Lens spaces in the Regge calculus approach to quantum cosmology

Danny Birmingham*
Universiteit van Amsterdam, Instituut voor Theoretische Fysica, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
(Received 6 June 1995)

We study the wave function for a universe which is topologically a lens space within the Regge calculus approach. By restricting the four-dimensional simplicial complex to be a cone over the boundary lens space, described by a single internal edge length, and a single boundary edge length, one can analyze in detail the analytic properties of the action in the space of complex edge lengths. The classical extrema and convergent steepest descent contours of integration yielding the wave function are found. Both the Hartle-Hawking- and Linde-Vilenkin-type proposals are examined and, in all cases, we find wave functions which predict a Lorentzian oscillatory behavior in the late universe. The behavior of the results under subdivision of the boundary universe is also presented.

PACS number(s): 04.60.Nc, 98.80.Hw

I. INTRODUCTION

The study of simplicial approaches to the quantization of gravity is generally divided into those based either on Regge calculus [1], for reviews see [2–5], or dynamical triangulations [6]. In the latter, one typically restricts attention to a given simplicial topology with fixed lengths assigned to the edges (one-simplices) of the simplicial complex. The metric is then taken to be generated by summing over various triangulations of the given topology. These triangulations can be obtained by applying a set of local moves to a starting complex, and with the specification of a Boltzmann weight, one can thus simulate the quantum path integral for the topology in question.

In the Regge calculus approach, on the other hand, the simplicial complex which models the topology of interest is taken to be fixed, while the squared edge length assignments become the dynamical variables. Thus, the quantum theory is defined by a summation over edge lengths, which serves to model the continuum integration over the metric tensor. This approach enjoys some advantages; in particular one can analyze with ease both classical and semiclassical issues, and such calculations are often useful in determining the viability of any approach. It is also possible to study these models within a minisuperspace of edge lengths, whereby one truncates the allowed set of dynamical variables to a smaller more manageable set, known as simplicial minisuperspace. Such information is not so readily extracted from the dynamical triangulation framework.

In [7], the application of the Regge calculus approach to quantum cosmology was initiated through the study of the Hartle-Hawking [8] wave function. The particular example considered there was to take the spatial universe to be topologically S^3. The space of complex valued edge lengths was restricted to that consisting of a single internal edge length and a single boundary edge length. A steepest descent contour of constant imaginary action which yielded a convergent path integral was found. The resulting wave function was shown to behave in the desired way, namely a Lorentzian oscillatory behavior, for values of the boundary edge length greater than a certain critical value. For values less than the critical value, the wave function displayed Euclidean behavior. In fact, it was also observed that a closed contour of integration existed in this model, which had the property that it was deformable to the steepest descent contours for all values of the boundary edge length. As discussed in [9], one can adopt the viewpoint that the integral should be defined through a contour specification which is independent of the arguments of the wave function. The closed contour found in [7] satisfies this criterion, and as such one may regard it as a contour prescription for the model.

An important observation made in [7] was that while an integration over real valued Euclidean geometries may yield a convergent result for the path integral, it would not predict oscillatory behavior of the wave function in the late universe. As a result, one is obliged to study the path integral in the space of complex valued edge lengths. General criteria for defining the wave function of the universe were explored in [10], and explicit computations in continuum minisuperspace models were performed in [9,11–13]. A calculation in three-dimensional Regge calculus was presented in [14].

The purpose of the present investigation is to study nontrivial topological and cobordism effects within this Regge calculus approach to quantum cosmology. In particular, we obtain the wave function for a universe which is topologically a lens space $L(p,1)$, $p \geq 2$. This is achieved by considering the four-dimensional simplicial complex to be given by the cone over the boundary universe $L(p,1)$. We again restrict attention to a minisuperspace consisting of a single internal edge length and a single boundary edge length. In fact, such spatial universes correspond to those present in the Eguchi-Hanson [15] and associated Gibbons-Hawking [16,17] series of gravitational instantons, although their four-dimensional topol-

*Electronic address: Dannyb@phys.uva.nl

0556-2821/95/52(10)/5760(13)/$06.00

ogy is not captured by the cone structures considered here.

We analyze the wave function of the Hartle-Hawking [8] and Linde-Vilenkin [18,19] type, and in the cases considered we find convergent steepest descent contours of integration which satisfy their requirements. Once the convergent contours are identified, one can proceed to evaluate the full path integral, or appeal to a semiclassical analysis. The interesting physical outcome of our investigation is that for large values of p, classical extrema of the Regge action are present in the Lorentzian regime for smaller values of the bounding edge length, as compared to S^3. Thus, the critical value of the boundary edge length separating the Lorentzian and Euclidean regimes in these cases is smaller than the corresponding critical value for S^3. In particular, we find that for $p \geq 5$, there are Lorentzian extrema for all positive values of the boundary edge length. Consequently, we find in such cases that the wave function exhibits Lorentzian oscillatory behavior for all physical values of the boundary edge length, without the presence of a Euclidean regime. For all p, wave functions which exhibit Lorentzian oscillatory behavior in the late universe can be obtained. In addition, for the case of Hartle-Hawking wave functions, one can establish the existence of a closed contour of integration which is independent of the arguments of the wave function. The structure of this contour is identical to that found in [7].

Finally, we examine the behavior of the results under lattice subdivisions of the boundary universe. By appealing to the lattice subdivision moves of Pachner [20], it is shown that by remaining within the initial minisuperspace, one can readily read off the results on finer triangulations.

The outline of this work is as follows. In the following section, we list for reference the relevant equations in the general formalism of Regge calculus. We then present in Sec. III the construction of simplicial complexes for the lens space $L(p,1)$, due to [21]. We discuss the symmetries of these complexes and describe some properties of their associated cones. Section IV is devoted to a study of the analytical structure of the Regge action in the simplicial minisuperspace, and the classical extrema are obtained. This allows us in Sec. V to investigate the steepest descent contours in the space of complex edge lengths, thus yielding the wave function. In Sec. VI, we present a brief study of the behavior of the results under subdivisions of the triangulations.

II. GENERAL FORMALISM

The basic idea in the Regge calculus approach to the quantization of gravity is to take the spacetime manifold to be modeled by a simplicial complex. If one restricts attention to a fixed topology, then one can capture this chosen topology by complexes which are combinatorially equivalent. The dynamical variables in the theory are now given by the assignment of squared edge length variables to the edges (one-simplices) of the complex. Thus, the path integration over the metric tensor is replaced by an integration over a finite number of edge length variables.

Given a simplicial complex M with boundary ∂M, possibly containing several disjoint components, the corresponding amplitude of quantum gravity will thus take the form

$$\Psi_0(s_b) = \int_C d\mu(s_i)\, \exp[-I(s_b, s_i)] \; . \tag{1}$$

Here, the variables s_b specify the edge lengths of the boundary and the integration is over the internal dynamical variables s_i. The form of the measure μ, the contour of integration C, along with the action I, are required to complete the specification of the model.

The Euclidean Einstein action with cosmological term for a manifold with boundary is given by

$$I = -\frac{1}{16\pi G}\int_M d^4x\sqrt{g}R + \frac{2\Lambda}{16\pi G}\int_M d^4x\sqrt{g} -\frac{2}{16\pi G}\int_{\partial M} d^3x\sqrt{h}K \; , \tag{2}$$

where R is the scalar curvature of the metric g, Λ is the cosmological constant, and K is the extrinsic curvature scalar of the induced metric h on the boundary. The simplicial analog of this action is the corresponding Regge action [1,22], which takes the form

$$I = -\frac{2}{l^2}\sum_{\sigma_2\subset \mathrm{int}(M)} V_2(\sigma_2)\theta(\sigma_2) + \frac{2\Lambda}{l^2}\sum_{\sigma_4\subset \mathrm{int}(m)} V_4(\sigma_4) -\frac{2}{l^2}\sum_{\sigma_2\subset \partial M} V_2(\sigma_2)\psi(\sigma_2) \; . \tag{3}$$

where the Planck length, in units where $\hbar = c = 1$, is $l = (16\pi G)^{1/2}$. The various terms in (3) are described as follows. The Einstein term is represented by a summation over internal two-simplices $\sigma_2 \subset \mathrm{int}(M)$ (also known as hinges). An internal hinge is any two-simplex of the complex which contains at least one internal vertex, and the notation $\mathrm{int}(M)$ is used to denote this set. The form of the Einstein action involves the volume of the hinge $V_2(\sigma_2)$ and the associated deficit angle $\theta(\sigma_2)$. Similarly, the boundary term is given in terms of the boundary two-simplices and their associated deficit angles denoted by $\psi(\sigma_2)$. The cosmological term is simply represented as a sum over the volumes $V_4(\sigma_4)$ of the four-simplices σ_4 of the complex. As mentioned above, the dynamical variables in the Regge calculus approach are the edge length assignments. Thus, the above action should be expressed in terms of these variables. In fact, the Regge action is expressible in terms of the squared edge lengths, and for details of the procedures involved we refer to [1–5].

For ease of reference, we collect here some of the relevant formulas which will be useful in the following.

The internal deficit angle is given by

$$\theta(\sigma_2) = 2\pi - \sum_{\sigma_4\supset\sigma_2} \theta_d(\sigma_2,\sigma_4) \; , \tag{4}$$

where the summation is over all four-simplices containing the two-simplex σ_2. The dihedral angle θ_d can then be expressed in terms of the squared edge lengths, see below. The corresponding deficit angle for the boundary two-simplex is

$$\psi(\sigma_2) = \pi - \sum_{\sigma_4 \supset \sigma_2} \theta_d(\sigma_2, \sigma_4) \ . \tag{5}$$

It remains only to give the explicit formulas for the various volumes and dihedral angles in terms of the squared edge lengths. The volume of the two-simplex $\sigma_2 = [i,j,k]$ is given by

$$V_2([i,j,k]) = \frac{1}{2!} \operatorname{Det}^{1/2}[M_2([i,j,k])] \ , \tag{6}$$

where

$$M_2([i,j,k]) = \begin{pmatrix} s_{ij} & \frac{1}{2}(s_{ij} + s_{ik} - s_{jk}) \\ \frac{1}{2}(s_{ij} + s_{ik} - s_{jk}) & s_{ik} \end{pmatrix} . \tag{7}$$

Here, s_{ij} is the squared edge length between the vertices i and j. Similarly, the volume of the three-simplex $[i,j,k,l]$ is

$$V_3([i,j,k,l]) = \frac{1}{3!} \operatorname{Det}^{1/2}[M_3([i,j,k,l])] \ , \tag{8}$$

where

$$M_3([i,j,k,l]) = \begin{pmatrix} s_{ij} & s_{ijk} & s_{ijl} \\ s_{ijk} & s_{ik} & s_{ikl} \\ s_{ijl} & s_{ikl} & s_{il} \end{pmatrix} , \tag{9}$$

and $s_{ijk} = \frac{1}{2}(s_{ij} + s_{ik} - s_{jk})$. Also, the four-volume assumes the form

$$V_4([i,j,k,l,m]) = \frac{1}{4!} \operatorname{Det}^{1/2}[M_4([i,j,k,l,m])] \ , \tag{10}$$

with

$$M_4([i,j,k,l,m]) = \begin{pmatrix} s_{ij} & s_{ijk} & s_{ijl} & s_{ijm} \\ s_{ijk} & s_{ik} & s_{ikl} & s_{ikm} \\ s_{ijl} & s_{ikl} & s_{il} & s_{ilm} \\ s_{ijm} & s_{ikm} & s_{ilm} & s_{im} \end{pmatrix} . \tag{11}$$

Finally, the dihedral angle of the two-simplex $\sigma_2 = [i,j,k]$ is expressible as

$$\theta_d([i,j,k],[i,j,k,l,m]) = \arccos\left\{ \left(\frac{1}{3!}\right)^2 \operatorname{Det}[M_{33}([i,j,k],[i,j,k,l,m])] / \{V_3([i,j,k,l]) V_3([i,j,k,m])\} \right\} , \tag{12}$$

where

$$M_{33}([i,j,k],[i,j,k,l,m]) = \begin{pmatrix} s_{ij} & s_{ijk} & s_{ijm} \\ s_{ijk} & s_{ik} & s_{ikm} \\ s_{ijl} & s_{ikl} & s_{ilm} \end{pmatrix} . \tag{13}$$

From the above relations, the explicit form of the action for a given simplicial complex as a function of the squared edge lengths can be obtained. In the sequel, we will be interested in analyzing the analytic properties of this action in the space of complex valued edge lengths, for the purposes of computing the amplitudes of interest.

III. TRIANGULATIONS OF LENS SPACES AND THEIR CONES

Our task now is to obtain explicit triangulations for the lens spaces which serve to model the spatial universe. Once this is achieved, one can reality construct the corresponding cone over $L(p,1)$.

A particularly convenient construction of simplicial complexes for lens spaces, with a small number of simplices, has been given in [21]. We present here the essential ingredients in this construction. These triangulations are interesting in their use of a small number of simplices, and the accompanying dihedral automorphism group. The triangulation of $L(k-2,1)$ for $k \geq 4$, is denoted by S_{2k}, and it has dihedral automorphism group D_k. The number N_i of i-simplices in each dimension is given by

$$\begin{aligned} N_0 &= 2k + 3 \ , \\ N_1 &= 2k^2 + 4k + 3 \ , \\ N_2 &= 4k^2 + 4k \ , \\ N_3 &= 2k^2 + 2k \ . \end{aligned} \tag{14}$$

In particular, the resulting triangulation of $L(2,1) \cong RP^3$ with 11 vertices is the smallest number possible for this manifold [23].

For $k \geq 4$, let $Z_{2k} = \{0,1,\ldots,2k-1\}$ denote the additive set of integers mod$2k$, and consider the permutations of that set defined by

$$\begin{aligned} c(i) &= i + 1 \bmod 2k \ , \\ a(i) &= i + 2 \bmod 2k \ , \\ b(i) &= 2k - i \bmod 2k \ . \end{aligned} \tag{15}$$

We denote by $C_{2k} = gr(c)$ the group generated by the permutation c, and its action can be naturally extended to the set of all simplices with vertices taking values in the set Z_{2k}. The automorphism group of the triangulation is the dihedral group generated by a and b, although for the purpose of obtaining the simplicial complex, only the elements c and a are required.

The first step in the construction is to introduce the set of three-simplices

$$\Delta_i = [0, 1, k-i, k-i+1] \ \text{ with } i = 0, 1, \ldots, k-3 \ , \tag{16}$$

and the corresponding orbit under the action of C_{2k} is written as $C_{2k}(\Delta_i)$. Each of these orbits contains $2k$ elements, save for $C_{2k}(\Delta_0)$ which contains k elements. A simplicial complex containing $k(2k-5)$ three-simplices is now given by

$$H_{2k} = C_{2k}(\Delta_0) \cup \cdots \cup C_{2k}(\Delta_{k-3}) \ . \tag{17}$$

Here, we use the $\cup$ notation to describe the union of simplices, but when the need arises we will specify precisely the relative orientations of the three-simplices involved.

Consider now the following collection of simplices:

$$\begin{aligned} F_1 &= [0,1,3] \ , \\ F_2 &= [0,2,3] \ , \\ F_3 &= [1,2,4] \ , \\ F_4 &= [1,3,4] \ , \\ E_1 &= [0,2] \ , \\ E_2 &= [1,3] \ . \end{aligned} \tag{18}$$

The next step is to introduce three additional vertices (x, y, z), and consider the simplicial complex

$$\begin{aligned} K_{2k} = &[y, z, C_k(E_1)] \cup [x, z, C_k(E_2)] \cup [x, y, C_k(E_2)] \\ &\cup [y, C_k(F_1)] \\ &\cup [y, C_k(F_2)] \cup [z, C_k(F_3)] \cup [z, C_k(F_4)] \ , \end{aligned} \tag{19}$$

where the subgroup $C_k \subset C_{2k}$ of index two is given by $C_k = gr(a)$.

Finally, we obtain the simplicial complex

$$S_{2k} = H_{2k} \cup K_{2k} \ . \tag{20}$$

According to [21], S_{2k} is a D_k-symmetric triangulation of the lens space $L(k-2,1)$, for $k \geq 4$.

One can proceed and obtain the explicit set of three-simplices for a given case of interest, and we list here the resulting triangulation of $L(2,1)$. It is important to note that the relative orientations of the three-simplices in S_{2k} need to be specified in order to ensure a vanishing boundary. This yields

$$\begin{aligned} L(2,1) = &+[0,1,4,5] + [1,2,5,6] + [2,3,6,7] - [0,3,4,7] + [0,1,3,4] \\ &+[1,2,4,5] + [2,3,5,6] + [3,4,6,7] - [0,4,5,7] + [0,1,5,6] \\ &+[1,2,6,7] - [0,2,3,7] + [y,z,0,2] + [y,z,2,4] + [y,z,4,6] \\ &-[y,z,0,6] - [x,z,1,3] - [x,z,3,5] - [x,z,5,7] + [x,z,1,7] \\ &+[x,y,1,3] + [x,y,3,5] + [x,y,5,7] - [x,y,1,7] + [y,0,1,3] \\ &+[y,2,3,5] + [y,4,5,7] + [y,1,6,7] - [y,0,2,3] - [y,2,4,5] \\ &-[y,4,6,7] - [y,0,1,6] + [z,1,2,4] + [z,3,4,6] + [z,0,5,6] \\ &+[z,0,2,7] - [z,1,3,4] - [z,3,5,6] - [z,0,5,7] - [z,1,2,7] \ . \end{aligned} \tag{21}$$

In the sequel, we shall endow these simplicial complexes with a geometry by assigning edge length variables to the one-simplices, so it is useful to discuss the automorphism groups of these complexes, and the resulting simplicial geometry.

The automorphism group of S_{2k} is the dihedral group D_k generated by a and b of Eq. (15). This is the group of order $2k$ with generators satisfying the relations $a^k = 1$, $b^2 = 1$, and $ba = a^{-1}b$. In order to establish this, one extends the action of the generators to the vertices x, y, z as follows:

$$\begin{aligned} &a(x) = x \ , \quad a(y) = y \ , \quad a(z) = z \ , \\ &b(x) = x \ , \quad b(y) = z \ , \quad b(z) = y \ . \end{aligned} \tag{22}$$

For the purpose of illustration, let us discuss in some detail the symmetry properties of the above triangulation of $L(2,1)$. The general case will then follow quite straightforwardly.

The symmetry group of $L(2,1)$ is the dihedral group D_4 of order eight, and one can check explicitly that the form of the triangulation (21) is indeed invariant. First, one finds that the vertices split into four independent orbits under the group action: namely,

$$\begin{aligned} &\{0,2,4,6\} \ , \\ &\{1,3,5,7\} \ , \\ &\{x\} \ , \\ &\{y,z\} \ . \end{aligned} \tag{23}$$

One can define the notion of a homogeneous triangulation as being one for which the vertices form a single orbit with respect to the action of the symmetry group. For the triangulations of lens spaces considered here, we see that they are not homogeneous in this sense. Consequently, one notes that the number of one-simplices which emanate from vertices of distinct orbits can be different in general.

The one-simplices are divided into 11 orbits as follows:

$$\{01, 12, 23, 34, 45, 56, 67, 07\} \ ,$$

$$\{02, 24, 46, 06\} \ ,$$

$$\{03, 25, 47, 16, 05, 36, 14, 27\} \ ,$$

$$\{04, 26\} \ ,$$

$$\{13, 35, 57, 17\} \ ,$$

$$\{15,37\}\ ,$$
$$\{x1,x3,x5,x7\}\ ,$$
$$\{y0,y2,y4,y6,z0,z2,z4,z6\}\ ,$$
$$\{y1,y3,y5,y7,z1,z3,z5,z7\}\ ,$$
$$\{xy,xz\}\ ,$$
$$\{yz\}\ . \tag{24}$$

A natural geometry is defined as one for which independent edge lengths are assigned to each of these orbits, and the resulting simplicial geometry is easily seen to be anisotropic. The notion of anisotropy is defined as one for which independent edge length variables emanate from a given vertex.

For the general case of S_{2k}, we again note that there are four vertex orbits:

$$\{0,2,\ldots,2k-2\}\ ,$$
$$\{1,3,\ldots,2k-1\}\ ,$$
$$\{x\}\ ,$$
$$\{y,z\}\ . \tag{25}$$

Before determining the number of one-simplex orbits, let us recall that a simplicial complex is said to be two-neighborly if every pair of vertices form a one-simplex [23]. For this to be the case, we have $N_1 = N_0(N_0-1)/2$. The triangulations S_{2k} fail to be two-neighborly precisely because of the absence of one-simplices of the form

$$\{x0,x2,\ldots,x(2k-2)\}\ . \tag{26}$$

With this knowledge, it is easy to see that there are five orbits involving the vertices x,y,z, which are given by the extension of the last five entries in (24). In addition, there are $k+[k/2]$ orbits among the vertices $\{0,1,\ldots,2k-1\}$, where $[x]$ denotes the greatest integer less than or equal to x. In general then, the total number $(k+[k/2]+5)$ of one-simplex orbits depends on the particular lens space under consideration. As a result, if we choose a simplicial geometry described by $(k+[k/2]+5)$ independent edge length variables, the degree of anisotropy of $L(k-2,1)$ increases with k.

Given these simplicial complexes for the spatial universe, we can turn our hand to constructing a four-dimensional simplicial complex which has $L(p,1)$ as its boundary. One means of achieving this is to consider the complex known as the cone over $L(p,1)$ [24]. This simply involves the addition of a single extra vertex, the cone vertex denoted by c, and joining this to all vertices of the bounding lens space. Each four-simplex of the cone then takes the form $[c,\sigma_3]$, where σ_3 is a three-simplex of the boundary complex. It is typical to denote by $M_4 = c\star M_3$ the four-dimensional cone complex over its boundary M_3. With this orientation, the boundary of the cone M_4 is $+M_3$. The number of i-simplices contained in the cone are immediately evident:

$$N_0(M_4) = N_0(M_3)+1\ ,$$
$$N_1(M_4) = N_1(M_3)+N_0(M_3)\ ,$$
$$N_2(M_4) = N_2(M_3)+N_1(M_3)\ ,$$
$$N_3(M_4) = N_3(M_3)+N_2(M_3)\ ,$$
$$N_4(M_4) = N_3(M_3)\ . \tag{27}$$

As verified in [21], the above triangulations of lens spaces are in fact simplicial manifolds, satisfying the so-called manifold condition. Given an n-dimensional simplicial complex K, we recall that the star of a simplex σ in K is the collection of simplices which contain σ, together with all their subsimplices. The link of the simplex σ is then the set of simplices in the star of σ which do not contain σ. The simplicial complex K is said to be a simplicial n manifold if and only if the link of every k simplex is combinatorially equivalent to an $(n-k-1)$ sphere [25].

In particular then, because of the manifold condition, the Euler character of the lens space triangulations vanishes, $N_0 - N_1 + N_2 - N_3 = 0$. The fact that the complex is closed implies that $N_2 = 2N_3$; the three-simplices are glued together pairwise. One can then express the information in (27) in terms of the two independent quantities $N_1(M_3)$ and $N_3(M_3)$, for example.

IV. ANALYSIS OF THE REGGE ACTION AND ITS EXTREMA

Having chosen our four-dimensional spacetime to be represented by a cone over the bounding lens space $M_3 = L(p,1)$, we can now obtain the explicit form of the associated Regge action. The cone structure introduced is appealing in the sense that we can immediately identify a natural minisuperspace in which to study the model. Since the only internal vertex is the cone vertex, all the internal one-simplices are of the form $[c,b]$, where b is a vertex lying in the boundary. The truncation of edge length variables to the minisuperspace of interest can be effected by considering all internal edges to be described by a single internal edge length, denoted s_i. In addition, we assume that the boundary lens space is described in terms of a single bounding edge length s_b. Thus, we take all the independent one-simplex orbits to be described by a single edge length. This choice is appealing in the sense that we reduce the degree of anisotropy mentioned in the previous section, and also a more direct comparison can then be made with the results for S^3.

As a result, the Regge action will be a function of only two variables, and our task will be to perform the integration over the single internal edge length. In this respect, we will be able to appeal to the analysis performed for the case of S^3 boundary in [7]. It is perhaps worth noting that in the model discussed in [7], the simplicial complex with S^3 boundary was itself a simplicial manifold. In the case under study here, the four-dimensional cone does not satisfy the manifold condition, in particular because of the fact that the link of the cone vertex is a lens space rather than S^3. Nevertheless, it does provide a particularly simple cobordism with a boundary of non-trivial topology. Furthermore, the general framework of Regge calculus requires only that spacetime be represented by a simplicial complex. One might also note that the simplicial cones considered here belong to the class of simplicial conifolds discussed in [26].

It is convenient to introduce the scaled variables:

$$z = \frac{s_i}{s_b}\ , \quad S = \frac{H^2 s_b}{l^2}\ , \tag{28}$$

where $H^2 = l^2\Lambda/3$. The general formulas of Sec. II now yield the following results. For the cosmological term, we note that each four-simplex is of the form $[c, \sigma_3]$, where σ_3 is a three-simplex in the boundary, and has four-volume:

$$V_4 = \frac{1}{24\sqrt{2}} s_b^2 \left(z - \frac{3}{8}\right)^{1/2}\ . \tag{29}$$

The number of four-simplices is of course $N_3(M_3)$.

To evaluate the Einstein term, we observe that the internal two-simplices are all of the type $[c, \sigma_1]$, where σ_1 is a one-simplex on the boundary. Hence, there are $N_1(M_3)$ internal two-simplices each with volume

$$V_2(\text{int}) = \frac{2_b}{2}\left(z - \frac{1}{4}\right)^{1/2}\ . \tag{30}$$

The associated dihedral angle of this internal two-simplex is then

$$\theta_d(\text{int}) = \arccos\left(\frac{2z-1}{2(3z-1)}\right)\ . \tag{31}$$

Here, one uses the fact that the volumes of the internal and boundary three-simplices are

$$V_3(\text{int}) = \frac{1}{12} s_b^{3/2}(3z-1)^{1/2}\ ,$$

$$V_3(\text{bound}) = \frac{\sqrt{2}}{12} s_b^{3/2}\ . \tag{32}$$

Turning now to the boundary term in the action, we again have a single type of two-simplex. There are $2N_3(M_3)$ of these and the volume of each is

$$V_2(\text{bound}) = \frac{\sqrt{3}}{4} s_b\ . \tag{33}$$

The dihedral angle in this case is given by

$$\theta_d(\text{bound}) = \arccos\left(\frac{1}{2\sqrt{2}(3z-1)^{1/2}}\right)\ . \tag{34}$$

Assembling the various terms, we obtain the complete Regge action in the form

$$I(z,S) = [-SF(z) + S^2 G(z)]/H^2\ , \tag{35}$$

where

$$F(z) = a_1[\pi - 2\arccos(z_1)] + a_2(z - \tfrac{1}{4})^{1/2}[2\pi - a_3\arccos(z_2)]\ , \tag{36}$$

$$G(z) = a_4(z - \tfrac{3}{8})^{1/2}\ , \tag{37}$$

and we have introduced the variables

$$z_1 = \frac{1}{2\sqrt{2}(3z-1)^{1/2}}\ , \quad z_2 = \frac{2z-1}{2(3z-1)}\ . \tag{38}$$

The coefficients a_i appearing in the action are expressed in terms of the number of i-simplices of the boundary as follows:

$$a_1 = N_3(M_3)\sqrt{3}\ ,$$

$$a_2 = N_1(M_3)\ ,$$

$$a_3 = 6N_3(M_3)/N_1(M_3)\ ,$$

$$a_4 = N_3(M_3)/(4\sqrt{2})\ . \tag{39}$$

The values are collected in Table I for the cases $2 \le p \le 7$, and can be read off from equation (14) in general.

It is important to make the following observation regarding the factor of 2 appearing in the formula for the deficit angle of a boundary two-simplex [the first term in $F(z)$]. The bounding lens space is represented by a closed simplicial complex, and closure of the complex means that each two-simplex is contained in precisely two three-simplices. Thus, when we elevate this boundary complex to its associated cone, we immediately know that the number of four-simplices containing each boundary two-simplex is again precisely 2, each being of the form $[c, \sigma_3]$ with σ_3 belonging to the boundary. This fact becomes crucially relevant when we search for the extrema of the Regge action.

The value of the coefficient a_3 is obtained by determining the number of four-simplices which contain each internal two-simplex. Since each internal two-simplex is of the form $[c, \sigma_1]$, where σ_1 is a one-simplex in the boundary, we must equivalently determine the number of boundary three-simplices containing a given boundary one-simplex. This number depends on the individual one-simplex. However, since each three-simplex

TABLE I. The coefficients a_i appearing in the Regge action for the cone over the lens space $L(p,1)$, and the critical value S_{crit} of the boundary edge length.

M_3	a_1	a_2	a_3	a_4	S_{crit}
$L(2,1)$	$40\sqrt{3}$	51	240/51	$5\sqrt{2}$	3.537 26
$L(3,1)$	$60\sqrt{3}$	73	360/73	$15/\sqrt{2}$	1.463 91
$L(4,1)$	$84\sqrt{3}$	99	504/99	$21/\sqrt{2}$	0.109 891
$L(5,1)$	$112\sqrt{3}$	129	672/129	$14\sqrt{2}$	−0.842 155
$L(6,1)$	$144\sqrt{3}$	163	864/163	$18\sqrt{2}$	−1.54737
$L(7,1)$	$180\sqrt{3}$	201	1080/201	$45/\sqrt{2}$	−2.090 39

contains six one-simplices, the sum of the number of three-simplices containing all the one-simplices is clearly $6N_3(M_3)$. Therefore, the sum of the internal deficit angles takes the form given above, with the quoted values of a_2 and a_3.

The explicit form of the Regge action enables us to examine its analytic and asymptotic properties. On general grounds, as noted in [7], the action is an analytic function of the squared edge lengths, apart from the presence of certain branch surfaces and logarithmic infinities, which arise when the volumes of various simplices vanish. In the case at hand, we can explicitly identify the analytic nature of the action $I(z,S)$. In fact, because of our choice of cone complexes and simplicial minisuperspace, the analytic behavior of the action parallels that presented in [7] for the case of S^3.

There is a square root branch point at $z = \frac{3}{8}$, where the volume of the four-simplices vanishes. Similarly, one spots a square-root branch point when the volume of the internal two-simplices vanishes, at $z = \frac{1}{4}$. In addition, the vanishing of the volume of internal three-simplices is responsible for a square root branch point at $z = \frac{1}{3}$; this point also affords a logarithmic branch point. Using the representation,

$$\arccos(z) = -i \ln[z + \sqrt{z^2-1}] \ , \tag{40}$$

we find that in the neighborhood of $z = \frac{1}{3}$, the action has the behavior

$$I(z,S) \sim \frac{i}{H^2} S[2\sqrt{3}N_3(M_3)] \ln[3z-1] \ . \tag{41}$$

According to Eq. (11) the metric in each four-simplex is specified in terms of the squared edge lengths, and is real for real values of s_{ij}. In our specific minisuperspace, we have just a single type of four-simplex, and the signature of the metric depends on the value of the variable z, the corresponding eigenvalues being

$$\lambda = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 4(z - \tfrac{3}{8})) \ . \tag{42}$$

Thus, for real $z > \frac{3}{8}$, we have a regime of real Euclidean geometries, while for real $z < \frac{3}{8}$ there lies a regime of real geometries of Lorentzian signature.

Due to the presence of the branch points, we need to declare the location of the branch cuts, and the corresponding phases of the action on its various sheets will then be determined. One first notes that the function $\arccos(z)$ has branch points at -1, $+1$, and ∞, and conventionally the branch cuts are placed from $-\infty$ to -1, and from $+1$ to $+\infty$. With this choice, $\arccos(z)$ is real for real $-1 < z < +1$. The corresponding cuts for $\arccos(z_2)$ in the z plane then lie between the points $\frac{1}{3}$ to $\frac{3}{8}$, and $\frac{1}{4}$ to $\frac{1}{3}$, respectively. The branch cuts of $\arccos(z_1)$ lie between $\frac{1}{3}$ and $\frac{3}{8}$. In addition, because of the presence of the square-root branch points at $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{3}{8}$, it is convenient to define a first sheet for the action $I(z,S)$ with a branch cut extending from $\frac{3}{8}$ to $-\infty$.

With the z-plane cut in the above way, we note that for real $z > \frac{3}{8}$, we have real valued Euclidean signature action, with real volumes and real deficit angles. The action on the first sheet is given by Eqs. (35)–(38) with positive signs taken for the square-root factors.

As we have seen, for real $z < \frac{3}{8}$, we have a region of Lorentzian geometries. In particular, however, in the range real $z < \frac{1}{4}$, we find that the action is purely imaginary. On the first sheet, we have

$$F(z) = ia_1 \left[-2 \operatorname{arcsinh}\left(\frac{1}{2\sqrt{2}(1-3z)^{1/2}} \right) \right] + ia_2 \left(\frac{1}{4} - z \right)^{1/2} \left[2\pi - a_3 \arccos\left(\frac{2z-1}{2(3z-1)} \right) \right] \tag{43}$$

and

$$G(z) = ia_4(\tfrac{3}{8} - z)^{1/2} \ . \tag{44}$$

The identity

$$\pi - 2\arccos(iz) = 2\arcsin(iz) = 2i \operatorname{arcsinh}(z) \tag{45}$$

has been used in the above. It is here that we notice that the factor of 2 in the boundary deficit angle is crucial, so that the action in the range real $z < \frac{1}{4}$ is purely imaginary. As we shall see, this is important for the existence of Lorentzian signature solutions to the Regge equations of motion. One now sees that if the action is continued once around all the branch points at $z = \frac{1}{4}$, $\frac{1}{3}$, and $\frac{3}{8}$, we will reach a second sheet, and the value of the action is the negative of its value on the first sheet. Thus, continuing twice around all branch points returns the action to its initial value. This behavior can be established by simply noting that encircling all the branch points renders no change in $\arccos(z_2)$, while $\arccos(z_1) \to \arccos(-z_1) = \pi - \arccos(z_1)$.

The asymptotic behavior of the action is important when discussing the extrema of the action, and when searching for convergent contours of integration. For large $|z|$ on the first sheet, we have

$$F(z) \sim a_4 S_{\text{crit}} (z - \tfrac{1}{4})^{1/2} \tag{46}$$

and

$$G(z) \sim a_4 (z - \tfrac{3}{8})^{1/2} \ , \tag{47}$$

where

$$S_{\text{crit}} = \frac{a_2}{a_4}[2\pi - a_3 \arccos(\tfrac{1}{3})] \ . \tag{48}$$

The asymptotic behavior of the complete action on the first sheet then takes the form

$$I(z,S) \sim \frac{a_4}{H^2} S(S - S_{\text{crit}}) z^{1/2} \ . \tag{49}$$

Thus, it is clear that the behavior of the action is crucially dependent on whether the value of the boundary edge length is greater or less than the critical value, denoted by S_{crit}. The values of S_{crit} for the cases $2 \le p \le 7$ are listed in Table I. Using Eq. (14), one sees that S_{crit} remains negative for all $p \ge 5$, and tends to a value

$S_{\rm crit} = -6.237\,08$ as p approaches infinity.

We can turn now to a description and analysis of the classical extrema of the action. The Regge equation of motion in this model takes the simple form

$$\frac{d}{dz} I(z,S) = 0 \ . \tag{50}$$

This equation is to be solved for the value of z subject to fixed boundary data S, and via Eq. (28), the solution then determines a complete simplicial geometry. Our physical restriction on the chosen boundary data is that S should be real valued and positive. The equation of motion can be rewritten in the form

$$S = \frac{F'(z)}{G'(z)} = \frac{a_2}{a_4} \frac{(z-\frac{3}{8})^{1/2}}{(z-\frac{1}{4})^{1/2}} [2\pi - a_3 \arccos(z_2)] \ , \tag{51}$$

where the prime indicates a derivative with respect to z. It should be pointed out that the above form requires the relation $a_1\sqrt{3} = a_2 a_3/2$, which is the case for the models considered here, as can be seen from Eq. (39).

Writing $F' = f_1 + i f_2$ and $G' = g_1 + i g_2$, we see that S is real if and only if $f_1 g_2 = g_1 f_2$. In particular then, solutions exist when F' and G' are both purely real valued, or when both are purely imaginary. Thus, we can declare that classical extrema exist in two regions, for real $z > \frac{3}{8}$, and for real $z < \frac{1}{4}$. The physical acceptability of such solutions will require in addition that S is positive. The fact that these constitute all possible solutions to the constraint $f_1 g_2 = g_1 f_2$ can be seen, for example, by expanding the various factors in (51) in series expansions.

From (51), we see that the value of the boundary edge length is always $S < S_{\rm crit}$, for real $z > \frac{3}{8}$. Thus, for every $0 < S < S_{\rm crit}$, there exists a real Euclidean solution at real $z > \frac{3}{8}$. Similarly, for every positive S with $S > S_{\rm crit}$, there is a real Lorentzian solution at real $z < \frac{1}{4}$. It should also be noted that these solutions occur in pairs, in addition to those on the first sheet, there are corresponding solutions on the second sheet with opposite value of the action. Interestingly, as can be seen from the values of $S_{\rm crit}$ quoted in Table I, real solutions with Lorentzian signature exist for smaller values of the boundary edge length, as compared to the case of S^3 studied in [7].

Furthermore, the value of $S_{\rm crit}$ is negative for $p \leq 5$. Therefore, it appears that no Euclidean solutions exist in these models. However, one could contemplate encircling the branch points either at $z = \frac{3}{8}$ or $\frac{1}{4}$. This will cause a change of sign in one of the square-root prefactors in (51). However, in addition the value of $\arccos(z_2)$ will flip sign, and thus S will remain negative and less that $S_{\rm crit}$. Thus, no physically acceptable Euclidean solutions exist in these cases.

On a related matter, one should perhaps note that even in the case when $S_{\rm crit}$ is positive, Euclidean solutions do not exist for every value of $z > \frac{3}{8}$. This can be seen even in the case with S^3 boundary. At values of z close to $\frac{3}{8}$, for example, $\frac{3}{8} < z < 0.4$ for S^3, and $\frac{3}{8} < z < 0.889\,512$ for $L(2,1)$, one finds that $a_3 \arccos(z_2) > 2\pi$, resulting in a negative value of S.

The structure of the classical solutions are presented for the cases of $L(2,1)$ and $L(5,1)$ in Figs. 1 and 2.

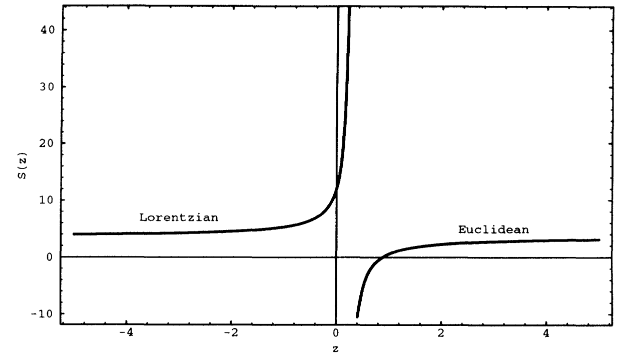

FIG. 1. The classical extrema of the Regge action for the cone over RP^3.

One can also examine the nature of the classical extrema of the Regge action for the case of vanishing cosmological constant Λ. In this case, the action consists solely of the F term (36), and the equation of motion then takes the form

$$F'(z) = \frac{a_2}{2(z-\frac{1}{4})^{1/2}} \left[2\pi - a_3 \arccos\left(\frac{2z-1}{2(3z-1)} \right) \right] = 0 \ . \tag{52}$$

Solutions exist for those values of z which satisfy

$$\arccos\left(\frac{2z-1}{2(3z-1)} \right) = \frac{2\pi}{a_3} \ . \tag{53}$$

Since $\cos^{-1}(z)$ is real for real $-1 < z < +1$, we see that a solution exists on the real axis at

$$z = \frac{[2\,\cos(2\pi/a_3) - 1]}{[6\,\cos(2\pi/a_3) - 2]} \ . \tag{54}$$

For the lens spaces $L(p,1)$, $p = 2,3,4$ considered here, the value of z lies on the real axis with $z > \frac{3}{8}$, thus corresponding to a Euclidean signature solution. Recall that $s_i = z s_b$, so that for each value of the boundary

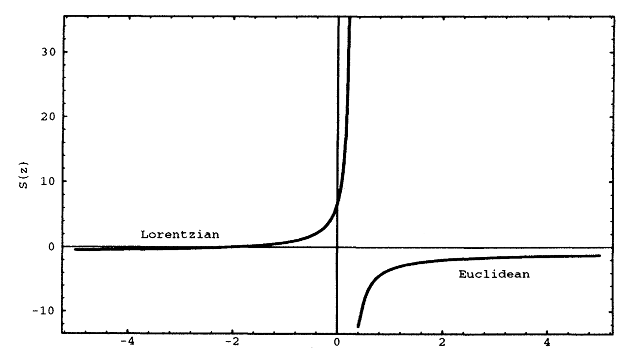

FIG. 2. The classical extrema of the Regge action for the cone over $L(5,1)$. In this case, solutions with Lorentzian signature are present for all physical values of the scaled boundary edge length squared S. There are no physical Euclidean signature solutions.

data $s_b > 0$, we have a full Euclidean signature solution for the Regge equations on the cone over $L(p,1)$. There are no Lorentzian signature solutions in these cases. On the other hand, for all $p \geq 5$, we find that z lies on the real axis with $z < \frac{1}{4}$, and thus we have a Lorentzian solution for all positive values of the boundary edge length s_b.

V. STEEPEST DESCENT CONTOURS FOR THE WAVE FUNCTION OF THE UNIVERSE

Armed with the classical solutions to the Regge equation of motion, we can proceed with our determination of the wave function of the universe. The remaining data needed is a specification of the measure μ, and the integration contour C. We shall take the measure in the form

$$d\mu(s_i) = \frac{ds_i}{2\pi i l^2} . \tag{55}$$

Therefore, the wave function is given as

$$\Psi_0(S) = \frac{S}{2\pi i H^2} \int_C dz \exp[-I(z,S)] . \tag{56}$$

We wish to determine if $\Psi_0(S)$ can be obtained in a form which exhibits oscillatory behavior for large values of the bounding edge length S. As shown in [7] for a universe with S^3 topology, this is indeed possible. We recall that there are two Lorentzian extrema when $S > S_{\text{crit}}$, one on the first sheet of the action, and one on the second sheet. At the corresponding values of real $z < \frac{1}{4}$, the action is purely imaginary and of opposite sign on the two sheets. The aim is then to identify a convergent steepest descent contour C of constant imaginary action which passes through one, or both, of these Lorentzian extrema.

As we have noted, the critical value S_{crit} for the lens spaces studied here is smaller that the corresponding value for spherical topology. Therefore, once we have succeeded in identifying the steepest descent contours, it means we can define an oscillating wave function for relatively smaller values of the bounding edge length. Indeed, we see that for $p \geq 5$, oscillatory behavior can be obtained for all positive values of S.

Based on our explicit knowledge of the analytic properties of the action and its extrema, we can determine the nature of the steepest descent contour by resorting to general argument, as reviewed in [7]. Consider, for example, the Lorentzian extremum lying at some value of real $z < \frac{1}{4}$; the action here is purely imaginary with value $\text{Im}(I) = I_{\text{ext}}$. By construction, a contour of constant imaginary action consists of two sections, one of steepest ascent, the other of steepest descent. Descending most steeply away from the extremum, one could in general end either at infinity, a singular point of the action, or at another extremum with the same value of $\text{Im}(I)$. From (41), we see that the only singular point is at $z = \frac{1}{3}$, where the value of $\text{Im}(I)$ diverges, so the contour cannot end there. Since the other extremum at this value of z lies on the second sheet, it has opposite value of $\text{Im}(I)$, so a contour cannot connect them. Thus, on general grounds, the steepest descent contour must be in infinite in extent, and passes from infinity to infinity through the extremum. Indeed, one can verify this explicitly, and an example for the universe with RP^3 topology is given in Fig. 3.

The convergence of this contour can be verified by recalling the asymptotic behavior of the action, as presented in the previous section. Beginning on the first sheet, at the Lorentzian extremum with real $z < \frac{1}{4}$, the locus of the contour on the upper half of this sheet is given asymptotically by

$$\frac{a_4}{H^2} S(S - S_{\text{crit}}) \,\text{Im}(z^{1/2}) = I_{\text{ext}} . \tag{57}$$

The asymptotic behavior of the real part of the action on this section of the contour is

$$\text{Re}[I(z,S)] \sim \frac{a_4}{H^2} S(S - S_{\text{crit}})|z|^{1/2} , \tag{58}$$

thus guaranteeing convergence.

Moving downwards from the extremum, we immediately cross the branch cut, and hence pass onto the second sheet. However, due to the alteration in sign of the action, one cannot proceed to infinity on the second sheet. Instead, one finds that the contour enjoys traversing the branch cut once more, moving onto a third sheet. For large values of S, this crossing point lies between $z = \frac{1}{4}$ and $\frac{1}{3}$. Having emerged onto the third sheet, the contour is asymptotic to

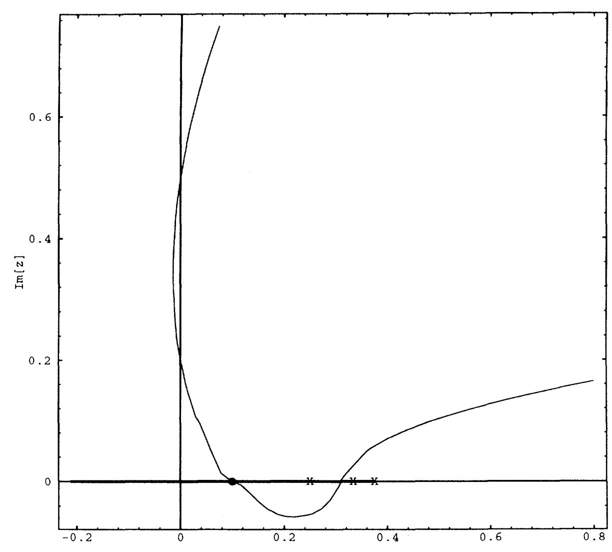

FIG. 3. A section of the steepest descent contour of integration for the cone over RP^3. The branch points at $z = \frac{1}{4}, \frac{1}{3}, \frac{3}{8}$ are indicated by the crosses, while the Lorentzian extremum at $\text{Re}[z] = 0.1$, corresponding to $S = 17.1246$, is marked by a bold dot. The value of the action on the first sheet at the extremum is $I = 1469.32i$ and the branch cut is highlighted by the solid line extending from $z = \frac{3}{8}$ to $-\infty$. The contour proceeds upwards from the extremum along the first sheet of the Regge action. Below the branch cut, the contour lies on the second sheet. Finally, it traverses the cut between the branch points $z = \frac{1}{4}$ and $\frac{1}{3}$, reaching a third sheet.

$$\frac{a_4}{H^2} S(S + \tilde{S}_{\text{crit}}) \,\text{Im}(z^{1/2}) = I_{\text{ext}} \ , \quad (59)$$

where

$$\tilde{S}_{\text{crit}} = \frac{a_2}{a_4}[2\pi + a_3 \arccos(\tfrac{1}{3})] \ . \quad (60)$$

Convergence is ensured due to the fact that

$$\text{Re}[I(z, S)] \sim \frac{a_4}{H^2} S(S + \tilde{S}_{\text{crit}})|z|^{1/2} \ . \quad (61)$$

For smaller values of S, the crossing point to the third sheet lies between $z = \frac{1}{3}$ and $\frac{3}{8}$. When the dust settles, however, the end result is that a convergent contour of integration for the wave function exists. Since it passes through the Lorentzian extrema, the desired oscillatory behavior in the late universe, i.e., when $S > S_{\text{crit}}$, is guaranteed. In fact the steepest descent contour just described has a complex conjugate partner with $\text{Im}(I) = -I_{\text{ext}}$, where one begins at the extremum on the second sheet.

It turns out that one can also find a steepest descent contour when $S < S_{\text{crit}}$. Beginning at a Euclidean extremum with real $z > \frac{3}{8}$ on the first sheet, the action is purely real, and hence a contour of constant $\text{Im}(I) = 0$ is required. Clearly, one contour extends along the real axis from $\frac{3}{8}$ to $+\infty$, corresponding to an integration over real Euclidean geometries. However, such a contour is of steepest ascent. The section of steepest descent can be found, and in fact is one which encircles all three finite branch points. The contours for the extremum on the second sheet are equally given.

The existence of convergent steepest descent contours allows us to perform a complete numerical integration yielding the wave function. For illustrative purposes however, it suffices to resort to the semiclassical approximation, and evaluate the wave function to first order.

To implement the Hartle-Hawking proposal, we wish to obtain a real valued wave function. This can be achieved by combining the two sections of the contour, passing through both Lorentzian, and both Euclidean extrema. The form of the wave function for $S > S_{\text{crit}}$ is therefore

$$\Psi_0(S) = \left(\frac{S^2}{2\pi H^4 \tilde{I}''_{\text{ext}}(S)}\right)^{1/2} 2\cos\left(\tilde{I}_{\text{ext}}(S) - \frac{\pi}{4}\right) \ , \quad (62)$$

where the action is written as $I = i\tilde{I}$, and $\tilde{I}_{\text{ext}}(S)$ and $\tilde{I}''_{\text{ext}}(s)$ are evaluated at the value of z corresponding to the extremum S. The fluctuation term, in a form valid for real $z < \frac{1}{4}$, is given by

$$\begin{aligned}\tilde{I}''(z,S) &= -\frac{i}{H^2}[-SF'' + S^2 G''] \\ &= \frac{S}{H^2}\left\{\frac{a_2}{4}\frac{1}{(\frac{1}{4}-z)^{3/2}}[2\pi - a_3 \arccos(z_2)] \right. \\ &\quad \left. + \frac{a_2 a_3}{8\sqrt{2}} \frac{1}{(\frac{1}{4}-z)(\frac{3}{8}-z)^{1/2}(1-3z)}\right\} \\ &\quad - \frac{S^2}{H^2}\frac{a_4}{4}\frac{1}{(\frac{3}{8}-z)^{3/2}} \ . \end{aligned} \quad (63)$$

For $S < S_{\text{crit}}$, the Hartle-Hawking wave function is also given by considering the two Euclidean extrema. However, because of the alternate signs, it is the extremum on the first sheet which gives the dominant contribution: namely,

$$\Psi_0(S) = -\left(-\frac{S^2}{2\pi H^4 I''_{\text{ext}}(S)}\right)^{1/2} \exp[-I_{\text{ext}}(S)] \ . \quad (64)$$

It is now a simple matter to plot these wave functions for the lens space universe. In Fig. 4, the Hartle-Hawking wave function is given for RP^3, in the semiclassical approximation. The corresponding wave function in a continuum minisuperspace model has been obtained in [27]. In Fig. 5, the case of $L(5,1)$ is presented. Here, we again remark that an oscillating wave function is allowed for all positive values of the boundary edge length, as is also the case for all $p > 5$. For all $L(p,1), p \geq 2$, we find an oscillating wave function for large values of the boundary edge length.

As shown in [7], when one considers both sections of the steepest descent contour passing through the Lorentzian minima, they can be joined to form a closed contour. This closed contour is one which encircles all three branch points twice. By similar argument, one can establish the closure of the contour for the lens space models considered here. This is appealing since one then has a contour prescription for the model, in the sense that the contour is independent of the argument of the wave function [9]. Furthermore, the closed contour can be deformed to the steepest descent contours for all values of S, thus accounting for both Lorentzian and Euclidean regimes. Recall that the form of the steepest descent contours obtained above depends on the particular minimum through which they run. Thus, in line with the arguments set forth in [9], it is the closed contour which should be taken as the defining contour for the Hartle-Hawking wave function in these particular simplicial minisuperspace models.

We also mention the wave functions of the Linde-Vilenkin [18,19] variety. The proposal here is to define the wave function to consist purely of outgoing waves. In other words, the wave function should be defined in terms

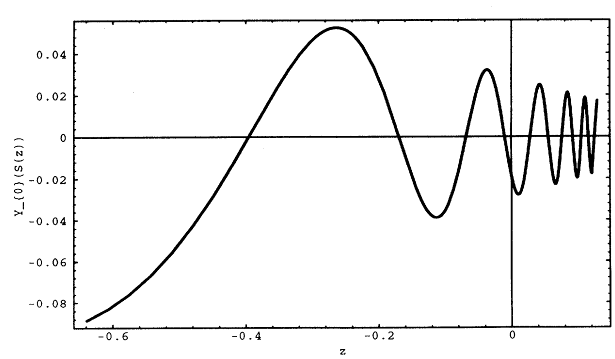

FIG. 4. A section of the semiclassical Hartle-Hawking wave function for a universe with RP^3 topology. The range of S values included is $6 < S < 20$, and $H^2 = 50$.

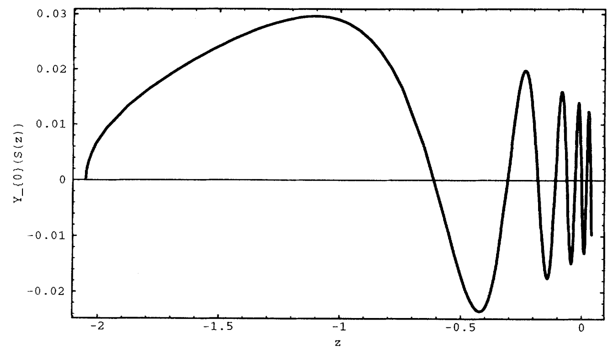

FIG. 5. The semiclassical Hartle-Hawking wave function for a universe with $L(5,1)$ topology, in the range $0 < S < 8$, with $H^2 = 50$.

of the contour passing through the Lorentzian extremum on the first sheet alone, giving it a form

$$\Psi_0(S) = \left(\frac{S^2}{2\pi H^4 \tilde{I}''_{\text{ext}}(S)}\right)^{1/2} \exp\left[-i\left(\tilde{I}_{\text{ext}}(S) - \frac{\pi}{4}\right)\right] . \quad (65)$$

Clearly then, wave functions satisfying the requirements of the Linde-Vilenkin proposal are possible for all the lens spaces studied here.

VI. BEHAVIOR UNDER BOUNDARY SUBDIVISION

The final topic of our investigation is to determine how the above results behave when the boundary universe undergoes a simplicial subdivision. While the triangulations presented in Sec. III do indeed capture the topology of the lens spaces, they are by no means the only available triangulation. Indeed, one can subject these complexes to various subdivision moves, which yield combinatorially equivalent triangulations.

For our purposes here, we shall appeal to a set of moves due to Pachner [20]; these are known as (k,l) subdivision moves, since they replace a set of k simplices by a set of l simplices. Of particular relevance to us is the fact that the triangulations of Sec. III are simplicial manifolds. According to the result of [20], the (k,l) moves are equivalent to the so-called Alexander moves [28] for closed simplicial manifolds. Therefore, all combinatorially equivalent triangulations of the lens spaces can be obtained via the (k,l) moves.

In the three-dimensional case of interest here, there are four (k,l) moves, with $k = 1,\ldots,4$, and $k+l=5$. The (1,4) move is described by adding a new vertex x to the center of the three-simplex [0,1,2,3], and linking it to the other four vertices. The original three-simplex is then replaced by four three-simplices:

$$[0,1,2,3] \to [x,1,2,3] - [x,0,2,3] + [x,0,1,3] - [x,0,1,2] . \quad (66)$$

The (2,3) move involves replacing two three-simplices which share a common two-simplex [0,1,2] by three three-simplices sharing a common one-simplex $[x,y]$:

$$[x,0,1,2] - [y,0,1,2] \to [x,y,1,2] - [x,y,0,2] + [x,y,0,1] . \quad (67)$$

The (3,2) and (4,1) moves are inverse to the above.

If we denote by ΔN_i, the increase in the number of i-simplices due to a (k,l) move, then it is straightforward to check that under the (1,4) move we have

$$\Delta N_0 = 1 ,\quad \Delta N_1 = 4 ,\quad \Delta N_2 = 6 ,\quad \Delta N_3 = 3 . \quad (68)$$

The changes under the (2,3) move are given by

$$\Delta N_0 = 0 ,\quad \Delta N_1 = 1 ,\quad \Delta N_2 = 2 ,\quad \Delta N_3 = 1 . \quad (69)$$

We recall that the Regge action in our simplicial minisuperspace was fixed in terms of the coefficients a_i, which in turn were related to the $N_i(M_3)$. When we perform a subdivision move of type (k,l) on our bounding complex, the resulting four-dimensional complex is still a cone over this subdivided boundary. However, the subdivision creates a number of new internal simplices, and also of course additional boundary simplices. If we now decide to maintain the nature of our simplicial minisuperspace, described entirely in terms of a single internal edge length, and a single boundary edge length, then we must endow the newly generated edges with those values.

With this declaration in place, the effects of the subdivision moves are easily established, the net result being an alteration in the values of the a_i coefficients. The changes can readily be extracted by combining Eqs. (68) and (69) with Eq. (39). In this way, we immediately obtain the Regge action for the cone with subdivided boundary, and most importantly the analytic structure of the action remains intact, so that the previous analysis carries through.

One interesting feature of these subdivisions is their influence on the critical value of the boundary edge length S_{crit}. If we perform n subdivision moves of type (1,4), we find

$$S_{\text{crit}} \to 4\sqrt{2}\frac{N_1(M_3)+4n}{N_3(M_3)+3n} \times \left[2\pi - \frac{6[N_3(M_3)+3n]}{N_1(M_3)+4n} \arccos\left(\frac{1}{3}\right)\right] . \quad (70)$$

In the limit of large n, the critical value tends to $S_{\rm crit} = 5.6106$. For the case of S^3 boundary, for example, this means one can extend the Lorentzian regime of extrema to smaller values of the boundary edge length, by performing such subdivisions.

Under a set of m moves of type (2,3), one finds that

$$S_{\rm crit} \to 4\sqrt{2}\frac{N_1(M_3)+n}{N_3(M_3)+n} \times \left[2\pi - \frac{6[N_3(M_3)+n]}{N_1(M_3)+n}\arccos\left(\frac{1}{3}\right)\right] . \quad (71)$$

The limiting value for large m in this case is $S_{\rm crit} = -6.237\,80$. Therefore, by performing such moves we can ensure that the region of Lorentzian extrema covers the full range of physically allowed values of S.

In Sec. III, we discussed the symmetry properties of the lens space triangulations and their resulting simplicial geometry. It is clear that the (k,l) moves affect this geometry, as can be seen from the relations (68) and (69). In general, one sees that new one-simplices are introduced as a result of these moves, and thus the degree of anisotropy may be altered by assigning independent edge lengths to them.

VII. CONCLUSIONS

We have studied the wave function for a universe which is topologically a lens space. The crucial element in the construction was to assume that the four-dimensional spacetime was modeled by the cone over the bounding lens space. By restricting attention to a simplicial minisuperspace, the Regge action simplified to the extent that one could investigate its relevant properties explicitly. Indeed, it should be observed that the coefficients a_i appearing in the action are determined solely in terms of the number of i-simplices of the boundary universe $N_i(M_3)$. This followed as a direct consequence of the choice of cone structure for the four-dimensional simplicial spacetime, and was not reliant on the boundary universe being of lens space topology. In particular then, the analysis presented here can be applied to spatial universes with arbitrary topology. Given a simplicial complex which models a universe with some general topology, the Regge action in the minisuperspace of interest here is completely fixed, and is of the form given in Eqs. (35)–(38), with the values of the a_i coefficients given by Eq. (39).

We may also be interested in studying the situation when the universe is given by the disjoint union of a number of components of varying topology. The wave function of the universe in this case is then a topology changing amplitude. Let us suppose that the universe consists of the disjoint union of a number of components, and we then construct the cone over this boundary.

Given this cone structure, we can immediately identify a convenient simplicial minisuperspace in which to study these amplitudes. Again, we let the simplicial geometry of each boundary component be described in terms of a single edge length. In addition, we allow an independent internal edge length to emanate from the cone vertex to each of the boundary components. With this choice of minisuperspace, the Regge action is given by a sum of independent terms, one for each of the boundary components, and consequently the wave function factorizes into a produce of single-component wave functions. However, such a factorization property is a direct consequence of the nature of the minisuperspace, and the cone type cobordism.

Of course, one can consider more sophisticated cobordism structures, and in particular it would be interesting to perform an analysis when the four-dimensional spacetime is itself a simplicial manifold.

Indeed, since the triangulations of [21] are themselves simplicial manifolds, they provide an ideal opportunity to study nontrivial cobordism effects in three dimensions. In particular, for example, if one removes the link of any vertex from the triangulation of $L(p,1)$, the resulting structure is a simplicial manifold with S^2 boundary. In this way, one could study the wave function for a universe with S^2 topology, cobordant to a variety of lens spaces, and compare notes with the wave function obtained when one takes the trivial cobordism, i.e., the three-disc.

ACKNOWLEDGMENTS

This work was supported by Stichting voor Fundamenteel Onderzoek der Materie (FOM).

[1] T. Regge, Nuovo Cimento **19**, 558 (1961).
[2] J. A. Wheeler, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).
[3] J. B. Hartle, J. Math. Phys. **26**, 804 (1985).
[4] R. M. Williams and P. A. Tuckey, Class. Quantum Grav. **9**, 1409 (1992).
[5] H. W. Hambler, in *Critical Phenomena, Random Systems, Gauge Theories*, Proceedings of the Les Houches Summer School 1984, edited by K. Osterwalder and R. Stora (North-Holland, Amsterdam, 1986).
[6] F. David, in *Gravitation and Quantizations*, Proceedings of the Les Houches Summer School, Les Houches, France, 1992, edited by J. Zinn-Justin and B. Julia, Les Houches Summer School Proceedings Vol. 57 (North-Holland, Amsterdam, 1995).
[7] J. B. Hartle, J. Math. Phys. **30**, 452 (1989).
[8] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[9] J. J. Halliwell and J. Louko, Phys. Rev. D **42**, 3997 (1990).
[10] J. J. Halliwell and J. B. Hartle, Phys. Rev. D **41**, 1815 (1990).
[11] J. J. Halliwell and J. Louko, Phys. Rev. D **39**, 2206

(1989).
[12] J. J. Halliwell and J. Louko, Phys. Rev. D **40**, 1868 (1989).
[13] J. J. Halliwell and R. C. Myers, Phys. Rev. D **40**, 4011 (1989).
[14] J. Louko and P. A. Tuckey, Class. Quantum Grav. **9**, 41 (1992).
[15] T. Eguchi and A. J. Hanson, Phys. Lett. **74B**, 249 (1978); Ann. Phys. (N.Y.) **120**, 82 (1979).
[16] S. W. Hawking, Phys. Lett. **60A**, 81 (1977).
[17] G. W. Gibbons and S. W. Hawking, Phys. Lett. **78B**, 430 (1978).
[18] A. Linde, Zh. Eksp. Teor. Fiz. **87**, 369 (1984) [Sov. Phys. JETP **60**, 211 (1984)]; Nuovo Cimento **39**, 401 (1984); Rep. Prog. Phys. **47**, 925 (1984).
[19] A. Vilenkin, Phys. Rev. D **30**, 509 (1984); **33**, 3560 (1986); **37**, 888 (1988).
[20] U. Pachner, Arch. Math. **30**, 89 (1978); Eur. J. Combinatorics **12**, 129 (1991).
[21] U. Brehm and J. Świątkowski, "Triangulations of Lens Space with Few Simplices," T. U. Berlin report, 1993 (unpublished).
[22] J. B. Hartle and R. Sorkin, Gen. Relativ. Gravit. **13**, 541 (1981).
[23] W. Kühnel, in *Advances in Differential Geometry and Topology*, edited by F. Tricerri (World Scientific, Singapore, 1990).
[24] J. Munkres, *Elements of Algebraic Topology* (Addison-Wesley, Menlo Park, 1984).
[25] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise Linear Topology* (Springer, Berlin, 1972).
[26] K. Schleich and D. M. Witt, Nucl. Phys. **B402**, 469 (1993).
[27] K. Schleich and D. M. Witt, Nucl. Phys. **B402**, 411 (1993).
[28] J. W. Alexander, Ann. Math. **31**, 292 (1930).