Global charges in Chern-Simons theory and the 2+1 black hole

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We use the Regge-Teitelboim method to treat surface integrals in gauge theories to find global charges in Chern-Simons theory. We derive the affine and Virasoro generators as global charges associated with symmetries of the boundary. The role of boundary conditions is clarified. We prove that for diffeomorphisms that do not preserve the boundary there is a classical contribution to the central charge in the Virasoro algebra. The example of anti–de Sitter 2+1 gravity is considered in detail.

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I. INTRODUCTION

Global charges in Chern-Simons theory can be understood by several independent methods. Perhaps the most popular one is found in the observation that the Chern-Simons action, defined on a manifold with boundaries, induces a Wess-Zumino-Witten theory at the boundary [1]. Then, at least for simple topologies, the symplectic structure associated with the degrees of freedom at the boundary is the affine extension of the Lie algebra considered [2], and their corresponding Virasoro generators can be constructed by the Sugawara construction. From the point of view of the classical theory, which will be our main interest here, this gives an infinite set of global charges that satisfy a well-defined (Dirac brackets) algebra. (The Virasoro generators are associated with the diffeomorphism invariance of the theory. They are not independent from the affine generators because, in Chern-Simons theory, the local gauge group contains the diffeomorphism group.)

A different approach leading to the same results was followed in Ref. [3]. In that reference, the authors studied the differentiability properties of the generators of gauge transformations. Imposing strong boundary conditions they ensure their differentiability and then they find a set of first class quantities (which are not zero on shell) satisfying an affine extension of the Lie algebra. The same procedure is then applied to gauge transformations that generate diffeomorphisms showing that an analogous set of first class quantities exists such that they satisfy the Virasoro algebra. Both sets of first class functions were shown to be related by the Sugawara construction. More recently, the asymptotic group of anti–de Sitter 2+1 gravity using the Chern-Simons formulation was studied in [4]. In that work, however, the advantages of the Chern-Simons formulation are not fully explored because the boundary conditions are read off from the metric formulation.

In this paper we apply the Regge-Teitelboim [5] method to treat boundary terms and boundary conditions in gauge field theory. This method, although closely related to the one used in [3], improves it in several ways. (1) It deals only with the generators of gauge transformations and their associated charges, it is not necessary to introduce other first class quantities. (2) It provides a natural interpretation for global charges as the generators of the residual gauge group (in the Dirac brackets) after the gauge is fixed. (This point was not properly examined in [3].) (3) The boundary conditions are somehow dictated by the theory and not imposed from the outside. This allows us to relax the boundary conditions such that a *classical* central charge in the Virasoro algebra will appear. (This boundary condition allows for diffeomorphisms that have a nonzero component normal to the boundary.)

In Sec. II we study the differentiability of the generators of gauge transformations and define the charges that regularize them for a generic choice of boundary conditions. Then we consider the two particular cases leading to the affine and Virasoro algebras.

The strategy to define global charges in gauge theories is standard [5]. Given a set of first class constraints $G_a \approx 0$ satisfying

$$\{G_a, G_b\} = f^c_{\ ab} G_c, \tag{1}$$

one considers the smeared generator (supplemented by a charge Q that makes it differentiable)

$$G(\eta) = \int \eta^a G_a + Q(\eta) \approx Q(\eta). \tag{2}$$

Since $G(\eta)$ is differentiable, one can compute its Poisson brackets with $G(\lambda)$. The question is whether or not the smeared generators satisfy a closed algebra. The answer to this question is intimately related to the issue of boundary conditions. Indeed, the value of the charge Q crucially depends on them. Normally, one first chooses a set of boundary conditions, with precise falloff conditions for the fields, and then finds the most general set of gauge transformations that leave them invariant. This set of parameters defines what we call global symmetries. The smeared generator (2) is then supplemented with the extra condition that the parameter of the transformation

(η), at the boundary, must be a global symmetry. It turns out that global symmetries are not generated by the constraints but by the complete generator (2) which is not weakly zero. The conserved charge associated with this symmetry is the charge Q, being also the generator of the global symmetry after the gauge is fixed. Global symmetries form, in general, a subgroup of the complete gauge group. For example, the global group of asymptotically flat 3+1 gravity is the Poincaré group, which, of course, does not contain the whole of the diffeomorphism invariance.

Another problem related to the issue of boundary conditions is the gauge-fixing procedure in gauge field theory. Since, in general, the constraints are differential functions of the canonical variables, it is not possible to completely fix the gauge (determine the Lagrange multipliers) by a canonical gauge condition. One is left, instead, with a differential equation for the Lagrange multipliers whose solution contains "integration functions" at the boundary. These functions represent the freedom to make "improper" or global gauge transformations, even after the gauge is fixed. In Sec. II B we describe how to fix the gauge in the simple case in which the (spatial) manifold has the topology of a disk. We shall see that the connection is fixed up to an arbitrary function at the boundary. Also, there is a nontrivial residual symmetry at the boundary.

Once the boundary conditions are chosen and the group of global symmetries is known we seek a canonical realization for this group. It turns out that the canonical realization of the algebra of global symmetries gives, in general, a central extension of its algebra [6]. The possibility of a central charge cannot be ruled out by a general principle. An example of this "classical anomaly" is asymptotically anti–de Sitter 2+1 gravity [7]. The group of global charges in that case is the pseudoconformal group and its canonical realization has a central charge proportional to the inverse of the square root of the cosmological constant. We shall see in Sec. II that this "anomaly" is also present in Chern-Simons theory for a general group. Both the affine and Virasoro algebras contain nonzero central terms providing new examples of the results reported in [6,7].

The particular case of 2+1 gravity with a negative cosmological constant is considered in Sec. III. We give in Sec. III a simple derivation of the results reported in [7] using the Chern-Simons formulation of 2+1 gravity. In simple words, since the Lie algebra so(2,2) is a direct product of two copies of so(2,1), there are two commuting affine currents, one for each so(2,1) copy. Thus, by the (classical) Sugawara construction, one finds two copies of the Virasoro algebra, i.e., the conformal group. The main point of Sec. III is the discussion of the necessary boundary conditions to actually obtain the two commuting Virasoro algebras and to clarify the origin of the classical central charge in them.

It is worth mentioning here that global charges have been shown to play an important role in the definition of black hole entropy. It is already generally accepted that the black hole entropy is given by a boundary term in the action (at the horizon) which is added in order to make the action differentiable [8]; i.e., it can be understood as a global charge lying at the horizon. An even more interesting connection between global charges and entropy has been recently proposed by Carlip [9] in the case of 2+1 dimensions. In simple words, he computed the number of states associated with a Wess-Zumins-Witten (WZW) theory defined at the black hole horizon proving that the logarithm of this number, at least in the limit $k \to \infty$, gives the correct value for the 2+1 black hole entropy. The key point in Carlip's analysis is the assumption that the horizon has to be treated as a boundary when computing path integrals. Whether this assumption is correct is not yet clear, however, Carlip's result shows that those degrees of freedom associated with the presence of boundaries might be of central importance in the understanding of black hole physics.

Apart from the computational simplicity of using the Chern-Simons formulation for 2+1 gravity, the main advantage of that formalism is the possibility to quantize the resulting algebras of global charges. This is not an easy task when one works in the Arnowitt-Deser-Misner (ADM) formulation. We mention some aspects of the quantization of global charges throughout the paper.

II. GLOBAL CHARGES IN CHERN-SIMONS THEORIES

A. Chern-Simons action

We start with the Chern-Simons action defined in a manifold with the topology $\Sigma \times \Re$ written in Hamiltonian form:

$$I = \frac{k}{4\pi}\int_{\Re} dt \int_{\Sigma} d^2x \epsilon^{ij} g_{ab}(\dot{A}_i^a A_j^b + A_0^a F_{ij}^b) + B, \qquad (3)$$

where B is a boundary term that has to be included in order to ensure gauge invariance of the action and depends on boundary conditions. The curvature is defined by $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f^a_{\ bc} A_i^b A_j^c$, where $f^a_{\ bc}$ are the structure constants for the Lie algebra considered here. The metric g_{ab} is defined by $g_{ab} = x\mathrm{Tr}(J_a J_b)$ where x is some real number depending on the representation. All local indices $(a, b, c, ...)$ are raised and lowered with this metric which is assumed to be invertible. We deal only with integrals over the two-dimensional (spatial) manifold Σ and its boundary, the one-dimensional manifold $\partial\Sigma$. Hence all the integrands are assumed to be two-forms and one-forms, respectively. We work in fundamental units $\hbar = 1$ and k is dimensionless.

From (3) we learn that A_0^a is a Lagrange multiplier and the A_i^a are the dynamical fields satisfying the basic Poisson brackets

$$\{A_i^a(x), A_j^b(y)\} = \frac{2\pi}{k} g^{ab} \epsilon_{ij} \delta(x, y) \qquad (4)$$

where ϵ_{ij} is defined as the inverse matrix of ϵ^{ij} by $\epsilon^{ij}\epsilon_{ik} = \delta^j_k$; no metric is needed. The Poisson brackets of any two functions $G(A_i)$ and $H(A_i)$ of the canonical variables are then given by

$$\{G,H\} = \frac{2\pi}{k}\int \frac{\delta G}{\delta A^a_i}\epsilon_{ij}g^{ab}\frac{\delta H}{\delta A^b_j}. \tag{5}$$

The variation of (3) with respect to A^a_0 gives the constraint

$$G_a \equiv \frac{k}{4\pi}g_{ab}\epsilon^{ij}F^b_{ij} \approx 0, \tag{6}$$

which satisfies the Poisson brackets algebra

$$\{G_a(x), G_b(y)\} = f^c_{ab}G_c(x)\delta(x,y). \tag{7}$$

Hence, the G_a are the generators of gauge transformations acting on phase space.

All of this is well known. The question to be addressed now is whether or not the smeared generator (supplemented with a boundary term Q that makes it differentiable[1])

$$G(\eta^a) = \int_\Sigma \eta^a G_a + Q(\eta) \tag{8}$$

satisfies a first class algebra. It turns out that the smeared generators provide a *central* extension of the algebra of global charges.

Varying (8) with respect to the field A_i, assuming that the parameter η does not depend on A_i in the interior, we have

$$\delta G = \frac{k}{2\pi}\int_\Sigma \epsilon^{ij}\eta_a D_i\delta A^a_j \;+\; \delta Q(\eta) \tag{9}$$

where $D_i v^a \equiv \partial_i v^a + f^a_{\ bc}A^b_i v^c$ is the covariant derivative acting on a Lie-algebra vector in the adjoint representation. Integrating the first term by parts we find

$$\delta G = -\frac{k}{2\pi}\int_\Sigma \epsilon^{ij}D_i\eta_a\,\delta A^a_j + \frac{k}{2\pi}\int_{\partial\Sigma}\eta_a\delta A^a_k dx^k + \delta Q. \tag{10}$$

Therefore, if we demand the charge Q to be given by

$$\delta Q = -\frac{k}{2\pi}\int_{\partial\Sigma}\eta_a\,\delta A^a_i\,dx^i, \tag{11}$$

the surface terms cancel out, making the functional derivative of G well defined:

$$\frac{\delta G}{\delta A^a_i} = \frac{k}{2\pi}\epsilon^{ij}D_j\eta_a. \tag{12}$$

Using formula (12) and the definition of Poisson brackets (5), we can compute the Poisson brackets of two generators smeared with parameters η and λ, obtaining

$$\{G(\eta), G(\lambda)\} = \frac{k}{2\pi}\int_\Sigma D\eta_a{}_\wedge D\lambda^a. \tag{13}$$

Integrating by parts in the left-hand side one obtains

$$\{G(\eta), G(\lambda)\} = \int_\Sigma [\eta,\lambda]^a G_a + \frac{k}{2\pi}\int_{\partial\Sigma}\eta_a D\lambda^a \tag{14}$$

where $[\eta,\lambda]^a = f^a{}_{bc}\eta^b\lambda^c$ is the commutator in the Lie algebra. Here we have used the identity $D_\wedge Dv^a = f^a{}_{bc}F^b v^c$ and the definition of G_a [Eq. (6)].

The volume term in the right-hand side of (14) has the expected expression as the commutator of the two parameters times the constraint G_a. However, in order to reproduce the smeared generator, the volume term has to be supplemented by a surface term that makes it differentiable. This implies that the surface term present in the right-hand side of (14) has to be equal to the charge that regularizes the volume term, up to a possible central term [6], i.e.,

$$\frac{k}{2\pi}\int \eta_a D\lambda^a = Q(\sigma) + K(\eta,\lambda) \tag{15}$$

where Q is defined by Eq. (11), and σ is a parameter that depends on η and λ:

$$\sigma = \sigma(\eta,\lambda). \tag{16}$$

The precise form of the function σ will tell us the algebra of global charges. Of course, this depends crucially on the chosen boundary conditions. The term K, on the other hand, does not depend on the variables that are varied at the boundary and therefore is a central charge. Replacing (15) in (14) one finds

$$\{G(\eta), G(\lambda)\} = G(\sigma) + K(\eta,\lambda). \tag{17}$$

It turns out that, after the gauge is fixed, Eq. (17) is still valid provided the contraints are replaced by their corresponding charges, and the Poisson brackets are replaced by the Dirac brackets [5,11]:

$$\{Q(\lambda), Q(\eta)\}^* = Q(\sigma) + K(\lambda,\eta). \tag{18}$$

Therefore, if $K \neq 0$, the canonical realization of the algebra of global charges gives a central extension of its algebra.

The purpose of this section is to study the relations (11) and (15), extracting from them the form of the function $\sigma(\eta,\lambda)$ and the value of K. But first we have to fix the gauge and define the set of fields at the boundary which are not "pure gauge."

[1]In the particular case of gravity with the Hilbert action, the surface term is already present in the action [10]. This is because the Hilbert action contains second derivatives in the metric and therefore there is a natural surface term that is added to the action that cancels them. In Chern-Simons theory the action is of first order and there is no need to add any surface term apart from those originating from demanding its diferentiability.

B. Fixing the gauge

In a field theory with no degrees of freedom such as Chern-Simons theory, the only relevant degrees of freedom are holonomies or global charges. As we are interested in the definition of global charges we restrict ourselves to the special case in which the spatial manifold has the topology of a disk. In this section we briefly describe an appropriate way to fix the gauge in order to keep only the degrees of freedom at the boundary.

For a disk, we call r and ϕ the radial and polar coordinates, respectively. The constraint dictates that the connection has to be flat. On the disk, this has the general solution $A_i = U^{-1}\partial_i U$ where U is an arbitrary single valued group element. In order to fix the gauge we need to impose an extra condition. We use a gauge condition inspired by the WZW model,

$$A_{r,\phi} = 0, \tag{19}$$

which implies that U can be factorized in the form

$$U(r,\phi) = a(\phi)b(r), \tag{20}$$

where a and b belong to the group. Substituing (20) in the connection we have

$$A_r(r) = b^{-1}\partial_r b, \tag{21}$$

$$A_\phi(r,\phi) = b^{-1}A(\phi)\, b, \tag{22}$$

where $A(\phi) = a^{-1}\partial_\phi a$ is a function of ϕ only (and time).

The Lagrange multiplier A_0 is now found from the requirement that the gauge-fixing condition (19) has to be preserved by the time evolution. A simple calculation gives the condition over A_0:

$$D_r(\partial_\phi A_0) = 0, \tag{23}$$

where D_r is the covariant derivative along the r coordinate. Since the connection is flat, this equation has the general solution

$$A_0 = b^{-1}\lambda(\phi)b \tag{24}$$

where $\lambda(\phi)$ is an arbitrary function of the angular coordinate ϕ only (and time). The arbitrary function $\lambda(\phi)$ represents the residual gauge freedom. There is no way to fix it by a canonical gauge condition because the constraint is not algebraic but a differential function of the canonical variables. Note also that under the residual group, i.e., under those transformations with a parameter of the form (24), A_r is left invariant. This follows directly from (23) and the definition of gauge transformation.

For later reference we also consider here the case in which the Lagrange multiplier A_0 depends on the gauge field in the form:

$$A_0^a = -\xi^i A_i^a. \tag{25}$$

In this case, condition (24) on the allowed transformations gives the conditions over ξ^i:

$$\xi^r(\partial_r b)b^{-1} + \xi^\phi A(\phi) = \text{function of } \phi \text{ only}. \tag{26}$$

Since A and b are independent, this equation implies that ξ^ϕ has to be a function of ϕ only. Also, b is not an arbitrary function of r since the term $\xi^r(\partial_r b)b^{-1}$ cannot depend on r. This means that $(\partial_r b)b^{-1}$ depends on r only by a multiplicative function which is canceled by ξ^r. This dependence on r can be eliminated by an appropriate choice of the radial coordinate at the boundary. Hereafter we assume that the radial coordinate at the boundary is chosen so that

$$b = e^{\alpha r} \quad (\alpha \text{ is a constant Lie-algebra element}); \tag{27}$$

therefore $A_r = \alpha$, and ξ^r is a function of the angular coordinate only.

In summary, the gauge-fixed connection contains one arbitrary Lie-algebra value function of the angular coordinate (and time) $A(\phi)$ defined in (22). Moreover, this function transforms as a connection under the residual gauge group defined by the parameters of the form (24).

C. Field-independent gauge transformations. Affine Lie algebras

In this section we study the simplest boundary condition for which we can integrate the expression (11) for the charge. Suppose that the parameter does not depend on the fields that are varied at the boundary. In this case, the charge Q is trivially integrated from (11), obtaining

$$Q(\eta) = -\frac{k}{2\pi}\int \eta_a A^a + Q_0, \tag{28}$$

where Q_0 is a fixed arbitrary constant that we shall assume equal to zero. Now we go back to Eq. (15). The surface integral can be rewritten as

$$\frac{k}{2\pi}\int \eta_a D\lambda^a = \frac{k}{2\pi}\int \eta_a d\lambda^a + \frac{k}{2\pi}\int \eta_a [A,\lambda]^a \tag{29}$$

$$= \frac{k}{2\pi}\int \eta_a d\lambda^a + Q([\eta,\lambda]) \tag{30}$$

where Q is given in (28). Therefore, in view of (15), we find that in this case the function σ defined in (16) is simply the commutator of the two parameters

$$\sigma^a(\eta,\lambda) = f^a_{\ bc}\eta^b\lambda^c. \tag{31}$$

The central charge, on the other hand, is

$$K(\eta,\lambda) = \int_{\partial\Sigma} \eta_a d\lambda^a. \tag{32}$$

The Dirac brackets algebra of global charges is then

$$\{Q(\eta), Q(\lambda)\}^* = Q([\eta,\lambda]) + \frac{k}{2\pi}\int \eta_a\, d\lambda^a \tag{33}$$

where Q is the gauge-fixed expression for the charge:

$$Q = -\frac{k}{2\pi}\int \lambda(\phi)_a A^a(\phi) d\phi. \tag{34}$$

Here, $A^a(\phi)$ and $\lambda(\phi)$ are, respectively, the residual parts of the connection and gauge parameters not fixed by the gauge-fixing procedure. Thus, as expected, the canonical realization of the residual gauge group (global symmetries) gives a central extension of its algebra. It is interesting to note that the above analysis gives a geometrical meaning for the central term in (33). The charges Q generate the residual gauge transformations [5] acting on A^a. If one computes the Poisson brackets between A^a and the gauge-fixed generator Q, one finds that A^a transforms as a connection, as it should, only if the central term in (33) is present.

The algebra (33) can be put in a more familiar form in terms of Fourier modes. The field $A^a(\phi)$ at the boundary can be decomposed as

$$A^a(\phi) = \frac{1}{k}\sum_{n=-\infty}^{\infty} T_n^a e^{in\phi} \tag{35}$$

where the coefficients T_n^a satisfy the "classical" affine algebra

$$\{T_n^a, T_m^b\}^* = -f^{ab}{}_c T_{n+m}^c + ikng^{ab}\delta_{n+m}. \tag{36}$$

Therefore, for gauge transformations whose parameters do not depend on the gauge field, the associated algebra of global charges is the affine extension of the Lie algebra considered.

The quantum version of this algebra is obtained by replacing the Poisson brackets by $(-i)$ times the commutator:

$$[T_n^a, T_m^b] = if^{ab}{}_c T_{n+m}^c + kn\, g^{ab}\delta_{n+m} \tag{37}$$

where now the T_n^a are linear operators acting on a linear vector space. This algebra has received much study in the context of conformal field theory. The central charge k must be an integer in order to achieve gauge invariance of the path integral under large gauge transformations. It is remarkable that this quantization condition also ensures the existence of highest-weight unitary representations for the T_n^a.

In the context of this paper, the algebra (36) [or its quantum version (37)] has two different interpretations. First, it arises naturally as the algebra of an infinite set of conserved charges associated with a residual gauge symmetry at the boundary. A second less obvious but important interpretation is as the basic Dirac brackets (commutator) between the basic variables. Indeed, the basic variables after the gauge is fixed are the components of the residual connection $A(\phi)$ (or their Fourier components) whose Dirac brackets are given by (36). As we shall see in the next section, for a different choice of boundary conditions, the algebra (36) is no longer the algebra of conserved charges, but still the basic Dirac brackets for the gauge-fixed residual canonical variables.

D. Field-dependent gauge transformations. Virasoro algebra

In addition to gauge transformations, Chern-Simons theory is also invariant under diffeomorphisms. One may wonder if there exist different boundary conditions for which the charge Q represents the freedom of making a diffeomorphism at the boundary. This is indeed possible and it is the goal of this section.

It is well known that, in Chern-Simons theory, diffeomorphisms are contained in the gauge group when they act on the space of solutions of the equations of motion. Therefore one would not expect to have an independent set of conserved charges associated with that symmetry. Indeed, we shall see that the charges emerging from this new class of boundary conditions are completely determined by the ones found in the previous section. However, at the quantum level, normal ordering problems appear and the algebra of global charges associated with diffeomorphisms contains extra contributions to the central charge that make it worth studing.

The parameters of those gauge transformations that produce diffeomorphisms (up to the equation of motion) are of the form [12]

$$\eta^a = -\xi^i A_i^a \tag{38}$$

where ξ^i is an arbitrary vector. From this expression we see that the parameter depends explicitly on the gauge field and therefore, in this case, we cannot straightforwardly integrate the expression (11) for the charge. Since we are interested in global charges we assume that the parameter of the gauge transformation has the form (38) only at the boundary. As we are going to fix the gauge what happens in the interior is immaterial.

In this section we denote the charge (associated to diffeomorphisms at the boundary) by J. The variation of J is given by [see Eq. (11)]

$$\delta J = \frac{k}{2\pi}\int \xi^i A_i\, \delta A_j dx^j. \tag{39}$$

Denoting by r and ϕ the coordinates normal and tangential to the boundary, respectively, we obtain

$$\delta J = \frac{k}{2\pi}\int \left[\xi^r A_r \delta A_\phi + \frac{1}{2}\xi^\phi \delta A_\phi^2\right] d\phi. \tag{40}$$

In order to integrate this expression we need the boundary condition

$$\delta A_r^a = 0 \quad \text{(at the boundary)}. \tag{41}$$

This condition implies that A_r^a at the boundary is fixed. This is quite reasonable. Indeed, we already know that A_r at the boundary is a gauge-invariant quantity (see Sec. II B). We use the radial coordinate on which A_r^a is a constant Lie-algebra-value element:

$$A_r^a \equiv \alpha^a, \quad \alpha^a \text{ is a constant Lie-algebra-value element.} \tag{42}$$

In view of (41) we obtain for the charge J,

$$J(\xi^i) = \frac{k}{2\pi}\int \left[\xi^r \alpha A_\phi + \frac{1}{2}\xi^\phi A_\phi^2\right] d\phi + J_0, \tag{43}$$

where J_0 is an arbitrary constant whose variation vanishes and will be adjusted later. The formula (43) gives the value of J for a single transformation with parameter ξ^i. Now we have to go back to Eq. (15) and check that the boundary term

$$\frac{k}{2\pi}\int_{\partial\Sigma}(\xi^i A_i^a)D(\zeta^j A_j^b)g_{ab} \tag{44}$$

is equal to the charge J defined in (43) evaluated on some parameter $\sigma^i(\xi,\zeta)$, plus a possible central charge. The form of the function σ in terms of ξ and ζ defines the algebra of global charges associated with these boundary conditions.

To prove this we first note that the covariant derivative in the boundary term in (44) can be replaced by an ordinary derivative because $\mathrm{Tr}([A_i,A_j]A_k) = 0$ for $i,j,k = 1,2$. By a simple calculation one can prove the identity

$$\int(\xi^i A_i)\partial_\phi(\zeta^j A_j)d\phi$$
$$= \int\left([\xi,\zeta]^r\alpha A + \frac{1}{2}[\xi,\zeta]^\phi A^2\right)d\phi + \alpha^2\int \xi^r\,\partial_\phi\zeta^r d\phi \tag{45}$$

where we have already expressed A_r and A_ϕ in terms of their residual parts (see Sec. II B). The square brackets denote Lie brackets.

From (45) we see that the boundary term indeed contains the charge J evaluated on a parameter equal to the Lie brackets of the deformation vectors ξ^i and ζ^i. This is not at all surprising because we already knew that gauge transformations with parameters of the form (38) generate diffeomorphisms. What is more interesting is the term proportional to α^2 which, we stress, does not depend on the variable $A(\phi)$, and therefore, it is a central term in the algebra of global charges.

The algebra of global charges associated with these boundary conditions is then

$$\{J(\xi^i),J(\zeta^j)\}^* = J([\xi,\zeta]^i) + \frac{k\alpha^2}{2\pi}\int \xi^r\,\partial_\phi\zeta^r d\phi, \tag{46}$$

showing that, as stated above, the canonical realization of the symmetries at the boundary gives a central extension of its algebra.

The Virasoro algebra is contained in (46) in the special case in which the deformation vector appearing in (38) has the particular form

$$\xi^i = (-\partial_\phi\xi,\xi), \tag{47}$$

where $\xi = \xi(\phi)$ is an arbitrary function of ϕ. It is straightforward to check that the vectors (47) form a subalgebra in the Lie brackets. In this case the charge (43) depends only on one independent function $\xi(\phi)$,

$$J(\xi) = \frac{k}{4\pi}\int \xi(\phi)\left[\alpha^2 + 2\alpha A,_\phi + A^2\right] \tag{48}$$

where we have already adjusted the constant J_0 by

$$J_0 = \frac{k}{4\pi}\int \xi(\phi)d\phi. \tag{49}$$

Since A is a periodic function of ϕ we can make the Fourier expansion

$$\alpha^2 + 2\alpha A,_\phi + A^2 = \frac{1}{k}\sum_{-\infty}^{\infty} L_n e^{in\phi} \tag{50}$$

and it is easy to check that the algebra (46) gives the classical Virasoro algebra for the L_n,

$$\{L_n,L_m\}^* = i(n-m)L_{n+m} + \frac{ic_{\text{class}}}{12}n(n^2-1)\delta_{n+m} \tag{51}$$

where the central charge is given by

$$c_{\text{class}} = 12k\alpha^2. \tag{52}$$

We stress here that this central charge has a pure geometrical (classical) origin having nothing to do with quantum normal ordering ambiguities. In terms of Fourier modes the charge has the simple expression

$$J(\xi) = \sum_n L_n\xi^n \tag{53}$$

where the modes ξ^n are given by

$$\xi^n = \frac{1}{4\pi}\int d\phi\,\xi(\phi)e^{in\phi}. \tag{54}$$

A classical central charge in the canonical realization of global symmetries was first found in [7] in the context of 2+1 gravity with a negative cosmological constant. In the derivation given in [7] the ADM formalism was used, and therefore the asymptotic symmetries were defined as the most general set of Killing vectors that leave the boundary conditions invariant. The notion of a Killing vector requires a metric structure which is absent in our analysis. The *ad hoc* form of deformation vectors (47) used here can be derived from the Killing equations when a metric structure is present.

The derivation given here is general in the sense that no particular group has been assumed. The presence of the classical central charge is a feature that has to do only with the chosen boundary conditions.

It should be evident that the expression for the Virasoro generators given by (50) is nothing but the (modified) Sugawara construction. Indeed, by inverting relation (50) and expressing $A(\phi)$ also in the Fourier modes [see Eq. (35)] we obtain

$$L_n = \frac{1}{2k}\sum_m g_{ab}\,T_m^a T_{n-m}^b + in\alpha_a T_n^a + \frac{k}{2}\alpha^2\delta_n \tag{55}$$

which can be recognized as the (modified) Sugawara construction. The above relation is not at all surprising. The Virasoro generators generate diffeomorphisms at the boundary while the affine generators T^a_n generate gauge transformations. Relation (55) reflects the fact that diffeomorphisms in Chern-Simons theory can be expressed as gauge transformations. The role of the Sugawara construction in Chern-Simons theory was first found in [3].

The quantization of the Virasoro algebra has been extensively studied in the literature. We refer the reader to Ref. [13] for details. We define the quantum Virasoro operator

$$\hat{L}_n = \beta : L_n : + a\delta_n \tag{56}$$

where :: means normal order. The constant β and a are given by

$$\beta = \frac{2k}{2k+Q}, \qquad a = \frac{k}{2}\alpha^2\beta(\beta-1), \tag{57}$$

and $Qg^{ad} = f^a{}_{bc}f^{dbc}$ is the quadratic Casimir in the adjoint representation. The operator $\hat{L}_n$ satisfies the quantum Virasoro algebra

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} \tag{58}$$

with a total central charge

$$c = 12k\alpha^2\beta^2 + \beta N. \tag{59}$$

Here N is the dimension of the Lie algebra considered. The second term in this expression is the well-known central charge induced by the Sugawara construction and has a pure quantum origin. The first term, on the other hand, has to do only with the chosen boundary conditions. Note that the classical contribution appears in (59) multiplied by β^2.

III. SO(2,2) GRAVITY AND THE 2+1 BLACK HOLE

A. Chern-Simons formulation for 2+1 gravity

Three-dimensional gravity (with a negative cosmological constant) can be written as a Chern-Simons theory for the anti–de Sitter group [14,12]. The goal of this section is to describe how the results of the previous section can be applied to the case of 2+1 gravity. We shall be particularly interested in the quantization of the 2+1 black hole mass and angular momentum defined as global charges associated with symmetries of the boundary.[2]

The gauge field for 2+1 gravity with a negative cosmological constant is

$$A = e^a P_a + w^a J_a \tag{60}$$

where P_a and J_a satisfy the so(2,2) algebra

$$\begin{aligned}
[J_a, J_b] &= \epsilon_{ab}{}^c J_c,\\
[J_a, P_b] &= \epsilon_{ab}{}^c P_c,\\
[P_a, P_b] &= \frac{1}{l^2}\epsilon_{ab}{}^c J_c.
\end{aligned} \tag{61}$$

The one-forms e^a and w^a are the triad and spin connection, respectively. The generators $J^{\pm}_a = \frac{1}{2}(J_a \pm lP_a)$ both satisfy the algebra of so(2,1) and commute between them. The corresponding connections for each so(2,1) copy are

$$A^a_{\pm} = w^a \pm \frac{e^a}{l}. \tag{62}$$

In terms of these new fields the Chern-Simons action splits into the sum of two Chern-Simons actions:

$$I(e,w) = I^+(A_+) + I^-(A_-), \tag{63}$$

where

$$I^{\pm}(A^a) = \pm\frac{k}{4\pi}\int(\eta_{ab}A^a{\wedge}dA^b + (1/3)\epsilon_{abc}A^a{\wedge}A^b{\wedge}A^c) \tag{64}$$

with $\eta_{ab} = \mathrm{diag}(-1,1,1)$ and $\epsilon_{012} = 1$. We choose the coupling constant k equal to

$$k = \frac{l}{G} \tag{65}$$

in order to agree with the conventions followed in [17]. In fundamental units $\hbar = G = c = 1$ and k is dimensionless.

All the results described in previous sections can be applied to the present case which consists of two copies of the Chern-Simons action. It should be kept in mind that the sector $(-)$ has a negative coupling constant $(-k)$. The Hamiltonian is given by

$$\begin{aligned}
H(A_0) = &\int (A^+_0)^a\, G_a(A_+) + B^+\\
&- \int (A^-_0)^a\, G_a(A_-) - B^-
\end{aligned} \tag{66}$$

where $B^{\pm}$ are boundary terms that regularize each sector independently.

B. The 2+1 black hole

The Einstein equations in 2+1 dimensions with a negative cosmological term possess a black hole solution [17]. This solution, in the absence of electromagnetic fields, is characterized by its mass and angular momentum. The metric takes the form

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(N^{\phi}dt + d\phi)^2, \tag{67}$$

where the squared lapse $N^2(r)$ and the angular shift $N^{\phi}(r)$ are given by

$$\begin{aligned}
N^2(r) &= -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2},\\
N^{\phi}(r) &= -\frac{J}{2r^2},
\end{aligned}$$

[2] See [15] for an equivalent definition for these charges in terms of holonomies. Global charges in Chern-Simons gravity without a cosmological constant have been studied in [16].

with $-\infty < t < \infty$, $0 < r < \infty$, and $0 \leq \phi \leq 2\pi$.

The lapse function $N(r)$ vanishes for two values of r given by

$$r_{\pm} = l \left\{ \frac{M}{2} \left[1 \pm \sqrt{1 - \left(\frac{J}{Ml} \right)^2} \right] \right\}^{1/2}.$$

Of these, r_+ is the black hole horizon. The relation between $r_\pm$ and M, J can be inverted obtaining the useful formulas,

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad J = \frac{-2r_+ r_-}{l}. \tag{68}$$

In order for the horizon to exist one must have

$$M > 0, \qquad |J| \leq Ml. \tag{69}$$

In the extreme case $|J| = Ml$, both roots of $N^2 = 0$ coincide.

The metric (67) has two integration constants r_+ and r_-, or, equivalently, M and J. There are two additional parameters associated with the freedom of multiplying the lapse function N by an arbitrary constant N_∞, and the freedom to add to N^ϕ and arbitrary constant N^ϕ_∞. These freedoms are global symmetries whose conserved charges are precisely the mass M and the angular momentum J, respectively. In the ADM Hamiltonian variational principle [18] one keeps N_∞ and N^ϕ_∞ fixed at the boundary while M and J are varied. The purpose of this section is to translate the ADM variational principle to the Chern-Simons language. As we shall see, the Chern-Simons formulation allows for great simplifications.

For our purposes here it is useful to introduce a Rindler-like radial coordinate for the black hole metric. For $r \geq r_+$ we define a dimensionless radial coordinate ρ by

$$r^2 = r_+^2 \cosh^2 \rho - r_-^2 \sinh^2 \rho \qquad (\rho > 0). \tag{70}$$

This radial coordinate brings the metric into the simple form

$$\begin{aligned} ds^2 = &-\sinh^2 \rho [r_+ d\tau - r_- d\phi]^2 + l^2 d\rho^2 \\ &+ \cosh^2 \rho [-r_- d\tau + r_+ d\phi]^2 \end{aligned} \tag{71}$$

where we have defined the dimensionless time coordinate $\tau = t/l$. (The coordinate ρ can be extended to the black hole interior by replacing the hyperbolic functions by appropriate circular functions. Since we are interested in the metric at the outer boundary, we do not need to do this here.) It should be evident from (71) that the black hole differs from anti–de Sitter space only in its global properties [18].

From (71) we can read the form of the triad (up to local Lorentz rotations),

$$\begin{aligned} e^0 &= (r_+ d\tau - r_- d\phi) \sinh \rho, \\ e^2 &= (-r_- d\tau + r_+ d\phi) \cosh \rho, \\ e^1 &= l d\rho, \end{aligned} \tag{72}$$

and, solving the torsion equation $T^a = 0$, one finds the spin connection (up to a Lorentz rotation)

$$\begin{aligned} w^2 &= (1/l)(r_+ d\tau - r_- d\phi) \cosh \rho, \\ w^0 &= (1/l)(-r_- d\tau + r_+ d\phi) \sinh \rho, \\ w^1 &= 0. \end{aligned} \tag{73}$$

This solution corresponds to the "zero mode" solution considered before. Indeed, as discussed in Sec. II B, the general form of the solution for the constraints is $A_\phi = b^{-1} A(\phi) b$ and $A_r = \alpha$, where $b = e^{\alpha\rho}$ is a group element that only depends on the radial component. The above black hole solution corresponds to the simple case when the Lie-algebra-value function $A(\phi)$ is constant. The right (A^+) and left (A^-) functions associated with the black hole can be easily calculated. They are

$$A^+ = \frac{r_+ - r_-}{l} J_2^+, \tag{74}$$

$$A^- = -\frac{r_+ + r_-}{l} J_2^-. \tag{75}$$

On the other hand, the group elements $b_\pm$ are given by

$$b_\pm = e^{\pm \rho J_1^\pm}; \tag{76}$$

therefore, $A_r^\pm \equiv \alpha_\pm = \pm J_1^\pm$.

C. The black hole charges. Mass and angular momentum

In this section we shall prove that the mass (M) and the angular momentum (J) of the black hole solution are contained in the charges defined in Sec. II when one goes to the SO(2,2) group.

The black hole charges make use of the fact that the Lie algebra so(2,2) splits into two copies of so(2,1). Indeed, the Chern-Simons variational principle that gives rise to the "zero mode" black hole solution allows for independent diffeomorphisms in each copy of the Chern-Simons action (63).

In other words, we can consider the boundary condition for the Lagrange multiplier $A_0 = A_0^+ + A_0^-$:

$$A_0^\pm = \mp \xi_\pm^i A_i^\pm \qquad \text{(at the boundary)}, \tag{77}$$

where ξ_+^i and ξ_-^i are two independent vectors of the form (47). It should be stressed here that these boundary values for A_0 double the number of parameters in the global gauge group.

These boundary values for A_0 give us two copies of the Virasoro algebra given by [see Eq. (48)]

$$L_n^\pm = \frac{k}{4\pi} \int e^{-in\phi} \left(\alpha_\pm^2 + 2\alpha_\pm \partial_\phi A_\pm + A_\pm^2 \right), \tag{78}$$

which both satisfy the classical Virasoro algebra with a central charge equal to

$$c_\pm = 12 k \alpha_\pm^2. \tag{79}$$

The values of $A_\pm$ for the black hole are summarized in (74) and (75). Since for the black hole $A_\pm$ does not depend on ϕ the only nonzero charge is the zero mode. By direct replacement we find

$$L_0^\pm = \frac{k}{2}(1 + M \pm J/l) \qquad (80)$$

where M and J in terms of $r_\pm$ are given in (68). Therefore, we have

$$M + 1 = (L_0^+ + L_0^-)/l, \qquad (81)$$

$$J = L_0^+ - L_0^-. \qquad (82)$$

Hence we see that indeed the black holes charges M and J are related to the zero-mode charges studied in Sec. II. Similarly, the zero-mode components of the functions $\xi_\pm(\phi)$ are related to the ADM "lapse" (N_∞) and "shift" (N_∞^ϕ) functions by

$$\xi_\pm^0 = N_\infty \pm l N_\infty^\phi. \qquad (83)$$

We remember here that in the ADM Hamiltonian variational principle N_∞ and N_∞^ϕ are held fixed while the mass and angular momentum are varied [18]. In the present Chern-Simons variational principle we have kept fixed the deformation vectors $\xi_\pm$ while the function $A(\phi)$ is varied. Thus both variational principles are completely equivalent.

It should be noted that our expression for M in terms of the zero-mode Virasoro generators differs from the one exhibited in [18] by the additive constant +1. The convention followed in this paper was to adjust the zero point of the energy so that the charges satisfy the Virasoro algebra with the standard expression for the central charge, i.e., having an SL(2,$\Re$) subalgebra in its center.

The conformal group of asymmptotic motions for anti–de Sitter 2+1 gravity (two copies of the Virasoro algebra) and the presence of a classical central charge were first found in [7] using the ADM formalism. The simplicity of our derivation based in the Chern-Simons formulation for 2+1 gravity should be noted.

D. Quantization

Lastly we consider the quantization of the above algebra. As discussed before, the classical Virasoro algebra cannot be quantized straightforwardly because of normal ordering problems. The Virasoro generators (78) have to be modified by

$$\hat{L}_n^\pm = \beta_\pm : L_n^\pm : + a_\pm \delta_n \qquad (84)$$

where both generators satisfy the Virasoro algebra with central charge equal to

$$c_\pm = \beta_\pm N + 12k\beta_\pm^2 \alpha_\pm^2. \qquad (85)$$

In our normalizations the quadratic Casimir is given by $Q = -2$. Therefore, the constants $\beta_\pm$ and $a_\pm$ are given by [see Eq. (57)]

$$\beta_\pm = \frac{k}{k \mp 1}, \qquad a_\pm = \frac{k^2}{2(k \mp 1)^2}. \qquad (86)$$

The relation between the zero-mode operators $L_0^\pm$ with the mass and angular momentum of the black hole [Eqs. (81) and (82)] suggests that only highest-weight representations for the Virasoro generators should be physically relevant. These representations have a lower bound state $|\omega\rangle$ satisfying

$$\hat{L}_n|\omega\rangle = 0, \qquad T_n^a|w\rangle = 0 \qquad (n > 0). \qquad (87)$$

The value of the operators $\hat{L}_0^\pm$ on this state are

$$\hat{L}_0^\pm = \frac{k^2}{2(k \mp 1)^2}[k + (k \mp 1)(M \mp J/l)]. \qquad (88)$$

Because of the shift $a_\pm$ in (84), a reasonable condition for the lowest eigenvalue of $\hat{L}_0^\pm$ is

$$\hat{L}_0^\pm - a_\pm \geq 0. \qquad (89)$$

This condition gives the following bounds for the black hole mass and angular momentum:

$$M + 1 \geq |J/l|. \qquad (90)$$

The black hole horizon exists when $M > |J/l|$; therefore condition (90) is not enough to ensure cosmic censorship. In the context of this paper this is irrelevant because one can always add a constant to L_0, shifting its zero point. This shift, however, alters the form of the central charge. The criterion followed in this paper has been to keep the usual form of the central charge in the Virasoro algebra obtaining the bound (90). As discussed in [17], the region $M > 0$ corresponds to the black hole spectrum (with a causal singularity [18] at the origin), and $M = 0$ is the black hole vacuum. On the other hand, the region $-1 < M < 0$ corresponds to the point particles (with conical singularities) discussed in [19]. Finally, the lower state, $M = -1$, is anti–de Sitter space which has no singularities.

Similar bounds for the possible values of M and J have been discussed in [20,21] using supersymmetric techniques. A proof of an energy theorem in 2+1 gravity is given in [22].

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