Critical exponents and stability at the black hole threshold for a complex scalar field

Eric W. Hirschmann*
Department of Physics, University of California, Santa Barbara, California 93106-9530

Douglas M. Eardley†
Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030
(Received 30 June 1995)

This paper continues a study on Choptuik scaling in gravitational collapse of a massless complex scalar field at the threshold for black hole formation. We perform a linear perturbation analysis of the previously derived complex critical solution, and calculate the critical exponent for black hole mass, $\gamma \approx 0.387\,111 \pm 0.000\,003$. We also show that this critical solution is unstable via a growing oscillatory mode.

PACS number(s): 04.70.Bw, 04.60.Kz

I. INTRODUCTION

The recent discovery by Choptuik and others [1,2] of new and unexpected critical behavior in gravitational collapse has sparked efforts, both numerical and analytic, to understand these phenomena. One would especially like to understand the nature of the black hole scaling relation, and how the observed echoing arises out of the Einstein equations. That these phenomena appear for a variety of different matter fields suggests that they may reflect universal properties of the Einstein equations. Choptuik originally conjectured that the exponent γ in the threshold scaling relation for black holes mass,

$$M_{\rm BH}(p) \propto \begin{cases} 0, & p \le p^*, \\ (p-p^*)^\gamma \quad (\gamma \approx 0.37), & p > p^*, \end{cases} \tag{1}$$

was universal for collapse of a scalar field, and presented considerable evidence. Here p is some parameter which characterizes the strength of the initial conditions, and p^* is the threshold value, i.e., the value for the critical solution. Work by Abrahams and Evans [2] on vacuum collapse in axial symmetry, and Evans and Coleman [3] on collapse of a radiation fluid, suggested a broader universality or near universality of γ (see also [4]) among different matter models. All these results were obtained by numerical relativity, so that accuracy of γ was at best 10^{-3}. However, Maison [5] showed that fluid collapse models with an equation of state given by $p = k\rho$ have a critical exponent γ that depends strongly on the parameter k in the range $0 \le k \le 0.88$. (Evans and Coleman had examined only the case for radiation fluid, $k = 1/3$.) Maison used an accurate perturbation theory method, as suggested by Evans and Coleman, and carried out by Koike, Hara, and Adachi [4]. Perturbation theory affords both a calculation of γ, and a test of stability.

Just at the critical value $p = p^*$, at the "black hole threshold," a critical solution of the field equation appears, a "Choptuon." A Choptuon is a delicately poised dynamical state of the fields, a collapsing, shrinking, radiating ball of field energy. Its self-gravity gives rise to spacetime curvature that partially traps field energy, but only partially: Energy continually leaks outward and becomes outgoing radiation. By careful tuning of p, the rate of collapse exactly balances the radiation rate, so that the collapse becomes self-similar, and the Choptuon shrinks all the way down to the Planck scale. This situation is of course intrinsically unstable: If self-gravity is a little too weak, the Choptuon will soon dissipate completely; if a little too strong, it will soon form an event horizon and undergo terminal gravitational collapse into a black hole much larger than the Planck mass. Construction of the Choptuon implies stabilization, by careful tuning of a parameter p, against this threshold instability toward either subcritical dissipation or supercritical black hole formation. We will call this instability the "black hole instability" for short, bearing in mind that its other side is dissipation. The critical exponent γ measures the strength of the black hole instability.

Then the question of stability of the Choptuon is this: Does the Choptuon have any *other* instabilities, aside from the inevitable black hole instability? The previous evidence is that Choptuons are in fact stable in this sense [1,2].

In this paper we return to the gravitational collapse of a massless complex scalar field in spherical symmetry, as a good theoretical laboratory in which to address these issues. In [6], hereinafter called paper I, we derived a new continuously self-similar Choptuon at the threshold of black hole formation, and constructed the complete spacetime. The discrete echoing seen in the original Choptuon [1] reappeared as continuous phase oscillations of the complex scalar field. However, the Choptuon by itself does not yield the black hole scaling relation; one needs to perturb the Choptuon. In addition, perturbation theory can probe important questions about the stability of Choptuons, and the nature of the relative

*Electronic address:ehirsch@dolphin.physics.ucsb.edu
†Electronic address:doug@itp.ucsb.edu

attractor in the space of all initial conditions. This paper considers only spherically symmetric perturbations, the relevant ones for Eq. (1), and the most likely ones to be unstable. It would also be interesting to carry out a nonspherical perturbation analysis of any of the known Choptuons, to check Choptuik's "no hair" conjecture [1].

The properties of the critical solution are described in paper I, and will not be repeated here. Since there is confusion in the literature on this point, we emphasize again that the self-similar hypothesis, whether discrete [1] or continuous [3,6,7], is *not* unphysical for gravitational collapse in asymptotically flat spacetime. No self-similar spacetime is asymptotically flat when extended to infinity (except flat spacetime); however this is no problem, because the self-similar spacetime can be matched onto an asymptotically flat spacetime at some large radius, and within this radius the self-similar approximation is a good one; in fact, it becomes increasingly good and natural as the collapse proceeds to small scales at threshold [1,3,6].

Section II reviews the calculation of critical exponents, while Secs. III and IV give the perturbation equations and their boundary conditions. Section V deals with gauge modes, while the numerical methods are outlined in Sec. VI.

Our main results are given in Sec. VII. Upon perturbing the critical solution for the complex scalar field, we find two growing modes. The first mode is real; as expected it corresponds to formation of black holes off threshold, and it yields a value $\gamma \approx 0.387\,111 \pm 0.000\,003$. The second growing mode, actually a conjugate pair of modes, is oscillatory and must correspond to an instability of the critical solution on threshold.

Finally Sec. VIII discusses the significance of the results. We do not attempt to follow the growth of the second growing mode outside the range of validity of linear perturbation theory, but plausibly it would develop into the Choptuik discretely self-similar solution, the original Choptuon [1].

II. CRITICAL EXPONENTS AND PERTURBATION THEORY

We briefly review the perturbation theory method [3–5] and then apply it to our model. Begin with a continuously self-similar Choptuon. Perturb it. Quite generally, perturbed self-similar solutions to partial differential equations show power-law behavior of modes[1] in time t, tantamount to exponential behavior in logarithmic time $\tau \equiv \ln(-t)$. Consider a perturbation mode with power-law behavior $\epsilon(-t)^{-\kappa}$, where ϵ ($\propto p - p^*$) is a small constant, and t is a suitable time coordinate, in this case proper time along the time axis $r = 0$ before collapse at $t = 0$. Assume it develops into a small black hole. The actual formation of the black hole is outside the scope of perturbation theory, but we can still determine the black hole mass by perturbation theory, as follows. By scale invariance of the Choptuon, the formation of black holes belonging to different values of ϵ will be strictly homothetic to each other: i.e., the full spacetimes will be related to each other by exact scale transformations. Determine the mass M of the black hole as follows: Pick a fiducial amplitude of the perturbation mode, say small δ, which is still within the range of validity of perturbation theory. Then imagine evolving further by an exact calculation — but the exact calculation need not be done, because it will just scale. The black hole mass $M_{\rm BH}$ will be proportional to the value t_1 at which

$$\epsilon(-t_1)^{-\kappa} = \delta \tag{2}$$

or

$$M_{\rm BH} = \text{const} \times (-t_1) = \text{const}' \times \epsilon^{1/\kappa}. \tag{3}$$

The critical exponent, Eq. (1), can then be read off as

$$\gamma = 1/\kappa. \tag{4}$$

To obtain the critical exponent γ, it thus suffices to find the appropriate unstable mode in perturbation theory, and its characteristic growth rate κ.

III. EQUATIONS OF MOTION

To perturb about a self-similar critical solution (Choptuon), it is convenient to adopt coordinates $\tau = \ln(-t), z = -r/t$ where r is the usual areal coordinate that measures the area of two-spheres in spherical symmetry, and t is an orthogonal time coordinate, chosen to agree with proper time along the time axis ($t < 0, r = 0$). With these coordinates, the time-dependent spherically symmetric metric can be written

$$ds^2 = e^{2\tau}\{(1+u)\left[-(b^2 - z^2)d\tau^2 + 2zd\tau dz + dz^2\right] + z^2 d\Omega^2\} \; . \tag{5}$$

To enforce regularity along the time axis $z = 0$ at the center of spherical symmetry, the metric functions $u(\tau, z)$ and $b(\tau, z)$ must obey boundary conditions

$$b(\tau, 0) = 1, \quad u(\tau, 0) = 0 \; . \tag{6}$$

The perturbed fields are then defined as

$$b(\tau, z) \approx b_0(z) + \epsilon b_1(\tau, z) \; , \tag{7a}$$

$$u(\tau, z) \approx u_0(z) + \epsilon u_1(\tau, z) \; , \tag{7b}$$

$$\phi(\tau, z) \approx e^{i\omega\tau}\phi_0(z) + \epsilon e^{i\omega\tau}\phi_1(\tau, z) \; , \tag{7c}$$

where $_0$ denotes the zeroth-order critical solution, $_1$ denotes the first-order perturbation, $\omega = 1.915\,444\,6$ is the (unique) eigenvalue of the unperturbed equations (paper I); and where $\epsilon > 0$ is an infinitesimal constant, a measure of how far away the solution is from the critical solution in the space of initial conditions. Using Choptuik's terminology, we consider the supercritical regime

[1] In degenerate cases there may be logarithmic corrections to power-law behavior, of the form $\epsilon(-t)^{-\kappa}\ln^n(-t)$, but these turn out to be absent in the problem at hand.

for infinitesimal

$$\epsilon \propto p - p^*. \tag{8}$$

Matter consists of a free, massless,[2] complex scalar field ϕ that obeys the scalar wave equation

$$\Box\phi = 0 \tag{9}$$

while the Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \tag{10a}$$

$$= 8\pi \mathrm{Re}\left(\nabla_\mu\phi\nabla_\nu\phi^* - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi^*\right), \tag{10b}$$

paper I gives these equations in detail.

We now perturb these Einstein-scalar equations through first order in ϵ, to obtain a set of linear partial differential equations for the perturbed fields b_1, u_1, ϕ_1, in the independent variables τ, z. Following the standard approach, we Fourier transform the first-order fields with respect to the ignorable coordinate $\tau = \log(-t)$:

$$\hat{u}_1(\sigma, z) = \int e^{i\sigma\tau}u_1(\tau, z)d\tau, \tag{11a}$$

$$\hat{b}_1(\sigma, z) = \int e^{i\sigma\tau}b_1(\tau, z)d\tau, \tag{11b}$$

$$\hat{\phi}_1(\sigma, z) = \int e^{i\sigma\tau}\phi_1(\tau, z)d\tau, \tag{11c}$$

throughout, a caret will denote such a Fourier transform. The transform coordinate σ is in general complex. The first-order field equations now become ordinary differential equations (ODE's) in z, and under appropriate boundary conditions, become an eigenvalue problem for σ. Solutions of the eigenvalue problem are then normal modes of the critical solution. Eigenvalues in the lower half plane $\mathrm{Im}\sigma < 0$ belong to unstable (growing) normal modes. Eigenvalues in the upper half σ plane, which we will not explore in this paper, would correspond to quasinormal (dying) modes of the critical solution. The eigenvalue σ is related through Eq. (4) to the critical exponent by $\gamma = -1/\mathrm{Im}\sigma$.

Thereby, the Einstein-scalar equations reduce to the following set of ODE's in z. Define some auxiliary fields in zeroth order

$$q(z) = \phi_0(z)' , \tag{12}$$

$$p(z) = \frac{1}{b_0}[i\omega\phi_0(z) - z\phi_0(z)'] , \tag{13}$$

and, in first order,

$$r(\sigma, z) = \hat{\phi}_1(\sigma, z)' , \tag{14}$$

$$s(\sigma, z) = \frac{1}{b_0}\left(i(\omega - \sigma)\hat{\phi}_1 - z\hat{\phi}_1'\right) , \tag{15}$$

$$R(\sigma, z) = r(-\sigma^*, z)^* , \tag{16}$$

$$S(\sigma, z) = s(-\sigma^*, z)^* , \tag{17}$$

where an asterisk denotes complex conjugate. The field equations become (in notation which closely follows that of paper I), in zeroth order,

$$b_0' = \frac{b_0u_0}{z}, \tag{18a}$$

$$u_0 = 4\pi z^2\left(|q|^2 + |p|^2 + 2\frac{b_0}{z}\mathrm{Re}(qp^*)\right), \tag{18b}$$

$$u_0' = -4\pi b_0(1 + u_0)(qp^* + pq^*), \tag{18c}$$

$$q' = -\left(\frac{u_0 + 2}{z} + \frac{z\beta_+}{\Delta}\right)q + \frac{b_0\beta_-}{\Delta}p, \tag{18d}$$

$$p' = \frac{b_0\beta_+}{\Delta}q - \left(\frac{u_0}{z} + \frac{z\beta_-}{\Delta}\right)p, \tag{18e}$$

and in first order

$$r' = \left(\frac{2zb_0\hat{b}_1}{\Delta^2}\beta_+ - \frac{\hat{u}_1b_0^2}{z\Delta}\right)q + \left(-\frac{2\hat{b}_1b_0^2}{\Delta^2}\beta_- + \frac{i\sigma\hat{b}_1 + \hat{u}_1b_0}{\Delta}\right)p - \left(\frac{u_0 + 2}{z} + \frac{z(\beta_+ - i\sigma)}{\Delta}\right)r + \frac{b_0}{\Delta}(\beta_- - i\sigma)s, \tag{19a}$$

$$s' = \left(-\frac{2\hat{b}_1z^2}{\Delta^2}\beta_+ + \frac{\hat{u}_1b_0}{\Delta}\right)q + \left(\frac{2\hat{b}_1b_0z}{\Delta^2}\beta_- - \frac{z\hat{u}_1}{\Delta} - i\sigma\frac{z\hat{b}_1}{b_0\Delta}\right)p + \frac{b_0}{\Delta}(\beta_+ - i\sigma)r - \left(\frac{u_0}{z} + \frac{z(\beta_- - i\sigma)}{\Delta}\right)s, \tag{19b}$$

$$R' = \left(\frac{2zb_0\hat{b}_1}{\Delta^2}\beta_+^* - \frac{\hat{u}_1b_0^2}{z\Delta}\right)q^* + \left(-\frac{2\hat{b}_1b_0^2}{\Delta^2}\beta_-^* + \frac{i\sigma\hat{b}_1 + \hat{u}_1b_0}{\Delta}\right)p^* - \left(\frac{u_0 + 2}{z} + \frac{z(\beta_+^* - i\sigma)}{\Delta}\right)R + \frac{b_0}{\Delta}(\beta_-^* - i\sigma)S, \tag{19c}$$

$$S' = \left(-\frac{2\hat{b}_1z^2}{\Delta^2}\beta_+^* + \frac{\hat{u}_1b_0}{\Delta}\right)q^* + \left(\frac{2\hat{b}_1b_0z}{\Delta^2}\beta_- - \frac{z\hat{u}_1}{\Delta} - i\sigma\frac{z\hat{b}_1}{b_0\Delta}\right)p^* + \frac{b_0}{\Delta}(\beta_+^* - I\sigma)R - \left(\frac{u_0}{z} + \frac{z(\beta_-^* - i\sigma)}{\Delta}\right)S, \tag{19d}$$

$$\hat{b}_1' = \frac{1}{z}(\hat{u}_1b_0 + u_0\hat{b}_1), \tag{19e}$$

$$\hat{u}_1' = -\frac{\hat{u}_1(1 + u_0)}{z} + \frac{u_0'\hat{u}_1}{1 + u_0} + 4\pi(1 + u_0)z\left(-\frac{2\hat{b}_1}{b_0}|p|^2 + pS + p^*s + q^*r + qR\right). \tag{19f}$$

[2]In view of previous results [1] it is unlikely that a nonvanishing rest mass, or more generally a nonlinear self-coupling of the form $\Box\phi = V'(\phi)$ would make any difference below. However, other self-couplings such as in the axion-dilaton system [7] may well make a difference.

Here, the prime denotes d/dz and

$$\beta_\pm = i\omega + u_0 \pm 1, \quad (20a)$$

$$\Delta = b_0^2 - z^2. \quad (20b)$$

The equations for R and S are identical to those for r and s under the conjugacy $(q,p,r,s,\beta_\pm) \to (q^*,p^*,R,S,\beta_\pm^*)$. In the zeroth-order problem, the Bianchi identities allow us to determine u_0 as an algebraic expression, Eq. (18b). Similarly $u_1(z)$ can be written as a first-order algebraic expression involving other first-order variables:

$$\hat{u}_1 = \frac{4\pi z(u_0+1)}{u_0+1-i\sigma}\left(-\frac{2zb_1}{b_0}|p|^2 + z(qR+q^*r+pS+p^*s) + b_0(pR+p^*r+qS+q^*s)\right). \quad (21)$$

IV. BOUNDARY CONDITIONS

Now we specify boundary conditions for these equations. On the axis of spherical symmetry, $z = 0$, solutions must be regular. This demand leads to boundary conditions in zeroth order

$$b_0(0) = 1, \quad (22a)$$

$$u_0(0) = 0, \quad (22b)$$

$$q(0) = 0, \quad (22c)$$

$$p(0) = \text{free real const}, \quad (22d)$$

(22e)

and in first order

$$\hat{b}_1(0) = 0, \quad (23a)$$

$$\hat{u}_1(0) = 0, \quad (23b)$$

$$r(0) = 0, \quad (23c)$$

$$R(0) = 0, \quad (23d)$$

$$s(0) = \text{free complex const}, \quad (23e)$$

$$S(0) = \text{free complex const}. \quad (23f)$$

The global U(1) phase symmetry present in the unperturbed equations has allowed us to set $p(0)$ to be real. No such rotation can be performed for the time-dependent quantity ϕ_1, so $s(0)$ and $S(0)$ are in general complex.

The field equations have a regular singular point at the point $z = z_2$ where $\Delta(\equiv b_0^2 - z^2)$ vanishes. As discussed in paper I, this locus corresponds to a similarity horizon in spacetime, the past light cone of the spacetime singularity at the spacetime origin $(t,r) = (0,0)$. All fields must be regular on this horizon, because it lies within the domain of dependence of the initial hypersurface. This means in particular that the terms proportional to $1/\Delta$ in Eqs. (19a) and (19b) must cancel at z_2, giving a linear relation between $r(z_2)$, $s(z_2)$, $\hat{b}_1(z_2)$, and $\hat{u}_1(z_2)$; and similarly for $R(z_2)$ and $S(z_2)$ from Eqs. (19c) and (19d):

$$(u_0-1)[(\beta_+ - i\sigma)r - (\beta_- - i\sigma)s] = q\left(\frac{\hat{b}_1\beta_+}{z} + (i\omega+2)\hat{u}_1 + \hat{b}_1 u_0'\right) - p\left(\frac{u_0\hat{b}_1(\beta_- - i\sigma)}{z} + \frac{i\sigma\hat{b}_1}{z} + \hat{b}_1 u_0' + i\omega\hat{u}_1\right) + \hat{b}_1(\beta_+ q' - \beta_- p')\ , \quad (24a)$$

$$(u_0-1)[(\beta_+^* - i\sigma)R - (\beta_-^* - i\sigma)S] = q^*\left(\frac{\hat{b}_1\beta_+^*}{z} + (-i\omega+2)\hat{u}_1 + \hat{b}_1 u_0'\right) - p^*\left(\frac{u_0\hat{b}_1(\beta_-^* - i\sigma)}{z} + \frac{i\sigma\hat{b}_1}{z} + \hat{b}_1 u_0' - i\omega\hat{u}_1\right) + \hat{b}_1(\beta_+^* q^{*\prime} - \beta_-^* p^{*\prime}). \quad (24b)$$

These equations may conveniently be used to solve for $s(z_2)$ and $S(z_2)$ in terms of other boundary conditions as long as σ lies in the lower half plane; if one wished to explore the upper half plane this procedure would fail somewhere due to the vanishing of the coefficients of s and S at z_2. The resulting boundary conditions at $z = z_2$ are, in zeroth order,

$$b_0(z_2) = z_2 = \text{free real const}, \quad (25a)$$

$$p(z_2) = \text{free complex const}, \quad (25b)$$

and, in first order,

$$\hat{b}_1(z_2) = \text{free complex const}, \quad (26a)$$

$$\hat{u}_1(z_2) = \text{free complex const}, \quad (26b)$$

$$r(z_2) = \text{free complex const}, \quad (26c)$$

$$R(z_2) = \text{free complex const}. \quad (26d)$$

The zeroth-order equations, Eqs. (18), amount to five real ODE's in five real fields (b,p,q), where the complex fields p and q each count as two real fields. Correspondingly there are five free real constants: $(\omega, p(0), z_2, p(z_2))$, where the complex $p(z_2)$ counts twice.[3] Thus we have a well-posed eigenvalue problem on a finite domain $0 \le z \le z_2$, and we may expect a discrete spectrum.

The first-order equations, Eqs. (19), amount to six complex linear ODE's[4] in the six complex

[3]In paper I, we regarded u_0 as an unknown rather than solving for it from Eq. (18b); therefore the counting was slightly different.

[4]One could alternatively use the algebraic constraint, Eq. (21), for $\hat{u}_1(z)$ to reduce the number of complex equations to five. However, the denominator in Eq. (21) vanishes on a region of the negative imaginary σ axis, and causes numerical trouble.

fields $(\hat{b}_1, \hat{u}_1, r, s, R, S)$. At our disposal are $(\sigma, s(0), S(0), \hat{b}_1(z_2), \hat{u}_1(z_2), r(z_2), R(z_2))$, seven free complex constants. Do we have one too many free complex constants? No, because solutions to the equations must come in one-complex-dimensional linear families, due to linearity; that is, each solution may be freely scaled by an arbitrary complex factor. Thus we have a well-posed eigenvalue problem on a finite domain $0 \le z \le z_2$ which should show a discrete spectrum of solutions, apart from linearity. We note that the first-order system, Eqs. (19), with its boundary conditions, is not self-adjoint, at least not obviously so; therefore the eigenvalue spectrum of σ may be expected to be complex.

Once constructed on the domain $0 \le z \le z_2$, the first-order solution could be extended throughout the spacetime. In the terminology of paper I, Fig. 5, the first-order solution is first constructed in region I, and then it could be extended through regions II and III. We will have no need to carry out that extension in this paper.

V. COORDINATE AND GAUGE CONDITIONS

Since our field equations possess gauge invariance due to general coordinate invariance and global U(1) phase invariance, some unphysical pure gauge modes will appear at first order, to the extent that the gauge conditions implicit in our boundary conditions, Eqs. (22), fail to be unique.

Indeed there arises a pure gauge mode from an infinitesimal phase rotation $\phi \to e^{i\epsilon}\phi$ in the zeroth-order critical solution:

$$\hat{b}_1(z) = 0, \tag{27a}$$

$$\hat{u}_1(z) = 0, \tag{27b}$$

$$\hat{\phi}_1(z) = i\phi_0(z). \tag{27c}$$

This gives a time-independent solution of Eqs. (19) that satisfies the boundary conditions; hence it corresponds to an unphysical mode at $\sigma = 0$.

A second pure gauge mode arises upon adding an infinitesimal constant to time $t \to t + \epsilon$ at constant r in the zeroth-order critical solution. This is possible because our coordinate conditions, Eqs. (6), normalize t to proper time along the negative time axis ($t < 0, z = 0$), but the zero of time is not specified. Then the solution is perturbed by

$$b_1(\tau, z) = \left.\frac{\partial b_0}{\partial t}\right|_r = -(z/t)b'(z) = e^{-\tau} z b'(z), \tag{28a}$$

$$u_1(\tau, z) = \left.\frac{\partial u_0}{\partial t}\right|_r = -(z/t)u'(z) = e^{-\tau} z u'(z), \tag{28b}$$

$$\phi_1(\tau, z) = e^{-i\omega\tau} \left.\frac{\partial(e^{i\omega\tau}\phi_0)}{\partial t}\right|_r = e^{-\tau}[-i\omega\phi_0(z) + z\phi_0'(z)]. \tag{28c}$$

This pure gauge mode has time dependence $e^{-i\sigma\tau} = e^{-\tau}$ and so has negative imaginary $\sigma = -i$. There are no other pure gauge modes in the σ plane. These two modes should appear as numerical solutions, but are unphysical.

VI. NUMERICAL METHOD

To solve the first-order problem we used a Runge-Kutta integrator with adaptive stepsize as part of a standard two-point shooting method [8], shooting from $z = 0$ and from $z = z_2 = 5.0035380$, and matching in the middle $z = z_m = 2.5$. For convenience we solved the zeroth-order system, Eqs. (18), and the first-order system, Eqs. (19) simultaneously with the same steps in z. As discussed in paper I, the similarity horizon z_2 is a demanding place to enforce a boundary condition, and a second-order Taylor expansion of the regular solution was used for this purpose.

To solve the first-order system, we collected all the boundary values but σ into a complex six-vector $X \equiv (s(0), S(0), \hat{b}_1(z_2), \hat{u}_1(z_2), r(z_2), R(z_2))$. Because the equations are linear, the matching conditions at $z = z_m$ are likewise linear in the boundary values. A solution is found when the values at z_m of $(\hat{b}_1, \hat{u}_1, r, s, R, S)$ upon integrating from $z = 0$ match with those found by integrating from $z = z_2$, for some boundary values X. We can express this matching condition

$$A(\sigma)X = 0 \ , \tag{29}$$

where $A(\sigma)$ is a 6×6 complex matrix which is a nonlinear function of σ, constructed numerically by integrations of the first-order equations, Eqs. (19), for six linearly independent choices of boundary values X. The condition on σ for a solution is then

$$\det A(\sigma) = 0. \tag{30}$$

Once a value for σ was found that satisfies this condition, the corresponding boundary values X were found as a zero eigenvector of the matrix A; these come in one (complex) parameter families, as observed above. Solution of Eqs. (19) with boundary values X yields the normal mode itself.

Now, $A(\sigma)$ has been carefully constructed so that it is a complex *analytic* solution of σ. This follows from the fact that all equations leading to A contain σ but not σ^*, together with some standard theorems about ODE's. Moreover, $A(\sigma)$ has no singularities in the closed lower half σ plane. These properties allow us to use a number of ideas from scattering theory to study $\det A(\sigma)$. In particular, there is a theorem for counting the number N_C of zeros of $\det A$ within any closed contour $\mathcal{C}$ in the closed lower half σ plane:

$$\Delta_C \operatorname{Arg} \det A = 2\pi N_C \ , \tag{31}$$

where $\operatorname{Arg} \det A$ is the phase of $\det A$, and $\Delta_C \operatorname{Arg} \det A$ is the total phase wrap (in radians) around the closed contour $\mathcal{C}$, a result similar to Levinson's theorem for counting resonances in quantum scattering theory.

Furthermore, there holds a conjugacy relation

$$A^*(-\sigma^*) = A(\sigma), \tag{32}$$

which means that A need only be evaluated for $\mathrm{Re}\sigma \geq 0$ in the lower half plane.

The nonlinear equation $\det A(\sigma) = 0$ was solved by the secant variant of Newton's method [8]. The equation being complex analytic, the one-complex-dimensional realization of the method was used, and it performed well.

The following numerical checks are used in our work. First, the two pure gauge modes, Eqs. (27) and (28), appeared numerically at the expected places $\sigma = 0, -i$. Second, the constraint, Eq. (21), was confirmed numerically. Third, we numerically spotchecked the Cauchy-Riemann equations for $\det A(\sigma)$, to confirm that this function is analytic. Generally the accuracy was about 10^{-6}.

VII. RESULTS

There are five values of σ in the lower half plane at which $\det A$ vanishes, signaling five modes. All five are simple zeros; see Fig. 1. Two of these modes are pure gauge modes, Eqs. (27) and (28), which help check internal consistency of the work, but are unphysical; as expected these lie on the negative imaginary axis[5] at $\sigma = 0, -i$.

The third solution also lies on the negative imaginary axis, at

$$\sigma = -2.583\,24i \pm 0.000\,02i. \tag{33}$$

We interpret this mode as the instability toward forming an infinitesimal[6] black hole, Eq. (1). This corresponds through Eq. (4) to a critical exponent of $\gamma = 0.387\,111 \pm 0.000\,003$. This value differs from that found by Choptuik [1] for the real scalar field, further evidence

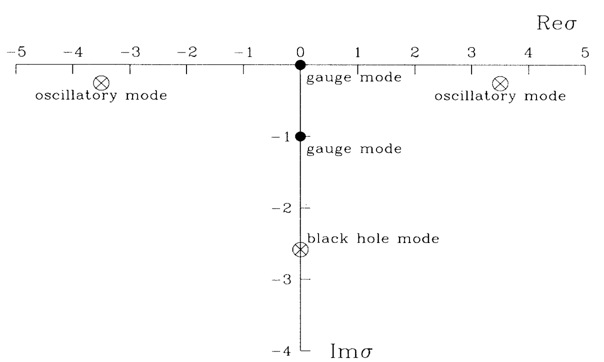

FIG. 1. The lower half complex σ plane. There are five zeros of $A(\sigma)$, representing five modes (solutions of the first-order perturbed field equations). • •, two unphysical gauge modes. ⊗ ⊗ ⊗, black hole mode, and pair of oscillatory unstable modes. See Sec. VII.

[5]To within $\mathrm{Re}\sigma = \pm 10^{-6}$.

[6]In classical gravity.

of nonuniversality among different matter fields.

The fourth and fifth solutions form a conjugate pair lying at

$$\sigma = \pm(3.502\,24 \pm 0.000\,01) - (0.258\,67 \pm 0.000\,01)i. \tag{34}$$

The presence of this pair of zeros in the lower half plane means that the continuously self-similar critical solution is unstable to perturbations, even among solutions "tuned" to the black hole threshold. These represent a pair of modes that are not just growing, but oscillating as well. These modes must grow until they become nonlinear, and then plausibly they go over to Choptuik's solution.

We evaluated $\det A(\sigma)$ around a large rectangular contour $\mathcal{C}$ bounded by $0 \leq \mathrm{Re}\sigma \leq 25$, $0 \geq \mathrm{Im}\sigma \geq -10$ (detouring around the three zeros on the imaginary axis) and used Eq. (31) to count the zeros lying within. The absence of other zeros in this large rectangle was thereby established. Asymptotically, we observed

$$\det A(\sigma) \sim \exp(\tau_\infty \sigma), \quad \tau_\infty \approx 6i\ , \tag{35}$$

in both the real and imaginary directions, and the outer boundaries of the large rectangle appeared to be well into the asymptotic regime. Thus it appears very likely that there are no zeros in the lower half σ plane other than the five described above.

A comment is in order concerning the error of our numerical solution. The error for the complex Choptuon constructed in paper I was a little better than 10^{-6}. To investigate how the error of the unperturbed solution affected the results in the perturbed problem, we varied the parameters of the unperturbed Choptuon by 10^{-5}. The effect on the above zeros was a variation of about 10^{-5}. We also varied the initial values for the perturbed problem (independent of the change in the unperturbed Choptuon) in order to study whether the locations of the zeros were sensitive to this variation. The resulting change in the values was on the order of 10^{-6}. This exercise also verified that the Taylor expansions at the end points 0 and z_2 are of adequate order.

Since our integrator uses an adaptive step size, we did not vary the step size directly; rather we varied the algorithmic error tolerance used to adjust the step size, and we allowed the initial step size ϵ and the location of the matching point, z_m, to be random numbers within a specific range:

$$10^{-7} < \epsilon < 10^{-6}, \quad 0.5 < z_m < 4.5\,. \tag{36}$$

The combined effect of varying the parameters of the original Choptuon by several times their quoted error, and of varying the initial values and other parameters for the perturbed problem, then gave the overall uncertainties quoted above.

Thus we conclude that the critical exponent obtained here, $\gamma = 0.387\,111 \pm 0.000\,003$, is indeed different from others that have been calculated to date, such as $\gamma_{\mathrm{rad}} = 0.355\,801\,9$, as found for radiation fluid collapse by Koike, Hara, and Adachi [4].

VIII. DISCUSSION AND CONCLUSION

We have perturbed the continuously self-similar complex Choptuon [6] and find the critical exponent for black hole mass, Eq. (1) to be

$$\gamma = 0.387\,111 \pm 0.000\,003, \qquad (37)$$

further evidence for nonuniversality of γ. We also find a further, oscillatory instability in this Choptuon.

In spherically symmetric collapse of a real scalar field, Choptuik [1] observed that the real, discretely self-similar Choptuon could be constructed by tuning only one (real) parameter p of the infinite number of parameters in the space $\mathcal{S}$ of initial conditions. That is, the real Choptuon is an attractor in a subspace of codimension 1 in $\mathcal{S}_{\text{real}}$. The results here for complex scalar fields imply that the continuously self-similar complex Choptuon would require tuning of some three (real) parameters p, q, r in the space $\mathcal{S}_{\text{complex}}$ of spherically symmetric initial conditions — one (p) to control the real black hole mode, two more ($q + ir$) to control the pair of oscillatory modes. Therefore the complex Choptuon appears to be an attractor of codimension 3 in $\mathcal{S}_{\text{complex}}$. On threshold, with only p tuned, it is highly plausible that the oscillatory instability found here in the complex Choptuon would evolve nonlinearly toward the original real Choptuon. Otherwise, it would have to evolve toward still a third Choptuon, for which there is currently no evidence.

Numerical relativity [1,9,10] is needed to follow this nonlinear evolution. In fact, the growth rate of the oscillatory mode (measured by $-\text{Im}\sigma$) is small, $\sim 10\%$ of the growth rate of the black hole mode, Eqs. (33) and (34), and consequently the oscillatory instability may be hard to isolate numerically. It would be interesting to study related models such as a complex scalar field with a conformal coupling ξ, or the axion-dilaton fields [7], to see what happens to the oscillatory growing mode. This mode might conceivably move into the upper half σ plane to become stable.

Choptuons thus show both similarities and differences compared to stationary black holes. For classical black holes, the uniqueness theorems show that "black holes have no hair," and perturbation theory proves that black holes are stable in vacuum, or when coupled to massless fields. On the other hand, classical Choptuons coupled to a massless complex scalar field show at least two alternative states, namely, the original Choptuon [1] and the complex Choptuon [6]. In this paper, perturbation theory has shown that the latter state is unstable, presumably toward the former.

ACKNOWLEDGMENTS

This research was supported in part by the National Science Foundation under Grant Nos. PHY89-04035 and PHY90-08502, and parts were carried out at the Aspen Center for Physics. We are grateful to Matt Choptuik, Chuck Evans, and Jim Horne for enlightening conversations and communications.

[1] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[2] A.M. Abrahams and C.R. Evans, Phys. Rev. Lett. **70**, 2980 (1993).
[3] C.R. Evans and J.S. Coleman, Phys. Rev. Lett. **72**, 1782 (1994).
[4] T. Koike, T. Hara, and S. Adachi, Phys. Rev. Lett. **6**, 5170 (1995).
[5] D. Maison, "Non-Universality of Critical Behaviour in Spherically Symmetric Gravitational Collapse," Report No. MPI-PhT/95-28, gr-qc/9504008, 1995 (unpublished).
[6] E.W. Hirschmann and D.M. Eardley, Phys. Rev. D **51**, 4198 (1995).
[7] D.M. Eardley, E.W. Hirschmann, and J.H. Horne, this issue, Phys. Rev. D **52**, R5397 (1995).
[8] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, England, 1986).
[9] C. Gundlach, R.H. Price, and J. Pullin, Phys. Rev. D **49**, 890 (1994).
[10] D. Garfinkle, Phys. Rev. D **51**, 5558 (1995).