge group defined as [17]

$$[T^a, T^b] = i\epsilon^{abc} T^c \ , \tag{2.5}$$

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab} \ . \tag{2.6}$$

In the light-cone frame approach, we set the coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \ , \tag{2.7}$$

and then rewrite all the quantities involved in the Lagrangian density (2.1) in terms of $x^{\pm}$ instead of the original coordinates x^0(time) and x^1(space). As usual in discretized light-cone quantization, we define x^+ as the line-cone "time," while x^- is the light-cone "space," which is restricted to a finite interval from $-L$ to L. Within the interval, we impose periodic and antiperiodic boundary conditions on the gluon field $A^a_\mu(x)$ and the quark field $\Psi(x)$, respectively: i.e.,

$$A^a_\mu(x^+, x^- + 2L) = A^a_\mu(x^+, x^-) \ , \tag{2.8}$$

$$\Psi(x^+, x^- + 2L) = -\Psi(x^+, x^-) \ . \tag{2.9}$$

In this way the Lagrangian density (2.1) is rewritten as

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}(F^a_{+-})^2 + \sqrt{2}(\Psi^\dagger_R i\partial_+\Psi_R + \Psi^\dagger_L i\partial_-\Psi_L) \\ & -m(\Psi^\dagger_L\Psi_R + \Psi^\dagger_R\Psi_L) - \sqrt{2}g(\Psi^\dagger_R T^a\Psi_R A^a_+ \\ & +\Psi^\dagger_L T^a \Psi_L A^a_-) \ ,\end{aligned} \tag{2.10}$$

where $\partial_\pm = \partial/\partial x^\pm$. In order to carry out the Hamiltonian formulation, we must compute the canonical momenta

$$\Pi^{+a}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+ A^a_+(x))} = 0 \ , \tag{2.11}$$

$$\Pi^{-a}(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+ A^a_-(x))} = F^a_{+-}(x) \ , \tag{2.12}$$

$$P_L(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+\Psi_L(x))} = 0 \ , \tag{2.13}$$

$$P^\dagger_L(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+\Psi^\dagger_L(x))} = 0 \ , \tag{2.14}$$

$$P_R(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+\Psi_R(x))} = \sqrt{2}i\Psi^\dagger_R(x) \ , \tag{2.15}$$

$$P^\dagger_R(x) = \frac{\partial\mathcal{L}}{\partial(\partial_+\Psi^\dagger_R(x))} = 0 \ . \tag{2.16}$$

On the other hand, we find that in the DLCQ approach, from the boundary conditions (2.8), the gluon field A^a_μ can be decomposed into a zero mode (ZM) and a particle mode (PM) as follows:

$$A^a_\mu(x) = \mathring{A}^a_\mu + \tilde{A}^a_\mu(x) \ , \tag{2.17}$$

where

$$\mathring{A}^a_\mu \equiv \frac{1}{2L}\int_{-L}^{L} dx^- A^a_\mu(x) \ , \tag{2.18}$$

$$\tilde{A}^a_\mu(x) \equiv A^a_\mu(x) - \mathring{A}^a_\mu \ . \tag{2.19}$$

Here $\mathring{A}^a_\mu$ and $\tilde{A}^a_\mu$ denote the zero modes and the particle modes of the gluon field, respectively. Similarly the canonical momenta (2.11) and (2.12) are decomposed into a ZM and PM, which leads to

$$\mathring{\Pi}^{+a} = 0 \ , \tag{2.20}$$

$$\tilde{\Pi}^{+a}(x) = 0 \ , \tag{2.21}$$

$$\mathring{\Pi}^{-a} = \mathring{F}^a_{+-} \ , \tag{2.22}$$

$$\tilde{\Pi}^{-a}(x) = \tilde{F}^a_{+-}(x) \ , \tag{2.23}$$

where

$$\begin{aligned}\mathring{F}^a_{+-} &\equiv \frac{1}{2L}\int_{-L}^{L} dx^- F^a_{+-}(x) \ , \\ &= \partial_+\mathring{A}^a_- - g\epsilon^{abc}\mathring{A}^b_+\mathring{A}^c_- \\ &\quad -\frac{g}{2L}\int_{-L}^{L} dx^- \epsilon^{abc}\tilde{A}^b_+(x)\tilde{A}^c_-(x) \ ,\end{aligned} \tag{2.24}$$

$$\begin{aligned}\tilde{F}^a_{+-}(x) &\equiv F^a_{+-}(x) - \mathring{F}^a_{+-} \ , \\ &= \partial_+\tilde{A}^a_- - \partial_-\tilde{A}^a_+ \\ &\quad -g\epsilon^{abc}(\tilde{A}^b_+\tilde{A}^c_- + \tilde{A}^b_+\mathring{A}^c_- + \mathring{A}^b_+\tilde{A}^c_-) \ .\end{aligned} \tag{2.25}$$

Following Dirac then [18] we can see that there are six primary constraints such as [19]

$$\mathring{\Pi}^{+a} \approx 0 \ , \tag{2.26}$$

$$\tilde{\Pi}^{+a}(x) \approx 0 \ , \tag{2.27}$$

$$P_L(x) \approx 0 \ , \tag{2.28}$$

$$P^\dagger_L(x) \approx 0 \ , \tag{2.29}$$

$$P_R(x) - \sqrt{2}i\Psi^\dagger_R(x) \approx 0 \ , \tag{2.30}$$

$$P^\dagger_R(x) \approx 0 \ , \tag{2.31}$$

where $\approx$ means the weak equality as usual. Substituting Eq. (2.17) into the Lagrangian density (2.10) and using Eqs. (2.11)–(2.16), the total light-cone Hamiltonian can be obtained:

$$
\begin{aligned}
H_T^{\rm LC} &= \int_{-L}^{L} dx^-[\Pi^{-a}\partial_+ A_-^a + P_R\partial_+\Psi_R - \mathcal{L}] \\
&= \int_{-L}^{L} dx^-[\tfrac{1}{2}(\mathring{\Pi}^{-a})^2 + \tfrac{1}{2}(\tilde{\Pi}^{-a})^2 + \tilde{\Pi}^{-a}\partial_-\tilde{A}_+^a + g\epsilon^{abc}(\mathring{\Pi}^{-a}\mathring{A}_+^b\mathring{A}_-^c + \tilde{\Pi}^{-a}\tilde{A}_+^b\tilde{A}_-^c \\
&\quad + \mathring{\Pi}^{-a}\tilde{A}_+^b\tilde{A}_-^c + \tilde{\Pi}^{-a}\mathring{A}_+^b\tilde{A}_-^c + \tilde{\Pi}^{-a}\tilde{A}_+^b\mathring{A}_-^c) - \sqrt{2}\Psi_L^\dagger i\partial_-\Psi_L + m(\Psi_L^\dagger\Psi_R + \Psi_R^\dagger\Psi_L) \\
&\quad + \sqrt{2}g\left(\Psi_R^\dagger T^a\Psi_R(\mathring{A}_+^a + \tilde{A}_+^a) + \Psi_L^\dagger T^a\Psi_L(\mathring{A}_-^a + \tilde{A}_-^a)\right) + u_0^a\mathring{\Pi}^{+a} + u_1^a\tilde{\Pi}^{+a} + u_2P_L + u_3P_L^\dagger \\
&\quad + u_4(P_R - \sqrt{2}i\Psi_R^\dagger) + u_5P_R^\dagger] \ ,
\end{aligned} \tag{2.32}
$$

where u_0^a and u_1^a (u_2, u_3, u_4, and u_5) are (Grassmann) Lagrange multipliers and we have used

$$\int_{-L}^{L} dx^-\tilde{A}_\pm^a(x) = \int_{-L}^{L} dx^-\tilde{\Pi}^{\pm a}(x) = 0 \ . \tag{2.33}$$

Once we obtain the expression for the Hamiltonian, we must investigate whether the primary constraints induce the secondary constraints by imposing the consistency conditions. As the result, we find there are four secondary constraints:

$$g\int_{-L}^{L} dx^-(\Pi^{-c}\epsilon^{abc}A_-^b + \sqrt{2}\Psi_R^\dagger T^a\Psi_R) \approx 0 \ , \tag{2.34}$$

$$\partial_-\tilde{\Pi}^{-a} - g(\Pi^{-c}\epsilon^{abc}A_-^b)_\sim - \sqrt{2}g(\Psi_R^\dagger T^a\Psi_R)_\sim \approx 0 \ , \tag{2.35}$$

$$\sqrt{2}i\partial_-\Psi_L - m\Psi_R - \sqrt{2}g\Psi_L T^a A_-^a \approx 0 \ , \tag{2.36}$$

$$(\sqrt{2}i\partial_-\Psi_L - m\Psi_R - \sqrt{2}g\Psi_L T^a A_-^a)^\dagger \approx 0 \ , \tag{2.37}$$

where

$$
\begin{aligned}
(A_1A_2\cdots A_n)_\sim &= A_1(x)A_2(x)\cdots A_n(x) \\
&\quad -\frac{1}{2L}\int_{-L}^{L} dx^- A_1(x)A_2(x)\cdots A_n(x) \ .
\end{aligned} \tag{2.38}
$$

We can show directly that constraints (2.34)–(2.37) do not generate new constraints further. Adding these constraints to the primary constraints, there exist ten constraints, which govern the dynamics of our system. What we have to do next is to classify these primary and secondary constraints to the first class or the second class constraints. A direct calculation shows that constraints (2.26) and (2.27) belong to the first class and the others to the second one. But this is not true. As indicated by some authors [19,20], the minimal set of second class constraints is found by combining constraints except for (2.26) and (2.27) appropriately and it is easy to show that this set is indeed given by

$$\Omega_0^a = \mathring{\Pi}^{+a} \ , \tag{2.39}$$

$$\Omega_1^a = \tilde{\Pi}^{+a}(x) \ , \tag{2.40}$$

$$
\begin{aligned}
\Omega_2^a = g\int_{-L}^{L} dx^-[&\Pi^{-c}\epsilon^{abc}A_-^b(x) - i(P_LT^a\Psi_L + \Psi_L^\dagger T^a P_L^\dagger \\
&+ P_RT^a\Psi_R + \Psi_R^\dagger T^a P_R^\dagger)(x)] \ ,
\end{aligned} \tag{2.41}
$$

$$
\begin{aligned}
\Omega_3^a = [&\partial_-\Pi^{-a}(x) - g\Pi^{-c}\epsilon^{abc}A_-^b(x) \\
&+ ig(P_LT^a\Psi_L + \Psi_L^\dagger T^a P_L^\dagger \\
&+ P_RT^a\Psi_R + \Psi_R^\dagger T^a P_R^\dagger)(x)]_\sim \ ,
\end{aligned} \tag{2.42}
$$

$$\chi_1 = P_L(x) \ , \tag{2.43}$$

$$\chi_2 = P_L^\dagger(x) \ , \tag{2.44}$$

$$\chi_3 = P_R(x) - \sqrt{2}i\Psi_R^\dagger(x) \ , \tag{2.45}$$

$$\chi_4 = P_R^\dagger(x) \ , \tag{2.46}$$

$$\chi_5 = \sqrt{2}i\partial_-\Psi_L(x) - m\Psi_R(x) - \sqrt{2}g\Psi_L T^a A_-^a(x) \ , \tag{2.47}$$

$$\chi_6 = [\sqrt{2}i\partial_-\Psi_L(x) - m\Psi_R(x) - \sqrt{2}g\Psi_L T^a A_-^a(x)]^\dagger \ , \tag{2.48}$$

where Ω_α^a ($\alpha = 0,1,2,3$) and χ_β ($\beta = 1$–6) denote the first- and second-class constraints, respectively. The first-class constraints satisfy the algebra

$$\{\Omega_\alpha^a(x), \Omega_\beta^b(y)\} = 0 \ , \tag{2.49}$$

which reflects the gauge invariance of the system. In order to eliminate all the constraints and quantize the system, we need to fix the gauge degrees of freedom and define the Dirac brackets along the usual prescription [19]. Here we shall give the gauge-fixing conditions as

$$\omega_0^a \equiv \mathring{A}_+^a \approx 0 \ , \tag{2.50}$$

$$\omega_1^a \equiv \tilde{\Pi}^{-a}(x) + \partial_-\tilde{A}_+^a(x) + g\epsilon^{abc}(A_+^b(x)A_-^c(x))_\sim \approx 0 \ , \tag{2.51}$$

$$\omega_2^a \equiv \mathring{A}_-^i \approx 0 \ \text{ for } i = 1,2 \ , \tag{2.52}$$

$$\omega_3^a \equiv \tilde{A}_-^a(x) \approx 0 \ . \tag{2.53}$$

Note here the following remarkable fact. As we see from the gauge-fixing conditions (2.52) and (2.53), we can not impose the traditional light-cone gauge $A_-^a = 0$ because we cannot put the third component of $\overset{\circ}{A}{}_-^a$ to be zero. SU(2) global color rotation symmetry always enables us to choose such gauge-fixing conditions [21]. That is why one of the zero modes of the gluon field $\overset{\circ}{A}{}_-^3$ becomes a dynamical variable, which might give insight to the nontrivial structures of the QCD light-cone vacuum [21].

Now we are coming in the stage of evaluating the Dirac brackets. After the straightforward but some tedious calculations, nonzero Dirac brackets are

$$\{\overset{\circ}{A}{}_-^3, \overset{\circ}{\Pi}{}^{-3}\}_{\rm DB} = \frac{1}{2L} \ , \tag{2.54}$$

$$\{\Psi_R(x), \Psi_R^\dagger(y)\}_{\rm DB} = \frac{i}{\sqrt{2}}\delta(x^- - y^-) \ . \tag{2.55}$$

As far as Dirac brackets have been used, we may set all the constraints and the gauge-fixing conditions to vanish strongly. The result is that total Hamiltonian (2.32) reduces to the form

$$H_T^{\rm LC} = \int_{-L}^{L} dx^- \left[\frac{1}{2}p^2 + m\Psi_R^\dagger \Psi_L + \frac{g}{\sqrt{2}}(\Psi_R^\dagger T^a \Psi_R)_\sim \tilde{A}_+^a\right] \ , \tag{2.56}$$

where $p \equiv \overset{\circ}{\Pi}{}^{-3}$ and $\Psi_L(x)$ and $\tilde{A}_+^a(x)$ are given as the functions satisfying following equations: i.e.,

$$\sqrt{2}i\partial_-\Psi_L(x) - m\Psi_R(x) - \sqrt{2}gq(x^+)T^3\Psi_L(x) = 0 \ , \tag{2.57}$$

$$\partial_-^2 \tilde{A}_-^a(x) + 2\epsilon^{ab3}gq(x^+)\partial_-\tilde{A}_+^a(x) - g^2q^2(x^+)(\tilde{A}_+^a - \delta^{a,3}\tilde{A}_+^3)(x) + \sqrt{2}g\Psi_R^\dagger T^a\Psi_R(x) = 0 \ , \tag{2.58}$$

where $q(x^+) \equiv \overset{\circ}{A}{}_-^3$. While $\Psi_R(x)$ has been chosen to satisfy charge neutrality condition

$$Q^3 \equiv g\int_{-L}^{L} dx^- \Psi_R^\dagger T^3 \Psi_R(x) = 0 \ . \tag{2.59}$$

This corresponds to the third component of ZM of Gauss law, which is necessary whenever the system is in a finite interval [22]. The fields Ψ_L and $\tilde{A}_+^a$ are expressed in terms of Ψ_R and $q(x^+)$ by solving Eqs. (2.57) and (2.58). As the result, we find that the physical degrees of freedom in this system are only the diagonal part of zero modes of the gluon field $q(x^+)$ and right-handed quark field Ψ_R.

III. QUANTUM THEORY: DYNAMICAL ZM EQUATION

In this section, we discuss the quantum aspects of the theory studied in the previous section at the classical level in detail. First we start by discussing eigenstates of the matter part in the Hamiltonian (2.56) in the fixed background gauge field and then we construct the full ground state including the ZM of the gauge field $q(x^+)$. Before doing that, we must solve Eqs. (2.57) and (2.58) for Ψ_L and $\tilde{A}_+^a$. Equation (2.57) for Ψ_L is easily solved:

$$\Psi_L^c(x) = \frac{m}{2\sqrt{2}L}\sum_{k=0}^{\infty}\int_{-L}^{L} dy^- \times \exp\left(-\frac{i\pi}{L}(k+\frac{1}{2})(x^- - y^-)\right)\tilde{\Psi}_R^c(y;k) \ , \tag{3.1}$$

where

$$\tilde{\Psi}_R^c(x;k) = \begin{pmatrix} \frac{\Psi_R^1(x)}{\frac{\pi}{L}(k+\frac{1}{2})-gq} \\ \frac{\Psi_R^2(x)}{\frac{\pi}{L}(k+\frac{1}{2})+gq} \end{pmatrix}^c \ , \tag{3.2}$$

with c being the color indices.

On the other hand, equations for $\tilde{A}_+^a$ consist of the three components

$$\partial_-^2\tilde{A}_+^1(x) + 2gq(x^+)\partial_-\tilde{A}_+^2 - g^2q^2(x^+)\tilde{A}_+^1 + \rho^1(x) = 0 \ , \tag{3.3}$$

$$\partial_-^2\tilde{A}_+^2(x) - 2gq(x^+)\partial_-\tilde{A}_+^1 - g^2q^2(x^+)\tilde{A}_+^2 + \rho^2(x) = 0 \ , \tag{3.4}$$

$$\partial_-^2\tilde{A}_+^3(x) + \rho^3(x) = 0 \ , \tag{3.5}$$

where $\rho^a(x) \equiv \sqrt{2}g\Psi_R^\dagger T^a\Psi_R(x)$. Clearly a solution for Eq. (3.5) is formally written of the form $\tilde{A}_+^3(x) = -(1/\partial_-^2)\rho^3(x)$. Thus, we will concentrate to the remaining equations (3.3) and (3.4). For brevity, we rewrite Eqs. (3.3) and (3.4) as

$$\begin{aligned} \frac{d^2f(x)}{dx^2} + 2a\frac{dg(x)}{dx} - a^2f(x) + \rho^1(x) = 0 \ , \\ \frac{d^2g(x)}{dx^2} - 2a\frac{df(x)}{dx} - a^2g(x) + \rho^2(x) = 0 \ , \end{aligned} \tag{3.6}$$

where we set

$$\begin{aligned} x &\equiv x^- \ , \\ a &\equiv gq(x^+) \ , \\ \tilde{A}_+^1(x) &\equiv f(x) \ , \\ \tilde{A}_+^2(x) &\equiv g(x) \ . \end{aligned} \tag{3.7}$$

It is easy to find that we can express Eq. (3.6) in matrix representation:

$$M^{ij}(x)\varphi^j(x) = -\rho^i(x) \ . \tag{3.8}$$

Here 2×2 matrix $M^{ij}(x)$ and vectors $\phi^i(x)$ and $\rho^i(x)$ are defined by

$$M^{ij}(x) = \begin{pmatrix} \frac{d^2}{dx^2} - a^2 & 2a\frac{d}{dx} \\ -2a\frac{d}{dx} & \frac{d^2}{dx^2} - a^2 \end{pmatrix}^{ij} ,$$
$$\varphi^i(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}^i , \quad \rho^i(x) = \begin{pmatrix} \rho^1(x) \\ \rho^2(x) \end{pmatrix}^i . \tag{3.9}$$

Using the usual Green function method, the form of $\varphi^i(x)$ would be given by

$$\varphi^i(x) = \int_{-L}^{L} dy G^{ij}(x,y)\rho^j(y) \ , \tag{3.10}$$

where $G^{ij}(x,y)$ is the Green function defined by

$$M^{ij}(x)G^{jk}(x,y) = -\delta^{ik}\delta(x-y) \ . \tag{3.11}$$

By solving Eq. (3.11) with $M^{ij}(x)$ given by (3.9), we obtain the explicit forms of $G^{ij}(x,y)$, $\tilde{A}^1_+(x)$, and $\tilde{A}^2_+(x)$ such that

$$G^{ij}(x,y) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \frac{e^{(\pi n i/L)(x-y)}}{(\pi n/L + a)^2(\pi n/L - a)^2} \begin{pmatrix} \left(\frac{\pi n}{L}\right)^2 + a^2 & \frac{2\pi i n a}{L} \\ -\frac{2\pi i n a}{L} & \left(\frac{\pi n}{L}\right)^2 + a^2 \end{pmatrix}^{ij} , \tag{3.12}$$

$$\tilde{A}^1_+(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_{-L}^{L} dy \frac{[(\pi n/L)^2 + a^2]\rho^1(y) + (2\pi i n a/L)\rho^2(y)}{(\pi n/L + a)^2(\pi n/L - a)^2} e^{(\pi n i/L)(x-y)} , \tag{3.13}$$

$$\tilde{A}^2_+(x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_{-L}^{L} dy \frac{[(\pi n L)^2 + a^2]\rho^2(y) - (2\pi i n a/L)\rho^1(y)}{(\pi n/L + a)^2(\pi n/L - a)^2} e^{(\pi n i/L)(x-y)} . \tag{3.14}$$

As a result, by substituting these equations and $\tilde{A}^3_+(x) = -(1/\partial_-^2)\rho^3(x)$ into the electrostatic Coulomb energy part in the Hamiltonian, we find that

$$\begin{aligned} H^{\mathrm{LC}}_{\mathrm{Coulomb}} &\equiv \int_{-L}^{L} dx^- \frac{g}{\sqrt{2}}(\Psi_R^\dagger T^a \Psi_R)_\sim \tilde{A}^a_+(x) \\ &= \frac{1}{2}\int_{-L}^{L} dx^- [\rho^1(x)\tilde{A}^1_+ + \rho^2(x)\tilde{A}^2_+] - \frac{1}{2}\int_{-L}^{L} dx^- \rho^3(x)\frac{1}{\partial_-^2}\rho^3(x) \\ &= \int_{-L}^{L} dx^- \int_{-L}^{L} dy^- \sum_{i=1,2}\rho^i(x)\rho^i(y)e^{-igq(x^+)T^3(x^- - y^-)} \\ &\quad \times \left(\frac{L}{2\sin^2(gqL)} + i(x^- - y^-)\cot(gqL) - |x^- - y^-|\right) - \frac{1}{2}\int_{-L}^{L} dx^- \rho^3(x)\frac{1}{\partial_-^2}\rho^3(x) \ . \end{aligned} \tag{3.15}$$

In order to quantize the Hamiltonian (2.56), we replace the Dirac brackets to the commutators. Quantization conditions for the fields $\Psi_R(x)$ and $q(x^+)$ are defined as

$$[\hat{q}(x^+), \hat{p}(x^+)] = i \ , \tag{3.16}$$

$$\{\hat{\Psi}^c_R(x), \hat{\Psi}^{c'\dagger}_R(y)\} = \frac{1}{\sqrt{2}}\delta^{c,c'}\delta(x^- - y^-) \ , \tag{3.17}$$

$$\{\hat{\Psi}^c_R(x), \hat{\Psi}^{c'}_R(y)\} = \{\hat{\Psi}^{c\dagger}_R(x), \hat{\Psi}^{c'\dagger}_R(y)\} = 0 \ , \tag{3.18}$$

where $\hat{A}$ means an operator and we have rescaled $2Lq \to q$. Thus, quantum light-cone Hamiltonian is composed of three parts:

$$\hat{H}^{\mathrm{LC}} = \hat{H}^{\mathrm{LC}}_{\mathrm{ZM}} + \hat{H}^{\mathrm{LC}}_F + \hat{H}^{\mathrm{LC}}_{\mathrm{Coulomb}} \ , \tag{3.19}$$

where

$$\hat{H}^{\mathrm{LC}}_{\mathrm{ZM}} = \int_{-L}^{L} dx^- \frac{1}{2}\hat{p}^2 \ , \tag{3.20}$$

$$\hat{H}_F^{\rm LC} = \int_{-L}^{L} dx^- m \hat{\Psi}_R^\dagger \hat{\Psi}_L \ , \tag{3.21}$$

$$\hat{H}_{\rm Coulomb}^{\rm LC} = \int_{-L}^{L} dx^- \int_{-L}^{L} dy^- \sum_{i=1,2} \hat{\rho}^i(x)\hat{\rho}^i(y) \exp\left(-\frac{ig}{2L}\hat{q}(x^+)(x^- - y^-)\right) \times \left(\frac{L}{2\,\sin^2(g\hat{q}/2)} + i(x^- - y^-)\cot\left(\frac{g\hat{q}}{2}\right) - |x^- - y^-|\right) - \frac{1}{2}\int_{-L}^{L} dx^- \hat{\rho}^3(x)\frac{1}{\partial_-^2}\hat{\rho}^3(x) \ , \tag{3.22}$$

and

$$\hat{\Psi}_L^c(x) = \frac{m}{2\sqrt{2}L}\sum_{k=0}^{\infty}\int_{-L}^{L} dy^- e^{-(i\pi/L)(k+1/2)(x^- - y^-)}\hat{\hat{\Psi}}{}_{Rk}^{c}(y) \ , \tag{3.23}$$

$$\hat{\rho}^a(x) = \sqrt{2}g : \hat{\Psi}_R^\dagger T^a \hat{\Psi}_R(x) : \ . \tag{3.24}$$

Here : : means a normal-ordered product.

In this stage, our treatment is still exact. Since we could obtain the complete forms of quantum light-cone Hamiltonian $\hat{H}^{\rm LC}$, we will next discuss eigenstates of $\hat{H}_F^{\rm LC} + \hat{H}_{\rm Coulomb}^{\rm LC}$. To do that, we first construct the ground state of our system in the presence of fixed background values of the ZM of the gauge field $q(x^+)$ [23]. This is corresponding to a kind of adiabatic approximation. Then we can easily find a fermionic part of the light-cone vacuum eigenstate as $|0\rangle_f$, defined by

$$\hat{a}_k^c|0\rangle_f = \hat{d}_k^c|0\rangle_f = 0 \ ,$$

for all c and k. Here the creation (annihilation) operator $\hat{a}_k^{c\dagger}$, $\hat{d}_k^{c\dagger}$ ($\hat{a}_k^c, \hat{d}_k^c$) is defined through the following mode expansion of $\hat{\Psi}_R(x)$, which comes from the antisymmetric boundary condition (2.9) and the anticommutation relation (3.17) and (3.18):

$$\hat{\Psi}_R^c(x) = \frac{1}{2^{1/4}\sqrt{2L}}\sum_{k=0}^{\infty}\{\hat{a}_k^c e^{-(i\pi/L)(k+1/2)x^-} + \hat{d}_k^{c\dagger} e^{(i\pi/L)(k+1/2)x^-}\} \ , \tag{3.25}$$

where

$$\begin{aligned} \{\hat{a}_k^c, \hat{a}_{k'}^{c'\dagger}\} &= \delta^{c,c'}\delta_{k,k'} = \{\hat{d}_k^c, \hat{d}_{k'}^{c'\dagger}\} \ , \\ \{\hat{a}_k^c, \hat{a}_{k'}^{c'}\} &= \{\hat{a}_k^{c\dagger}, \hat{a}_{k'}^{c'\dagger}\} = 0 \ , \\ \{\hat{d}_k^c, \hat{d}_{k'}^{c'}\} &= \{\hat{d}_k^{c\dagger}, \hat{d}_{k'}^{\dagger}\} = 0 \ . \end{aligned} \tag{3.26}$$

Of course, we can easily show that the states $|0\rangle_f$, satisfy the constraints (2.59): that is

$$\hat{Q}^3|0\rangle_f \equiv g\int_{-L}^{L} dx^- \hat{\Psi}_R^\dagger(x) T^3 \hat{\Psi}_R(x)|0\rangle_f = 0 \ . \tag{3.27}$$

This is nothing but the physical state condition. In physical meaning, this is saying that physical states be charge neutral as a whole.

As the result, the full ground state of the system can be written by

$$|\text{vacuum}\rangle \cong |0\rangle_f \otimes \Phi_0(q) \ . \tag{3.28}$$

$\Phi_0(q)$ is the zero-mode wave function in the q representation, satisfying the following Schrödinger equation for a free particle with a unit mass ($m = 1$):

$$\frac{1}{2}\left(-i\frac{d}{dq}\right)^2 \Phi_0(q) = \mathcal{E}\Phi_0(q) \ , \tag{3.29}$$

where $\mathcal{E} \equiv E/(2L)$ is an energy density.

This result seems to suggest that in a sense, the ground-state structure of the correctly normal-ordered light-cone QCD Hamiltonian is almost trivial in the adiabatic approximation. However, it is difficult to construct the full ground state beyond the adiabatic approximation. Rather, we are interested in how the effects of the ZM change the spectrum of the excited states. But we cannot answer this question in this paper.

Instead, we shall see the relation between our result and that of Kalloniatis *et al.* In order to do so, we shall neglect the fermion mass term (3.21) and replace the currents $\hat{\rho}^i(x)$ ($i = 1, 2$) in Eq. (3.22) with classical external source terms ρ^i independent of x [note that ρ^3 automatically vanishes because of the charge neutrality condition (2.59)]. They have assumed in their paper that only zero-mode external sources excite the ZM of the gauge fields. After some straightforward calculations of x and y integrations, we find that the light-cone Hamiltonian in the q representation would be of the form as

$$H^{\rm LC} = H_{\rm ZM}^{\rm LC} + 4L[(\rho^1 L)^2 + (\rho^2 L)^2] \times \left\{\frac{1}{gq}\left[\cot\left(\frac{gq}{2}\right) - 2\cot\left(\frac{gq}{2}\right)\cos(gq) - 4\,\sin(gq)\right] + \frac{4}{g^2q^2}\cos^2\left(\frac{gq}{2}\right)\right\} \ . \tag{3.30}$$

Note here that the functions

$$\frac{1}{x}\left[\cot\left(\frac{x}{2}\right)-2\cot\left(\frac{x}{2}\right)\cos x-4\sin x\right]+\frac{4}{x^2}\cos^2\left(\frac{x}{2}\right) \tag{3.31}$$

and

$$\frac{2}{x^2} \tag{3.32}$$

are almost the same in the interval $0 \le x \le 1$. Thus we can rewrite Eq. (3.30) as

$$\begin{aligned} H^{\rm LC} &\approx H^{\rm LC}_{\rm ZM} + 8L\left[(\rho^1 L)^2+(\rho^2 L)^2\right]\frac{1}{g^2q^2} \\ &= 2L\left[\frac{1}{2}\left(-i\frac{d}{dq}\right)^2+\frac{(2wL)^2}{2q^2}\right] , \end{aligned} \tag{3.33}$$

where

$$w^2 \equiv \frac{\rho_+\rho_-}{g^2}\ , \quad \rho_\pm \equiv \sqrt{2}(\rho^1 \pm i\rho^2)\ . \tag{3.34}$$

Therefore the Schrödinger equation for dynamical zero mode is now given by

$$\frac{1}{2}\left[-\frac{d^2}{dq^2}+\frac{(2wL)^2}{q^2}\right]\Phi_0(q)=\mathcal{E}\Phi_0(q)\ . \tag{3.35}$$

This is the same result Kalloniatis *et al.* have obtained in [12]. Thus we might say the results we have obtained here generalize their results.

IV. SUMMARY AND DISCUSSION

In this paper, we have studied QCD_{1+1} with fundamental fermions based on discretized light-cone quantization (DLCQ) formalism. We have discussed both classical and quantum aspects of the theory in detail and obtained the full ground-state wave function of the theory by the method of "separation of variables" mentioned in [23] and we could see the light-cone QCD ground state has almost trivial structure in the range of the adiabatic approximation we have used here. Also we could find the relation between the original work by Kalloniatis *et al.* and ours. The physical effects of the ZM for the essentially nonperturbative phenomena, e.g., chiral symmetry breaking and confinement, etc., however, remains unclear. More precise considerations for these would be future work. Moreover, what we would really like to understand is the QCD bound-state problem in 3+1 dimensions [6]. But as there are many difficulties to confront, especially a renormalization problem [10], it seems quite distant. Future works will concentrate on these points.

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APPENDIX: NOTATION AND CONVENTIONS

We describe here some notation and conventions in the light-cone formalism. They are essentially the same as those of Harada *et al.* [8]. The coordinates are set $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, where x^+ is taken as "time." The light-cone metric is given by

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g^{\mu\nu}, \quad \mu,\nu = +,-\ . \tag{A1}$$

The derivatives are also defined as

$$\partial_\pm \equiv \frac{\partial}{\partial x^\pm}\ , \tag{A2}$$

with $\partial_\pm = \partial^\mp$. γ matrices in a representation in which γ^5 is diagonal are

$$\begin{aligned} &\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ &\gamma^5 \equiv \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1-1 \end{pmatrix}\ , \\ &\gamma^+ = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}\ , \\ &\gamma^+\gamma^- = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \gamma^-\gamma^+ = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\ . \end{aligned} \tag{A3}$$

The SU(2) gauge fields are represented by

$$A_\mu \equiv A^a_\mu T^a, \quad T^a = \tfrac{1}{2}\sigma^a, \quad a = 1,2,3\ , \tag{A4}$$

where σ^a is ordinary Pauli matrices such that

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{A5}$$

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