Gravitational radiation from a particle in circular orbit around a black hole. VI. Accuracy of the post-Newtonian expansion

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A particle of mass μ moves on a circular orbit around a nonrotating black hole of mass M. Under the assumption $\mu \ll M$ the gravitational waves emitted by such a binary system can be calculated exactly numerically using black-hole perturbation theory. If, further, the particle is slowly moving, $v = (M\Omega)^{1/3} \ll 1$ (where v and Ω are, respectively, the linear and angular velocities in units such that $G = c = 1$), then the waves can be calculated approximately analytically, and expressed in the form of a post-Newtonian expansion. We determine the accuracy of this expansion in a quantitative way by calculating the reduction in signal-to-noise ratio incurred when matched filtering the exact signal with a nonoptimal, post-Newtonian filter. We find that the reduction is quite severe, approximately 25%, for systems of a few solar masses, even with a post-Newtonian expansion accurate to fourth order, $O(v^8)$, beyond the quadrupole approximation. Most of this reduction is caused by the post-Newtonian theory's inability to correctly locate the innermost stable circular orbit, which here is at $r = 6M$ (in Schwarzschild coordinates). Correcting for this yields reductions of only a few percent.

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Inspiraling compact binary systems, composed of neutron stars and/or black holes, have been identified [1,2] as the most promising source of gravitational waves for interferometric detectors such as LIGO (Laser Interferometer Gravitational-wave Observatory [3]) and VIRGO [4]. These systems evolve under radiation reaction, so that the gravitational-wave signal increases in amplitude as the frequency sweeps through the detector bandwidth, from approximately 10 to 1000 Hz.

Extraction of the information contained in the gravitational waves, most notably about the masses and spins of the companions, will necessitate the construction of accurate model signals, known as templates [5–8]. The extraction makes use of the well-known technique of matched filtering [9,10], in which the signal is integrated against a choice of templates in order to identify the true value of the source parameters. It has been established that these templates will need to reproduce the signal's *phasing* especially accurately [6,7,11]. This is because the signal undergoes a number of oscillations approximately equal to 16 000 (if the system is that of two neutron stars) as the frequency sweeps through the detector bandwidth. If the template loses phase with respect to the true signal, even by so little as one cycle, then the signal-to-noise ratio is severely reduced, and the information-extraction strategy severely impeded. Thus an accuracy of at least one part in 16 000 is required for the template's phasing.

The construction of accurate templates is currently the goal of many gravitational-wave theorists [12]. It is clear that the calculation must be based on some approximation to the equations of general relativity; the exact integration of the equations governing the evolution of compact binary systems is not currently amenable to numerical methods. The favored approach is post-Newtonian (PN) theory, which is based upon an assumption of slow motion: If v denotes the orbital velocity, M the total mass, and r the orbital separation, then it is assumed that the orbital evolution is such that $v \sim (M/r)^{1/2} \ll 1$. We set $c = G = 1$.

To date, templates have been calculated accurately through second post-Newtonian order [13–15], that is, order v^4 beyond the leading-order, quadrupole-formula expression. The question considered in this paper is whether this calculation is accurate enough for the purpose of information extraction. We shall see, as other authors have argued [16–18], that the answer is negative. The post-Newtonian expansion converges extremely slowly, if at all. Calculations must be pushed to a much higher order.

The exact treatment of the orbital evolution of a compact binary system whose companions have comparable masses is currently beyond reach. The question posed in the preceding paragraph cannot, therefore, be answered in this context. However, if instead we consider a system composed of a particle with small mass orbiting a black hole with much larger mass, then the question *can* satisfactorily be answered. This limiting case indeed allows for an exact treatment, which can then be compared with an approximate, slow-motion treatment. The degree of validity of the approximation can thereby be ascertained. Our hope is that our conclusions, based upon the small-mass-ratio limit, will remain qualitatively valid for systems with large mass ratios.

The gravitational waves generated by a particle in circular orbit around a black hole have been the topic of this series of papers [16,19–22]. If μ denotes the particle mass, and M that of the black hole (which is here assumed to be nonrotating), then the assumption $\mu/M \ll 1$ ensures

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that the gravitational perturbations produced by the orbiting particle are small and governed by Teukolsky's linear wave equation [23]. This equation can be integrated using straightforward numerical methods, and the gravitational waveform is thus determined exactly.

We will focus mainly on the phasing of the waves, which is determined by df/dt, the rate at which the gravitational-wave frequency changes with time. For circular orbits $f = \Omega/\pi$, where $\Omega = d\phi/dt$ is the angular velocity, and df/dt can be expressed as

$$\frac{df}{dt} = \frac{dE/dt}{dE/df}. \qquad (1)$$

Here, dE/dt is the rate at which gravitational waves remove energy from the system, and dE/df expresses the relation between orbital energy and wave frequency.

The solid curve in Fig. 1 is a plot of

$$P(v) \equiv \frac{dE/dt}{(dE/dt)_N}, \qquad (2)$$

where

$$v = (\pi M f)^{1/3} \qquad (3)$$

is the orbital velocity and $(dE/dt)_N = -32\mu^2 v^{10}/5M^2$ the Newtonian, quadrupole-formula expression for the gravitational-wave luminosity. The plot was obtained by numerically integrating the Teukolsky equation, along the lines described in Ref. [16]. An expression for dE/df can be obtained by integrating the geodesic equations for circular orbits. The orbital energy is defined as $E = -p_t$, where p^α is the particle's four-momentum; it is a constant of the motion in the absence of radiation reaction. We obtain

$$Q(v) \equiv \frac{dE/df}{(dE/df)_N} = (1 - 6v^2)(1 - 3v^2)^{-3/2}, \qquad (4)$$

where $(dE/df)_N = -\pi\mu M/3v$ is the Newtonian expression. Equation (4) and the solid curve in Fig. 1 give through Eq. (1) an *exact* representation of df/dt.

To produce the solid curve in Fig. 1 the Teukolsky equation had to be integrated numerically. If, however, the small-mass-ratio approximation is combined with a slow-motion approximation ($v \ll 1$), then the Teukolsky equation can be integrated analytically [18,19,22,24]. The resulting approximate result for $P(v)$ is

$$\begin{aligned} P(v) = {} & 1 - 3.711v^2 + 12.56v^3 - 4.928v^4 - 38.29v^5 \\ & + (115.7 - 16.30 \ln v)v^6 - 101.5v^7 \\ & - (117.5 - 52.74 \ln v)v^8 + \cdots ; \end{aligned} \qquad (5)$$

the various coefficients can be found in their analytic form in Ref. [18]. Equation (5) takes the form of a post-Newtonian expansion for $P(v)$. The first five terms, through $O(v^5)$, reproduce in the small-mass-ratio limit the post-Newtonian calculation of Refs. [13,25]; the remaining terms have not yet been calculated using post-Newtonian theory. A post-Newtonian expansion can also be given for $Q(v)$, by simply expanding Eq. (4) about $v = 0$. The result is

$$Q(v) = 1 - \tfrac{3}{2}v^2 - \tfrac{81}{8}v^4 - \tfrac{675}{16}v^6 - \tfrac{19845}{128}v^8 + \cdots. \qquad (6)$$

Equations (5) and (6) give through Eq. (1) an *approximate* (post-Newtonian) representation of df/dt.

Our goal in the sequel is to determine to which extent the post-Newtonian representation of df/dt reproduces the true phasing of the waves. We shall denote by $P(v)$ and $Q(v)$ the exact version of these functions, while $P_n(v)$ and $Q_n(v)$ will denote the post-Newtonian representations truncated to the nth power of v. For example, $P_4(v)$ represents the right-hand side of Eq. (5) with all terms of order v^5 and higher removed.

Figure 1 displays the various curves $P_n(v)$ for n ranging from 4 to 8. It is seen that for $v > 0.2$, these curves reproduce rather poorly the exact curve $P(v)$. Moreover, the convergence of the post-Newtonian expansion is also seen to be poor: adding a term in the expansion does not necessarily make the expression more accurate. Witness in particular the poor quality of $P_5(v)$ compared with that of $P_4(v)$. [Compare also $P_8(v)$ to $P_7(v)$.] We shall now attempt to determine, in a quantitative manner, how much of an obstacle this slow rate of convergence poses to information-extraction strategies.

The starting point of our analysis is an expression for the reduction in signal-to-noise ratio incurred when the matched filtering of a gravitational-wave signal is carried out with a nonoptimal filter. If $h(t)$ denotes the true gravitational waveform, then optimal filtering is achieved

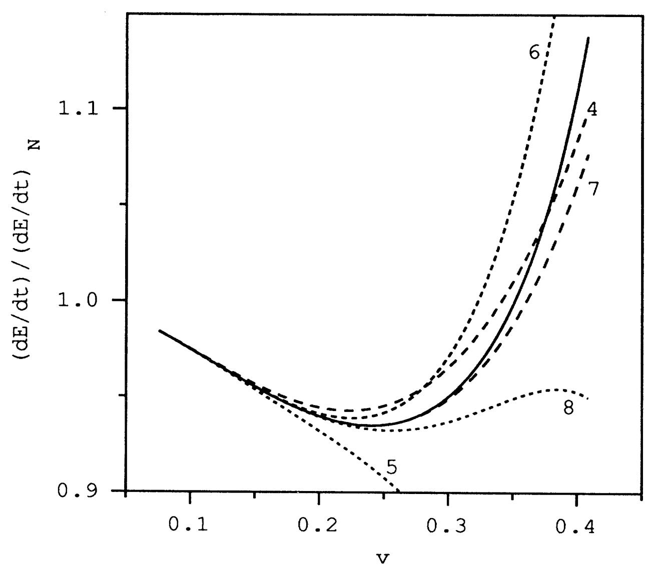

FIG. 1. Various representations of $(dE/dt)/(dE/dt)_N$ as a function of orbital velocity $v = (M/r)^{1/2} = (\pi M f)^{1/3}$. The solid curve represents the exact result $P(v)$, as calculated numerically. The various broken curves represent the post-Newtonian approximations $P_n(v)$, for $n = \{4, 5, 6, 7, 8\}$. The smallest value of v corresponds to an orbital radius r of $175M$; the largest value of v corresponds to $r = 6M$, the innermost stable circular orbit.

by using $h(t)$ as a template. This is well known to yield $S/N|_{\rm max}$, the largest possible value of the signal-to-noise ratio [9,10]. If, instead, we adopt the post-Newtonian approximation $h_n(t)$ as a template, then filtering is not optimal, and $S/N|_{\rm actual}$, the actual value of the signal-to-noise ratio, is smaller than the maximum possible value. The reduction in signal-to-noise ratio can be calculated to be [26]

$$\mathcal{R}_n \equiv \frac{S/N|_{\rm actual}}{S/N|_{\rm max}} = \frac{|(h|h_n)|}{\sqrt{(h|h)(h_n|h_n)}}, \tag{7}$$

where the inner product $(\cdot|\cdot)$ will be defined presently.

We pause here to remark that in our calculations, we shall not allow the mass parameters μ and M to take different values in $h(t)$ and $h_n(t)$. The problem considered here is *not* that of maximizing the signal-to-noise ratio over the source parameters in order to estimate their true value. We therefore do *not* address the issue of the systematic errors introduced in parameter estimation when using templates which are only approximations to the true general-relativistic signal [27]. Our considerations are less ambitious: the source parameters are assumed to be given, and the accuracy of the post-Newtonian templates is quantitatively measured by $\mathcal{R}_n$, the relative reduction in signal-to-noise ratio. This measure, we believe, is more satisfactory than the mere counting of the number of wave cycles contributed by various terms in the post-Newtonian expansion of the wave's phasing [16–18].

We now return to the definition of the inner product. Let $\tilde{h}(f)$ and $\tilde{h}_n(f)$ denote, respectively, the Fourier transforms of the functions $h(t)$ and $h_n(t)$, with the convention $\tilde{g}(f) = \int g(t)e^{2\pi i f t}\,dt$ for Fourier transforms. Let also $S(f)$ be the spectral density of the detector noise, assumed to be a stationary, Gaussian random process. For detectors of the advanced-LIGO type it is appropriate to set [7]

$$S(f) = \tfrac{1}{5}S_0\Big[(f_0/f)^4 + 2 + (f/f_0)^2\Big] \tag{8}$$

for $f > 10$ Hz; for $f < 10$ Hz we take $S(f) = \infty$. In Eq. (8), S_0 is a normalization constant irrelevant for our purposes, and f_0 is the frequency at which $S(f)$ is minimum; we set $f_0 = 70$ Hz. The inner product introduced in Eq. (7) is then given by [7]

$$(g|h) = 2\int_0^\infty \frac{\tilde{g}^*(f)\tilde{h}(f) + \tilde{g}(f)\tilde{h}^*(f)}{S(f)}\,df, \tag{9}$$

where an asterisk denotes complex conjugation.

We must now specify $\tilde{h}(f)$ and $\tilde{h}_n(f)$, the gravitational waveforms in the frequency domain. We use an approximation [7] in which the *amplitude* of the waveform is described accurately to Newtonian order, while its *phase* is described exactly in the case of $\tilde{h}(f)$, and by a post-Newtonian expansion in the case of $\tilde{h}_n(f)$. Consequently, we have [7,8]

$$\tilde{h}(f) = \mathcal{A} f^{-7/6}\exp\big[i\psi(f)\big],$$
$$\tilde{h}_n(f) = \mathcal{A} f^{-7/6}\exp\big[i\psi_n(f)\big], \tag{10}$$

where $\mathcal{A}$ is a constant. The phase functions $\psi(f)$ and $\psi_n(f)$ are constructed as follows.

We consider $\psi(f)$; $\psi_n(f)$ is dealt with similarly. Our starting point is the wave's phasing in the time domain, which is determined by df/dt given above. Combining Eqs. (1)–(4) we obtain $df/dt = 96\mu v^{11}P(v)/5\pi M^3 Q(v)$. Integration then yields

$$t(v)/M = t_i/M + \frac{5M}{32\mu}\int_{v_i}^{v} \frac{Q(v')}{v'^9 P(v')}\,dv' \tag{11}$$

for time as a function of velocity, and

$$\Phi(v) = \Phi_i + \frac{5M}{16\mu}\int_{v_i}^{v} \frac{Q(v')}{v'^6 P(v')}\,dv' \tag{12}$$

for the phase $\Phi = \int 2\pi f dt$. In Eqs. (11) and (12), v_i is an arbitrary reference point, and $t_i = t(v_i)$, $\Phi_i = \Phi(v_i)$. The frequency-domain phase function $\psi(f)$ is obtained via the stationary phase approximation [7,8,28], according to which $\psi(f) = 2\pi f t(v) - \Phi(v) - \pi/4$. We therefore find

$$\psi(f) = 2(t_i/M)v^3 - \Phi_i - \pi/4 + \frac{5M}{16\mu}\int_{v_i}^{v} \frac{(v^3 - v'^3)Q(v')}{v'^9 P(v')}\,dv', \tag{13}$$

where, we recall, $v \equiv (\pi M f)^{1/3}$. Equations (10) and (13), together with the fact that the gravitational-wave signal must be cut off at a frequency $f_{\rm isco}$ corresponding to the innermost stable circular orbit, completely specify the Fourier transform of the waveform. We use $\pi M f_{\rm isco} = (M/r_{\rm isco})^{3/2} = 6^{-3/2}$, where $r_{\rm isco} = 6M$ is the radius of the innermost stable circular orbit of the Schwarzschild spacetime.

The calculation of $\mathcal{R}_n$ can now be carried out. Straightforward manipulations, using Eqs. (7)–(10), yield

$$\mathcal{R}_n = I^{-1}\int_{1/7}^{x_{\rm isco}} \frac{x^{-7/3}\cos\Delta\psi}{x^{-4} + 2 + 2x^2}\,dx, \tag{14}$$

where $x = f/f_0$, $x_{\rm isco} = f_{\rm isco}/f_0$, and

$$\Delta\psi = \psi - \psi_n. \tag{15}$$

The constant I in Eq. (14) is given by

$$I = \int_{1/7}^{x_{\rm isco}} \frac{x^{-7/3}}{x^{-4} + 2 + 2x^2}\,dx, \tag{16}$$

and ensures that $\mathcal{R}_n = 1$ if $\psi(f) = \psi_n(f)$.

The numerical value of $\mathcal{R}_n$, for given n, depends on the value of the constants t_i and Φ_i which appear in $\psi(f)$, and on the value of the constants t_{ni} and Φ_{ni} which appear in $\psi_n(f)$. To maximize $\mathcal{R}_n$, we set $t_i = t_{ni}$ and choose $\Phi_i - \Phi_{ni}$ such that $\Delta\psi$ vanishes at the value of f for which $x^{-7/3}(x^{-4} + 2 + 2x^2)^{-1}$ is maximum. It is easy to check that this occurs at $x_{\rm max} \simeq 0.6654$, so that $f_{\rm max} \simeq 46.58$ Hz. A simple calculation also shows that

$v_{\rm max} \equiv (\pi M f_{\rm max})^{1/3} \simeq 0.089\,66(M/M_\odot)^{1/3}$, where $M_\odot$ denotes the mass of the Sun. With these choices, $\Delta\psi$ becomes

$$\Delta\psi = \frac{5M}{16\mu}\int_{v_{\rm max}}^{v} \frac{v^3 - v'^3}{v'^9}\left[\frac{Q(v')}{P(v')} - \frac{Q_n(v')}{P_n(v')}\right] dv'. \quad (17)$$

It is a straightforward numerical problem [29] to compute $\Delta\psi$ for a given $v = (\pi M f)^{1/3}$ and to then evaluate the integral to the right-hand side of Eq. (14).

The calculation presented above is, strictly speaking, only applicable to binary systems with small mass ratios. In this limit, the only place in which μ/M appears is as an overall multiplicative factor in Eq. (17). In this limit, therefore, the phase lag $\Delta\psi$ scales exactly as M/μ. As the mass ratio is allowed to increase, our results for $P(v)$, $Q(v)$, and their post-Newtonian analogues must be corrected [28]. For example, the constant coefficients in Eqs. (5) and (6) would acquire μ/M-dependent corrections [13]. Our expressions for $(dE/dt)_N$ and $(dE/df)_N$, however, stay valid for large mass ratios, provided that μ is then interpreted as the system's *reduced* mass, and M as the system's *total* mass.

In the following we will let μ/M become large, without modifying our expressions for $P(v)$, $Q(v)$, and their post-Newtonian analogues. We will *assume*, without justification, that the qualitative behavior of these functions is not appreciably affected by the finite-mass-ratio corrections [which are not known, apart from the terms of lowest order in $P_n(v)$ and $Q_n(v)$]. We will therefore apply our formalism to binary systems with comparable masses, in the hope that our conclusions based on the $\mu/M \to 0$ limit will be qualitatively valid also in the large-mass-ratio case. We note, in accordance with our previous observation, that in Eq. (17) M is to be interpreted as the total mass and μ as the reduced mass.

Our results are summarized in Table I. We consider three types of binary systems. The first (system A) consists of two neutron stars, each with a mass equal to $1.4M_\odot$. The second (system B) consists of one neutron star ($1.4M_\odot$) and one black hole ($10M_\odot$). The third (system C) consists of two black holes ($10M_\odot$ each).

The second column in Table I lists $\mathcal{R}_n$ as defined in Eqs. (14) and (17), for n ranging from 4 to 8. This shows that for system A, the signal-to-noise ratio is always smaller than 0.7651 times the maximum possible value, with the best result obtained when $n = 7$, which corresponds to dE/dt accurate to 3.5PN order, and dE/df accurate to 3PN order. The reduction in signal-to-noise ratio is therefore quite severe, even at such a high order in the post-Newtonian expansion. The corresponding results for systems B and C can also be obtained from Table I.

The reduction in signal-to-noise ratio is due to the fact that $P_n(v)$ is only an approximation to $P(v)$, see Fig. 1, and that $Q_n(v)$ is only an approximation to $Q(v)$. It is interesting to ask how much of the signal-to-noise ratio could be recovered if only $P(v)$ were approximated by a post-Newtonian expansion, while $Q(v)$ were kept exact. To answer this question amounts to repeating the calculation presented previously, but with $Q(v)$ substituted in place of $Q_n(v)$ in Eq. (17). The results are shown in the third column of Table I. Not surprisingly, we see that keeping dE/df exact gives much better results. For system A, $\mathcal{R}_n$ can become as large as 0.9864 (for $n = 7$), and is already as large as 0.9454 for $n = 6$ (which corresponds to dE/dt accurate to 3PN order). The corresponding results for systems B and C can also be obtained from Table I.

TABLE I. Reduction in signal-to-noise ratio incurred when matched filtering with approximate, post-Newtonian templates. For each of the considered binary systems, the first column lists the order n of the approximation, the second column lists $\mathcal{R}_n$ as calculated using the post-Newtonian approximation $Q_n(v)$ for dE/df, the third column lists $\mathcal{R}_n$ as calculated using the exact expression $Q(v)$ for dE/df, and the fourth column lists $\mathcal{R}_n$ as calculated using the alternative post-Newtonian approximation $Q'_n(v)$ for dE/df.

n	$Q_n(v)$	$Q(v)$	$Q'_n(v)$
System A ($1.4\ M_\odot$ + $1.4\ M_\odot$):			
4	0.5796	0.4958	0.4326
5	0.4646	0.5286	0.6492
6	0.7553	0.9454	0.9300
7	0.7651	0.9864	0.9819
8	0.7568	0.9695	0.9709
System B ($1.4\ M_\odot$ + $10\ M_\odot$):			
4	0.8478	0.3954	0.2916
5	0.2413	0.2922	0.4012
6	0.6097	0.6788	0.6051
7	0.6023	0.9966	0.8528
8	0.5734	0.8744	0.8942
System C ($10\ M_\odot$ + $10\ M_\odot$):			
4	0.8232	0.6919	0.4910
5	0.3515	0.4422	0.5866
6	0.7107	0.8088	0.7417
7	0.7201	0.9997	0.9310
8	0.6730	0.9152	0.9420

Why does the exact expression for dE/df give such better results? The answer is that while the exact expression for dE/df correctly vanishes at $v = v_{\rm isco} = 6^{-1/2}$ (at the innermost stable circular orbit), its post-Newtonian analogue fails to do so. For example, $Q_8(v)$ goes to zero at $v \simeq 0.4236 > 6^{-1/2}$, corresponding to a radius $r \simeq 5.572M$.

To establish that this is indeed the reason, we consider an alternative expression for $Q(v)$, obtained from Eq. (4) by expanding only the $(1 - 3v^2)^{-3/2}$ factor:

$$Q'(v) = (1 - 6v^2)\Big(1 + \tfrac{9}{2}v^2 + \tfrac{135}{8}v^4 + \tfrac{945}{16}v^6 + \tfrac{25515}{128}v^8 + \cdots\Big). \quad (18)$$

This expression, together with its truncated versions $Q'_n(v)$, manifestly go to zero at $v = 6^{-1/2}$. We have repeated the calculations with $Q'_n(v)$ substituted in place of $Q_n(v)$ in Eq. (17). The results are shown in the fourth column of Table I. It is evident that, generally speaking, using $Q'_n(v)$ gives much larger values of $\mathcal{R}_n$ than using $Q_n(v)$. This is especially remarkable for system A, since the relative contribution to the signal-to-noise ratio coming from large values of v is very small: For $v = 6^{-1/2}$ and $M = 2.8M_\odot$ the corresponding frequency is 1570 Hz, at which $S(f)$ is 200 times its minimum value S_0.

These results suggest that accurate knowledge of the location of the innermost stable circular orbit (where dE/df goes to zero) might significantly improve the performance of the post-Newtonian templates, through a factorization of the kind shown in Eq. (18). It is possible that in the case of binary systems with large mass ratios, such information could be obtained by numerically solving the initial value problem of general relativity [30,31], which would be far less laborious than solving the full dynamical problem. Another approach would be to use the Kidder-Will-Wiseman "hybrid" equations of motion [32], which give a better approximation to the innermost stable circular orbit than the standard post-Newtonian equations.

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