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Gravitational waves from the inspiral of a compact object into a massive, axisymmetric body with arbitrary multipole moments

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The gravitational waves, emitted by a compact object orbiting a much more massive central body, depend on the central body's spacetime geometry. This paper is a first attempt to explore that dependence. For simplicity, the central body is assumed to be stationary, axially symmetric (but rotating), and reflection symmetric through an equatorial plane, so its (vacuum) spacetime geometry is fully characterized by two families of scalar multipole moments M_l and S_l with $l = 0, 1, 2, 3, \ldots$, and it is assumed not to absorb any orbital energy (e.g., via waves going down a horizon or via tidal heating). Also for simplicity, the orbit is assumed to lie in the body's equatorial plane and to be circular, except for a gradual shrinkage due to radiative energy loss. For this idealized situation, it is shown that several features of the emitted waves carry, encoded within themselves, the values of all the body's multipole moments M_l, S_l (and thus, also the details of its full spacetime geometry). In particular, the body's moments are encoded in the time evolution of the waves' phase $\Phi(t)$ (the quantity that can be measured with extremely high accuracy by interferometric gravitational-wave detectors); and they are also encoded in the gravitational-wave spectrum $\Delta E(f)$ (energy emitted per unit logarithmic frequency interval). If the orbit is slightly elliptical, the moments are also encoded in the evolution of its periastron precession frequency as a function of wave frequency, $\Omega_\rho(f)$; if the orbit is slightly inclined to the body's equatorial plane, then they are encoded in its inclinational precession frequency as a function of wave frequency, $\Omega_z(f)$. Explicit algorithms are derived for deducing the moments from $\Delta E(f)$, $\Omega_\rho(f)$, and $\Omega_z(f)$. However, to deduce the moments explicitly from the (more accurately measurable) phase evolution $\Phi(t)$ will require a very difficult, explicit analysis of the wave generation process—a task far beyond the scope of this paper.

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I. INTRODUCTION

For some years, Thorne [1] has been arguing that it should be possible to extract, from the gravitational waves produced by a small object spiraling into a massive black hole, a map of the massive hole's spacetime geometry. This paper is a first attempt to develop the mathematical foundations for such a map extraction. As we shall see, the key to the map extraction is a theorem (proved in this paper) that, at least in certain idealized circumstances, the waves emitted by a small object spiraling into a massive body carry, encoded in themselves, the values of all the body's multipole moments [2,3], which characterize the vacuum spacetime geometry outside any stationary body (black hole or otherwise).

A separate paper by this author, Finn, and Thorne [4] discusses semiquantitatively the implementation of this paper's results in the analysis of future gravitational-wave data. As is discussed there, the goals of such a data analysis would be (i) to extract from the observed waves the values of the central body's lowest few multipole moments, (ii) to see whether those moments are in accord with the black-hole "no-hair" theorem (which states that the hole's spacetime geometry and thence all its moments are fully determined by its mass and its spin angular momentum), and (iii) via observed violations of the no-hair theorem, to search for unexpected types of massive, compact bodies (e.g., soliton stars and naked singularities) into which are spiraling small objects (white dwarfs, neutron stars, or small-mass black holes).

Such interesting observational studies can be carried out with moderate precision by the Earth-based network of laser-interferometer gravitational-wave detectors [Laser Interferometer Gravitational Wave Observatory (LIGO), VIRGO, GEO600, TAMA] [5], which is now under construction and which can study central bodies with masses up to $\sim 300 M_\odot$. Much higher precision will be achieved by the Laser Interferometer Space Antenna (LISA) [6], which is likely to fly in 2014 or sooner and can study central bodies with masses $\sim 3\times10^5$ to $3\times10^7 M_\odot$. See Ref. [4] for details.

For this paper's first analysis of extracting the central body's moments from gravitational-wave data, we make the following idealizing assumptions.

(i) The central body has a vacuum, external gravitational field which is stationary, axisymmetric, reflection symmetric across the equatorial plane, and asymptotically flat. Correspondingly, the body's multipole moments turn out to be scalars: The spacetime geometry can be characterized by mass multipole moments M_l and mass-current multipole moments S_l [3], and the odd-M moments and even-S moments vanish, i.e., the nonvanishing moments are the mass $M_0 \equiv M$, the mass quadrupole moment M_2, M_4, M_6, $\ldots$, and the spin angular momentum S_1, the current octopole moment S_3, S_5, S_7, $\ldots$.

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(ii) The inspiraling object is sufficiently compact and has a sufficiently small mass that its orbit evolves slowly and adiabatically from one geodesic orbit to another; and on the time scale of one orbital period, the orbit can be regarded as geodesic.

(iii) The geodesic orbits, through which the inspiral evolves, lie in the equatorial plane, or very nearly so, and are circular, or very nearly so. (For the $M \lesssim 300M_\odot$ central bodies that can be studied by Earth-based interferometers, radiation reaction is likely, in fact, to have circularized the orbit long ago; but for the $M \sim 10^6 M_\odot$ central bodies studied by LISA, the orbit is likely to be highly noncircular due to recent perturbations by other orbiting objects [7]. This should be a warning that the analysis of this paper is only a first treatment of what must ultimately be a much more complicated problem.)

(iv) The central body does not absorb any of the inspiraling object's orbital energy; i.e., we can neglect any energy that goes down the central body's horizon (if it has a horizon), and we can neglect tidal heating. This implies that all of the energy lost from the orbit gets deposited into outgoing gravitational waves.

For a system that satisfies our idealizing circular-orbit assumption (iii), the gravitational waves are emitted primarily (but not solely) at twice the orbital frequency, and correspondingly the dominant gravitational "spectral line" is at the frequency

$$f = \frac{2\Omega}{2\pi} = \frac{\Omega}{\pi}, \tag{1}$$

where Ω is the orbital angular frequency.

As time passes, radiation reaction will cause the orbit to shrink gradually; and correspondingly, f will be a slowly varying function of time t. There will also be emissions at frequencies $\frac{1}{2}f$, $\frac{3}{2}f$, $2f$, ...

In this paper we shall focus on aspects of the waves that can be computed without facing any serious complications of the theory of wave emission. We avoid analyzing wave emission in detail because, for a body with arbitrary multipole moments, such an analysis will be very complex. Fortunately, we can make considerable progress by focusing almost solely on gravitational-wave quantities that depend only on the properties of the central body's circular geodesic orbits.

One such quantity is a gravitational-wave spectrum $\Delta E(f)$, defined as follows: During a short interval of time when the waves' principal frequency is evolving from f to $f + df$, we take all the energy emitted into the principal spectral line, plus all being emitted into all the other lines nf with $n = \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, \ldots$; and we add all that energy together to obtain a total emitted energy $dE_{\rm wave}$. By our idealizing assumption (iv), this is equal to the energy lost from the orbit $-dE$ as the orbital angular frequency varies from $\Omega = \pi f$ to $\Omega + d\Omega = \pi(f + df)$. The quantity $\Delta E(f)$ is the corresponding amount of gravitational-wave energy per logarithmic interval of frequency:

$$\Delta E \equiv f\frac{dE_{\rm wave}}{df} = -\Omega\frac{dE}{d\Omega} . \tag{2}$$

Two other gravitational-wave quantities that can be computed without facing the complications of wave-emission theory [as well as without requiring assumption (iv) above] are the frequencies of wave modulation that result from orbital precession. There are two types of precession and corresponding two wave modulations: (i) if the orbit is slightly elliptical, then the ellipse can precess (a "precession of the orbit's periastron") at some angular frequency Ω_ρ that depends in some way on the orbital radius and thence on the waves' primary frequency f; (ii) if the orbit is slightly inclined to the central body's equator, then the orbital plane will precess at some angular frequency Ω_z that also depends on f. These orbital precessions will modulate the emitted waves at the angular frequencies $\Omega_\rho(f)$ and $\Omega_z(f)$.

In Sec. III of this paper we shall develop algorithms for computing these three gravitational-wave quantities, ΔE, Ω_ρ, and Ω_z, as power series in f, or equivalently in the dimensionless parameter

$$v = (\pi M f)^{1/3} = (M\Omega)^{1/3} . \tag{3}$$

In the Newtonian limit, v is the orbiting object's linear velocity.

In Sec. II [Eqs. (17)–(19)] we will write down the first few terms of those power series. As is suggested by the forms of those explicit series, our algorithms enable us to express the power series' coefficients entirely in terms of the central body's multipole moments M_l and S_l. Moreover, if (via idealized measurements) we could learn any one of the wave functions $\Delta E(f)$, $\Omega_\rho(f)$, or $\Omega_z(f)$, then by expanding that function as a power series in $v = (\pi M f)^{1/3}$ and examining the numerical values of the coefficients, we would be able to read off the values of all the multipole moments M_l, S_l.

This result is not of great practical interest, because a system of interferometers can achieve only a modest accuracy in any attempt to measure the functions $\Delta E(f)$, $\Omega_\rho(f)$, and $\Omega_z(f)$ (and also because of the idealizing assumptions that have been made). Of greater practical interest will be measurements of the time evolution $\Phi(t)$ of the waves' phase, since via the method of "matched filters" this quantity can be measured with very high accuracy ($\sim 1/10^4$ to $\sim 1/10^7$ depending on the system [4,8]). This phase evolution $\Phi(t)$ actually contains contributions from all the waves' spectral lines as well as from precessional modulations. In discussing $\Phi(t)$, we shall assume, for simplicity, that the orbit is precisely circular and equatorial so there are no precessions; and we shall focus solely on the portion of $\Phi(t)$ that is associated with the primary frequency, $\Phi_2(t) = 2\pi \int f\, dt = 2\int \Omega\, dt$. A knowledge of this primary phase evolution is equivalent to a knowledge of the number of cycles ΔN that the primary waves spend in a logarithmic interval of frequency:

$$\Delta N(f) \equiv \frac{f^2}{df/dt} = \frac{f\Delta E(f)}{dE_{\rm wave}/dt} . \tag{4}$$

Here $dE_{\rm wave}/dt$ is the gravitational-wave luminosity, or equivalently the rate of loss of orbital energy, $-dE/dt$.

To compute $dE_{\rm wave}/dt$ fully, even with our idealizing assumptions, would require dealing with all the complexities of wave-emission theory. Fortunately, however, we can compute the leading-order contribution of each central-body multipole M_l or S_l to $dE_{\rm wave}/dt$ using fairly elementary wave-generation considerations. We do so in Sec. IV, and we then use Eq. (4) to deduce each multipole's leading-order contribution to the power-series expansion of $N(f)$ [Eq. (57) below]. Just as was the case for our other three wave functions $\Delta E(f)$, $\Omega_\rho(f)$, and $\Omega_z(f)$, each multipole appears first at a different order in the series: M_l at order v^{2l} (beyond where $M_0 = M$ enters at leading order), and S_l at v^{2l+1}. This guarantees that, from the power series expansion of the (accurately measurable) phase evolution $\Delta N(f)$, one (in principle) can read off the values of all the central-body multipole moments. However, to produce a full algorithm for doing so would require dealing with the full complexities of wave-emission theory.

Our derivation and presentation of these results is organized as follows. In Sec. II we write down the spacetime metric for the central body; we derive equations describing the metric's nearly equatorial and nearly circular geodesic orbits, through which the inspiraling object moves; we use those orbital equations to derive expressions for our gravitational-wave functions ΔE, Ω_ρ, and Ω_z [Eqs. (17)–(19)] in terms of the central body's metric; and we state (with the proof to follow in Sec. III) the first few terms of the expansions of these quantities in powers of $v = (\pi M f)^{1/3}$ with coefficients depending on the central body's multipole moments. In Sec. III we briefly review key portions of the Ernst formalism for solving the axisymmetric, vacuum Einstein field equations and of the Geroch-Hansen multipole-moment formalism [2,3] by which the resulting solutions can be expressed in terms of multipole moments; and then we devise algorithms for computing the power series expansions of ΔE, Ω_ρ, and Ω_z. The explicit power series of Sec. II are derived from those algorithms. In Sec. IV we digress briefly from the main thread of the paper, to discuss an issue of principle that can be delicate: how to deduce the mass M from the power series expansion $\Omega_z(v)$. Finally, in Sec. V, we use elementary wave-generation arguments to compute the leading-order contribution of each central-body multipole to the gravitational luminosity, and thence to the waves' phase-evolution function $\Delta N(f)$.

II. FUNCTIONS OF THE MULTIPOLE MOMENTS

In this section, we will review the foundations for analyzing the three functions $\Delta E(f)$, $\Omega_\rho(f)$, and $\Omega_z(f)$ that contain full information of the multipole moments of the central body. The metric produced by the central body, ignoring the effects of the much less massive orbiting object, can be written in terms of (t,ϕ,ρ,z) as (units where $G = c = 1$ are used throughout)

$$ds^2 = -F(dt - \omega d\phi)^2 + \frac{1}{F}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right], \tag{5}$$

where F, ω, and γ are functions of ρ and $|z|$. Instead of specifying these functions, it is more convenient to classify the metric by the Geroch-Hansen [2,3] multipole moments associated with it. Because of the axisymmetry, specifying the $2l+1$ independent components of the l-th tensor multipole moment is equivalent to specifying the scalar multipole moment formed by the product of the tensor moment with l symmetry axis vectors, and then dividing by $l!$. As discussed and defined in Hansen [3], these scalar multipole moments can be classified into two families, corresponding to mass and mass current (i.e., momentum density), parametrized by integer values of $l \geq 0$. Because of the reflection symmetry across the equatorial plane, the mass multipole moments can be nonzero only for even l: M, M_2, M_4, ... , M_l, ... The mass monopole moment is the mass itself, so the "0" subscript of M_0 is omitted. Similarly, the current multipole moments can be nonzero only for odd l: S_1, S_3, ... , S_l, ... For example, the Kerr metric with mass m and spin a has $M_l + iS_l = m(ia)^l$ (Ref. [3], Eq. (3.14)). The letters M and S are used here to refer to the multipole moments of the central body alone, as opposed to the letters I and J which will be used in Sec. V when discussing the multipole moments of the entire system, including the orbiting object.

When radiation reaction is neglected, the orbit of the small object is governed by three conservation laws. The first follows from the standard normalization condition for the object's four-velocity:

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2g_{t\phi}\left(\frac{dt}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right) + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 + g_{zz}\left(\frac{dz}{d\tau}\right)^2. \tag{6}$$

The lack of t dependence in the metric implies that the energy per mass μ of the small object is a conserved quantity. It has a value

$$\frac{E}{\mu} = -g_{tt}\left(\frac{dt}{d\tau}\right) - g_{t\phi}\left(\frac{d\phi}{d\tau}\right). \tag{7}$$

Similarly, the "z component" of angular momentum per mass of the small object,

$$\frac{L_z}{\mu} = g_{t\phi}\left(\frac{dt}{d\tau}\right) + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right), \tag{8}$$

is conserved because of the lack of ϕ dependence in the metric.

If the object is moving in a circle along the equator $z = 0$, then the orbital angular velocity (or "angular frequency" as we shall call it) is

$$\Omega = \frac{d\phi}{dt} = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho}g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}}. \tag{9}$$

This is easily obtained from the geodesic equation and by imposing the conditions of constant orbital radius, that $d\rho/d\tau = 0$ and $d^2\rho/d\tau^2 = 0$.

A circular orbit also implies that $d\rho/d\tau = 0$ and $dz/d\tau = 0$ in Eq. (6), while $d\phi/d\tau = \Omega\ dt/d\tau$, so that solving for $dt/d\tau$ in Eq. (6) and substituting in Eq. (7) gives

$$\frac{E}{\mu} = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}. \tag{10}$$

Similarly, a circular orbit implies, from Eq. (8), that

$$\frac{L_z}{\mu} = \frac{g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}. \tag{11}$$

The orbit might also be slightly different from a circle in the equatorial plane: it might be slightly elliptical or slightly out of the equatorial plane. In this case, Eqs. (7) and (8) can be solved for $dt/d\tau$ and $d\phi/d\tau$, which can be inserted into Eq. (6) to get

$$-1 + \left(\frac{g_{\phi\phi}}{\rho^2}\right)\frac{E^2}{\mu^2} + 2\left(\frac{g_{t\phi}}{\rho^2}\right)\frac{EL_z}{\mu^2} + \left(\frac{g_{tt}}{\rho^2}\right)\frac{L_z^2}{\mu^2} = g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 + g_{zz}\left(\frac{dz}{d\tau}\right)^2, \tag{12}$$

where the fact that

$$\rho^2 = g_{t\phi}^2 - g_{tt}g_{\phi\phi} \tag{13}$$

was used. When the left-hand side of Eq. (12) is expanded in powers of z and of $\delta\rho \equiv$ (radial displacement from the value of ρ which, along with $z = 0$, maximizes the left-hand side), and when only the leading-order (quadratic) terms in z and $\delta\rho$ are kept, then Eq. (12) becomes the law of energy conservation for a two-dimensional harmonic oscillator. The vanishing of the mixed ρz derivative of the left-hand side (because of the reflection symmetry, taking a single z derivative gives zero) implies that the motions in the ρ and z directions are independent of each other. These motions correspond to the periastron precession and the orbital plane precession, which are at frequencies Ω_ρ and Ω_z, respectively. The precession frequencies are

$$\Omega_\alpha = \Omega - \left\{-\frac{g^{\alpha\alpha}}{2}\left[(g_{tt} + g_{t\phi}\Omega)^2\left(\frac{g_{\phi\phi}}{\rho^2}\right)_{,\alpha\alpha} - 2(g_{tt} + g_{t\phi}\Omega)(g_{t\phi} + g_{\phi\phi}\Omega)\left(\frac{g_{t\phi}}{\rho^2}\right)_{,\alpha\alpha} + (g_{t\phi} + g_{\phi\phi}\Omega)^2\left(\frac{g_{tt}}{\rho^2}\right)_{,\alpha\alpha}\right]\right\}^{1/2}, \tag{14}$$

where α is ρ or z, and the expression is evaluated at $z = 0$. The "$_{,\alpha\alpha}$" signifies double partial differentiation with respect to the α index. Equation (14) was derived by evaluating the second derivative, with respect to either ρ or z, of the left-hand side of Eq. (12). Then, the values of E and L_z were substituted from Eqs. (10) and (11). This substitution is valid only in the limit of small deviations of the orbit from a circle in the equatorial plane. The second derivatives were then used to determine the frequencies of the harmonic oscillators in the ρ and z directions which, when subtracted from Ω, give the precession frequencies of Eq. (14).

The metric functions and their derivatives, when evaluated at $z = 0$, can all be expressed as power series in $1/\rho$. From Eq. (9), Ω can be expressed as

$$\Omega = (M/\rho^3)^{1/2}(1 + \text{series in } \rho^{-1/2}) \tag{15}$$

so that

$$\begin{aligned} 1/\rho &= (M/\Omega^2)^{1/3}(1 + \text{series in } \rho^{-1/2}) \\ &= (M/\Omega^2)^{1/3}(1 + \text{series in } \Omega^{1/3}). \end{aligned} \tag{16}$$

Since $\Delta E/\mu$, Ω_ρ/Ω, and Ω_z/Ω are all functions of $1/\rho$ and Ω, then they too can be expressed as power series in $\Omega^{1/3}$. We shall see that the coefficients of these power series can be used to obtain the moments.

These power series have the following forms, as can be derived by an algorithm described in Sec. III below. Listing just the first few terms, which are functions of the lowest three mass moments M, M_2, and M_4 and the lowest two current moments S_1 and S_3, the functions are [using $v \equiv (M\Omega)^{1/3}$]

$$\begin{aligned} \frac{\Delta E}{\mu} = {} & \frac{1}{3}v^2 - \frac{1}{2}v^4 + \frac{20}{9}\frac{S_1}{M^2}v^5 + \left(-\frac{27}{8} + \frac{M_2}{M^3}\right)v^6 + \frac{28}{3}\frac{S_1}{M^2}v^7 + \left(-\frac{225}{16} + \frac{80}{27}\frac{S_1^2}{M^4} + \frac{70}{9}\frac{M_2}{M^3}\right)v^8 \\ & + \left(\frac{81}{2}\frac{S_1}{M^2} + 6\frac{S_1M_2}{M^5} - 6\frac{S_3}{M^4}\right)v^9 + \left(-\frac{6615}{128} + \frac{115}{18}\frac{S_1^2}{M^4} + \frac{935}{24}\frac{M_2}{M^3} + \frac{35}{12}\frac{M_2^2}{M^6} - \frac{35}{12}\frac{M_4}{M^5}\right)v^{10} \\ & + \left(165\frac{S_1}{M^2} + \frac{1408}{243}\frac{S_1^3}{M^6} + \frac{968}{27}\frac{S_1M_2}{M^5} - \frac{352}{9}\frac{S_3}{M^4}\right)v^{11} \\ & + \left(-\frac{45927}{256} - \frac{123}{14}\frac{S_1^2}{M^4} + \frac{9147}{56}\frac{M_2}{M^3} + \frac{93}{4}\frac{M_2^2}{M^6} + 24\frac{S_1^2M_2}{M^7} - 24\frac{S_1S_3}{M^6} - \frac{99}{4}\frac{M_4}{M^5}\right)v^{12} + \cdots, \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{\Omega_\rho}{\Omega} = {} & 3v^2 - 4\frac{S_1}{M^2}v^3 + \left(\frac{9}{2} - \frac{3}{2}\frac{M_2}{M^3}\right)v^4 - 10\frac{S_1}{M^2}v^5 + \left(\frac{27}{2} - 2\frac{S_1^2}{M^4} - \frac{21}{2}\frac{M_2}{M^3}\right)v^6 + \left(-48\frac{S_1}{M^2} - 5\frac{S_1M_2}{M^5} + 9\frac{S_3}{M^4}\right)v^7 \\ & + \left(\frac{405}{8} + \frac{2243}{84}\frac{S_1^2}{M^4} - \frac{661}{14}\frac{M_2}{M^3} - \frac{21}{8}\frac{M_2^2}{M^6} + \frac{15}{4}\frac{M_4}{M^5}\right)v^8 + \left(-243\frac{S_1}{M^2} - 16\frac{S_1^3}{M^6} + 4\frac{S_1M_2}{M^5} + 45\frac{S_3}{M^4}\right)v^9 \\ & + \left(\frac{1701}{8} + \frac{8443}{28}\frac{S_1^2}{M^4} - \frac{1545}{7}\frac{M_2}{M^3} - \frac{95}{8}\frac{M_2^2}{M^6} - \frac{85}{3}\frac{S_1^2M_2}{M^7} + 12\frac{S_1S_3}{M^6} + 30\frac{M_4}{M^5}\right)v^{10} + \cdots, \end{aligned} \tag{18}$$

$$\frac{\Omega_z}{\Omega} = 2\frac{S_1}{M^2}v^3 + \frac{3}{2}\frac{M_2}{M^3}v^4 + \left(7\frac{S_1^2}{M^4} + 3\frac{M_2}{M^3}\right)v^6 + \left(11\frac{S_1M_2}{M^5} - 6\frac{S_3}{M^4}\right)v^7$$
$$+ \left(\frac{153}{28}\frac{S_1^2}{M^4} + \frac{153}{28}\frac{M_2}{M^3} + \frac{39}{8}\frac{M_2^2}{M^6} - \frac{15}{4}\frac{M_4}{M^5}\right)v^8 + \left(26\frac{S_1^3}{M^6} + 31\frac{S_1M_2}{M^5} - 15\frac{S_3}{M^4}\right)v^9$$
$$+ \left(\frac{69}{7}\frac{S_1^2}{M^4} + \frac{69}{7}\frac{M_2}{M^3} + \frac{41}{2}\frac{M_2^2}{M^6} + \frac{389}{6}\frac{S_1^2M_2}{M^7} - 41\frac{S_1S_3}{M^6} - 15\frac{M_4}{M^5}\right)v^{10} + \cdots . \tag{19}$$

These expressions give some indication as to why all the multipole moments are obtainable from any one of the functions $\Delta E(v)$, $\Omega_\rho(v)$, or $\Omega_z(v)$ [with $v = (M\Omega)^{1/3} = (\pi M f)^{1/3}$]. The current moment S_l ($l = 1, 3, 5, \ldots$) always first appears in the coefficient of $\Omega^{(2l+3)/3}$ in $\Delta E/\mu$, and of $\Omega^{(2l+1)/3}$ in Ω_ρ/Ω and Ω_z/Ω. The mass moment M_l ($l = 2, 4, 6, \ldots$) always first appears in the coefficient of $\Omega^{(2l+2)/3}$ in $\Delta E/\mu$, and of $\Omega^{2l/3}$ in Ω_ρ/Ω and Ω_z/Ω. Since each multipole moment makes its first appearance at a different order, then one would expect that all the moments can be obtained from these functions.

In $\Delta E/\mu$, the first two powers of Ω have coefficients that involve only M, but to different powers. This allows not only for the determination of the mass, but also if $\Delta E/\mu$ is only measurable up to a proportionality constant (for example, because μ or the distance to the source is not known exactly), this constant can be determined. In Ω_ρ/Ω, the mass M can be determined from the first term. In Ω_z/Ω, there is no term that involves only the mass. If all the terms in the Ω_z/Ω expansion are zero (because $M_l = S_l = 0$ for $l \geq 1$), then the mass M cannot be determined at all from Ω_z/Ω. This case corresponds to the gravitational field of the more massive object being spherically symmetric, so that there is no orbital plane precession possible. If some of the terms in the Ω_z/Ω expansion are nonzero, then it is possible to determine M from this expansion, as we shall see in Sec. IV.

III. DETERMINATION OF THE MULTIPOLE MOMENTS

In this section we shall develop an algorithm by which the power series expansions (17)–(19) can be derived, to all orders; and we shall show that each moment S_l or M_l first appears in that expansion at the order described in Sec. II. The appearance of each moment at a unique order guarantees that the multipole moment can be determined from knowledge of the power series.

We will divide this presentation into five parts. In Sec. III A, we will review the Ernst potential and its relation to the metric. We will show that the Ernst potential is completely determined everywhere by a set of coefficients called a_{j0} and a_{j1} which describe the metric on the equatorial plane. In Sec. III B we will show that all the a_{j0} and a_{j1} can be determined from $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω. In Sec. III C the algorithm described in Sec. III B to do this will be summarized. In Sec. III D we will show how to go from the a_{j0} and a_{j1} to the multipole moments M_l and S_l. In Sec. III E, we will show how Eqs. (17)–(19) can be derived.

In Secs. III and IV we assume that any one of the dimensionless functions, $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω, is known exactly to all orders in Ω. In addition, in Sec. III, we assume that M is known—if $\Delta E/\mu$ or Ω_ρ/Ω is the known function, then M is easily extracted from the first term in either series (17) or (18); if Ω_z/Ω is the known function, then M can be determined from the algorithm described below in Sec. IV.

A. The Ernst potential

Fodor, Hoenselaers, and Perjés [9] give details of the computation of the multipole moments from the complex potential $\tilde{\xi}$, a function of ρ and z. This $\tilde{\xi}$ is related to the Ernst potential [10] $\mathcal{E}$ by

$$\mathcal{E} = F + i\psi = \frac{\sqrt{\rho^2 + z^2} - \tilde{\xi}}{\sqrt{\rho^2 + z^2} + \tilde{\xi}}, \tag{20}$$

where F is related to the metric by [see Eq. (5)]

$$g_{tt} = -F, \tag{21}$$

and ψ is related to the metric by (Ref. [11], Eq. (I.3b))

$$g_{t\phi} = -F \int_\rho^\infty \frac{\rho'}{F^2}\frac{\partial\psi}{\partial z}d\rho'\bigg|_{\text{constant } z} . \tag{22}$$

The Ernst potential $\mathcal{E}$ is powerful for generating stationary, axisymmetric solutions to the gravitational field equations. It contains all the information of the spacetime geometry in a single, complex function, and thus so also does $\tilde{\xi}$.

The potential $\tilde{\xi}$ has the property that it can be expanded as (Ref. [9], Eq. (15))

$$\tilde{\xi} = \sum_{j,k=0}^{\infty} a_{jk}\frac{\rho^j z^k}{(\rho^2 + z^2)^{j+k}}. \tag{23}$$

The a_{jk} can be nonzero only for non-negative, even j and non-negative k. Because of the reflection symmetry across the equatorial plane, a_{jk} is real for even k and imaginary for odd k.

Since the measured function, any one of $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω, is directly related to the metric in the region around the equatorial plane $z = 0$, then it is most convenient to convert the measured function into the coefficients that contain information of the equatorial plane metric, namely, a_{j0} and a_{j1}.

Assume for the moment that for any positive, even integer m, all the a_{j0} with $j = 0, 2, \ldots, m$ and the a_{j1} with $j = 0, 2, \ldots, m-2$ are known; and assume that for any positive, odd integer m, all the a_{j0} with $j = 0, 2, \ldots, m-1$ and a_{j1} with $j = 0, 2, \ldots, m-1$ are known.

From these a_{j0} and a_{j1}, all the a_{jk} for $j + k \leq m$ can be computed from (Ref. [9], Eq. (16))

$$\begin{aligned} a_{r,s+2} = \frac{1}{(s+2)(s+1)} \Bigg(& -(r+2)^2 a_{r+2,s} \\ & + \sum_{k,l,p,q} a_{kl} a^*_{r-k-p,s-l-q} \big[a_{pq}(p^2 + q^2 \\ & -4p - 5q - 2pk - 2ql - 2) \\ & + a_{p+2,q-2}(p+2)(p+2-2k) \\ & + a_{p-2,q+2}(q+2)(q+1-2l) \big] \Bigg). \end{aligned} \tag{24}$$

The sum is over all integer values of k, l, p, and q that give nonzero contributions, namely, $0 \leq k \leq r$, $0 \leq l \leq s+1$, $0 \leq p \leq r-k$, $-1 \leq q \leq s-l$, and k and p even.

All the coefficients a_{jk} that are within the summation sign in Eq. (24) have the property that $j + k < r + s + 2$. Thus, $a_{r,s+2}$ (with $s \geq 0$) is a function of the a_{j0} and $a_{j-1,1}$ with $j \leq r+s+2$, but no higher order a_{j0} or $a_{j-1,1}$. This shows explicitly that $\tilde{\xi}$, and thence also the entire spacetime metric, are fully determined by a knowledge of the a_{j0} and $a_{j-1,1}$, or equivalently a knowledge of the equatorial plane metric.

B. Computing a_{j0} and a_{j1}

The process [12] of determining the a_{j0} and a_{j1} from $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω occurs in iterations, each stage labeled by $n = 0, 1, 2, \ldots$. For now, assume that it is $\Delta E/\mu$ that is known, rather than Ω_ρ/Ω or Ω_z/Ω. Assume that the a_{j0} are known up to order $j = 2n$, and the a_{j1} are known up to order $j = 2n-2$. That is, a_{00}, a_{20}, $a_{40}, \ldots$, $a_{2n,0}$ and a_{01}, a_{21}, $a_{41}, \ldots, a_{2n-2,1}$ are known. (At the $n = 0$ stage, only $a_{00} = M$ is known.) All unknown a_{j0} and a_{j1} are set to zero at this nth stage. The goal of this nth stage is to figure out what $a_{2n+2,0}$ and $a_{2n,1}$ must be in order to reproduce the observed functional form for $\Delta E/\mu$.

From the known values of a_{j0} and a_{j1}, the metric functions g_{tt} and $g_{t\phi}$ on the equatorial plane can be computed with Eqs. (20)–(23). Then, the metric function $g_{\phi\phi}$ can be obtained from Eq. (13).

Therefore, with the a_{j0} known up to $j = 2n$, the a_{j1} known up to $j = 2n-2$, and all other a_{j0} and a_{j1} (temporarily) set to zero, the three metric functions g_{tt}, $g_{t\phi}$, and $g_{\phi\phi}$ can be expressed as power series in $1/\rho$ on the equatorial plane $z = 0$. Then, Ω can be computed as a power series in $1/\rho$ using Eq. (9). This series can be inverted to have $1/\rho$ as a series in Ω, so that the metric functions are power series in Ω. With Eqs. (2) and (10), we can compute $\Delta E/\mu$ as a power series in Ω; we will call this computed function $(\Delta E/\mu)_n$. The n subscript denotes the fact that this is as computed only using the known a_{j0} and a_{j1} at stage n, and setting all unknown a_{j0} and a_{j1} to zero. In particular, $a_{2n+2,0}$ and $a_{2n,1}$ were set to zero in calculating $(\Delta E/\mu)_n$, and we will remedy this situation below.

We can express these two functions, the actual $\Delta E/\mu$ that is being deciphered and the computed $(\Delta E/\mu)_n$, as power series in $\Omega^{1/3}$:

$$\Delta E/\mu = \sum_\alpha A_\alpha \Omega^{\alpha/3}, \tag{25a}$$

$$(\Delta E/\mu)_n = \sum_\alpha B_\alpha \Omega^{\alpha/3}. \tag{25b}$$

It is easy to verify that if $a_{2n,1}$ (which is unknown at this nth stage) were changed from zero to a nonzero value, then to leading order in Ω, $(\Delta E/\mu)_n$ would change by

$$-i\frac{16n+20}{9} a_{2n,1} M^{-(2n+1)/3} \Omega^{(4n+5)/3}. \tag{26a}$$

The $\Omega^{(4n+6)/3}$ term would not change if $a_{2n,1}$ were changed; however, if the $a_{2n+2,0}$ term were changed from zero to a nonzero value, then $(\Delta E/\mu)_n$ would change to lowest order by

$$-\frac{(4n+3)(4n+6)}{9} a_{2n+2,0} M^{-(2n+3)/3} \Omega^{(4n+6)/3}. \tag{26b}$$

Based on these facts, then the $a_{2n,1}$ and $a_{2n+2,0}$ terms can be computed at the nth iteration stage, by simply setting the $a_{2n,1}$ and $a_{2n+2,0}$ seen in Eqs. (26) to the values that would have made $(\Delta E/\mu)_n$ agree with $\Delta E/\mu$ to order $\Omega^{(4n+6)/3}$ (rather than setting $a_{2n,1}$ and $a_{2n+2,0}$ to zero as was done at the beginning of the nth stage): we set

$$a_{2n,1} = i\frac{9M^{(2n+1)/3}}{16n+20}(A_{4n+5} - B_{4n+5}), \tag{27a}$$

$$a_{2n+2,0} = -\frac{9M^{(2n+3)/3}}{(4n+3)(4n+6)}(A_{4n+6} - B_{4n+6}). \tag{27b}$$

Then the process can be repeated at the $(n+1)$th iteration stage.

Now, we will repeat the above argument of Sec. III B for what to do at the nth stage if instead of $\Delta E/\mu$, it is Ω_ρ/Ω that is known. A similar procedure as in the $\Delta E/\mu$ case can be followed, except that instead of Eq. (10), Eq. (14) must be used. To compute the $g^{\rho\rho}$ function that appears in this equation, it is necessary to compute the γ function that appears in the metric (5) evaluated on the equatorial plane (see, for example, Ref. [11], Eq. (I.4a) or Ref. [13], Eq. (7.1.26)):

$$\gamma = \frac{1}{4}\int_\rho^\infty \left[\frac{\rho'}{g_{tt}^2}\left(\frac{dg_{tt}}{d\rho'}\right)^2 - \frac{g_{tt}^2}{\rho'}\left(\frac{d(g_{t\phi}/g_{tt})}{d\rho'}\right)^2 \right] d\rho'. \tag{28}$$

Following a similar argument as in the $\Delta E/\mu$ case, at the iteration labeled by n,

$$(\Omega_\rho/\Omega)_n = \sum_\alpha D_\alpha \Omega^{\alpha/3} \tag{29a}$$

can be computed to order $\Omega^{(4n+4)/3}$ and compared to

$$\Omega_\rho/\Omega = \sum_\alpha C_\alpha \Omega^{\alpha/3}. \tag{29b}$$

It is easily verifiable that the leading-order effect of an $a_{2n,1}$ on $(\Omega_\rho/\Omega)_n$ is

$$i(2n+4)a_{2n,1}M^{-(2n+3)/3}\Omega^{(4n+3)/3} \tag{30a}$$

and $a_{2n,1}$ has no effect on the $\Omega^{(4n+4)/3}$ term. The leading-order effect of an $a_{2n+2,0}$ on $(\Omega_\rho/\Omega)_n$ is

$$(n+1)(2n+3)a_{2n+2,0}M^{-(2n+5)/3}\Omega^{(4n+4)/3}. \tag{30b}$$

From these facts, the next two coefficients should be set to

$$a_{2n,1} = -i\frac{M^{(2n+3)/3}}{2n+4}(C_{4n+3} - D_{4n+3}), \tag{31a}$$

$$a_{2n+2,0} = \frac{M^{(2n+5)/3}}{(n+1)(2n+3)}(C_{4n+4} - D_{4n+4}). \tag{31b}$$

If it is Ω_z/Ω that is known, then it is also necessary to compute the second derivatives of the metric functions $g_{tt,zz}$, $g_{t\phi,zz}$, and $g_{\phi\phi,zz}$, evaluated on $z=0$. These require the a_{j2} and a_{j3} terms, which can be obtained from Eq. (24). At the iteration labeled by n,

$$(\Omega_z/\Omega)_n = \sum_\alpha H_\alpha \Omega^{\alpha/3} \tag{32a}$$

can be computed to order $\Omega^{(4n+4)/3}$ and compared to

$$\Omega_z/\Omega = \sum_\alpha F_\alpha \Omega^{\alpha/3}. \tag{32b}$$

Following the same type of argument as in the case of $\Delta E/\mu$ and Ω_ρ/Ω, the effects of $a_{2n,1}$ and $a_{2n+2,0}$ on $(\Omega_z/\Omega)_n$ are

$$-i(2n+2)a_{2n,1}M^{-(2n+3)/3}\Omega^{(4n+3)/3}, \tag{33a}$$

$$-(n+1)(2n+3)a_{2n+2,0}M^{-(2n+5)/3}\Omega^{(4n+4)/3}, \tag{33b}$$

respectively, and $a_{2n,1}$ has no effect on the $\Omega^{(4n+4)/3}$ term. The next two coefficients therefore should be set to

$$a_{2n,1} = i\frac{M^{(2n+3)/3}}{2n+2}(F_{4n+3} - H_{4n+3}), \tag{34a}$$

$$a_{2n+2,0} = -\frac{M^{(2n+5)/3}}{(n+1)(2n+3)}(F_{4n+4} - H_{4n+4}). \tag{34b}$$

Whether analyzing $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω, this iteration can be repeated up to an indefinite order.

C. Summary of above

To summarize the iterative process that allows for the determination of the a_{j0} and the a_{j1}.

Stage n, Step 1. With the a_{j0} up to $j=2n$ and the a_{j1} up to $j=2n-2$, and the higher order a_{j0} and a_{j1} set to zero, use Eqs. (20)–(23) and (13) to compute g_{tt}, $g_{t\phi}$, and $g_{\phi\phi}$ as functions of $1/\rho$ on the equatorial plane.

Stage n, Step 2. From these g_{tt}, $g_{t\phi}$, and $g_{\phi\phi}$, compute $(\Delta E/\mu)_n$ [with the help of Eqs. (2) and (10)], $(\Omega_\rho/\Omega)_n$ [with Eqs. (14) and (28)], or $(\Omega_z/\Omega)_n$ [with Eqs. (14), (24), and (28)] as a function of Ω [with the aid of Eq. (9) to get $1/\rho$ as a function of Ω].

Stage n, Step 3. Set the values of $a_{2n,1}$ and $a_{2n+2,0}$ using Eqs. (27) for $\Delta E/\mu$, Eqs. (31) for Ω_ρ/Ω, or Eqs. (34) for Ω_z/Ω.

Stage n, Step 4. Go to Stage $n+1$, Step 1.

D. Computing the moments

After as many as desired of the a_{j0} and a_{j1} terms have been computed, the a_{jk} can be computed with Eq. (24). Then, using the algorithm in Ref. [9], the multipole moments can be computed from the a_{jk}: in terms of

$$\bar\rho = \frac{\rho}{\rho^2+z^2},\ \bar z = \frac{z}{\rho^2+z^2}, \tag{35}$$

the multipole moments are

$$M_l + iS_l = \left.\frac{S_0^{(l)}}{(2l-1)!!}\right|_{\bar\rho=0,\bar z=0}, \tag{36}$$

where these $S_a^{(n)}$, not to be confused with the S_l, are recursively computed by

$$S_0^{(0)} = \tilde\xi,\ S_0^{(1)} = \frac{\partial\tilde\xi}{\partial\bar z},\ S_1^{(1)} = \frac{\partial\tilde\xi}{\partial\bar\rho}, \tag{37}$$

$$\begin{aligned} S_a^{(n)} = \frac{1}{n}\Bigg[& a\frac{\partial}{\partial\bar\rho}S_{a-1}^{(n-1)} + (n-a)\frac{\partial}{\partial\bar z}S_a^{(n-1)} \\ & + a\left([a+1-2n]\gamma_1 - \frac{a-1}{\bar\rho}\right)S_{a-1}^{(n-1)} \\ & + (a-n)(a+n-1)\gamma_2 S_a^{(n-1)} + a(a-1)\gamma_2 S_{a-2}^{(n-1)} \\ & + (n-a)(n-a-1)\left(\gamma_1 - \frac{1}{\bar\rho}\right)S_{a+1}^{(n-1)} \\ & - \Big[a(a-1)\tilde R_{11}S_{a-2}^{(n-2)} + 2a(n-a)\tilde R_{12}S_{a-1}^{(n-2)} \\ & + (n-a)(n-a-1)\tilde R_{22}S_a^{(n-2)}\Big]\left(n-\frac{3}{2}\right)\Bigg], \end{aligned} \tag{38}$$

in which $\tilde R_{11}$, $\tilde R_{12}$, and $\tilde R_{22}$ are given by

$$\tilde R_{ij} = \left[(\bar\rho^2+\bar z^2)|\tilde\xi|^2 - 1\right]^{-2}\left(G_iG_j^* + G_i^*G_j\right), \tag{39}$$

with

$$G_1 = \bar{z}\frac{\partial\tilde{\xi}}{\partial\bar{\rho}} - \bar{\rho}\frac{\partial\tilde{\xi}}{\partial\bar{z}},\ G_2 = \bar{\rho}\frac{\partial\tilde{\xi}}{\partial\bar{\rho}} + \bar{z}\frac{\partial\tilde{\xi}}{\partial\bar{z}} + \tilde{\xi}, \tag{40}$$

and, from these $\tilde{R}_{ij}$,

$$\gamma_1 = (\bar{\rho}/2)(\tilde{R}_{11} - \tilde{R}_{22}),\ \gamma_2 = \bar{\rho}\tilde{R}_{12}. \tag{41}$$

Therefore, knowledge of the mass M and $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω allows for determination of the M_l and S_l.

We have seen that each a_{l0} and $a_{l-1,1}$ is determined from $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω by the value of a certain coefficient in the power series expansion. Then, with Eq. (24), all the a_{rs} with $r+s=l$ are determined, and with Eqs. (35)–(41), it can be verified that a variation of $\tilde{\xi}$ by $\sum_{r+s=l} a_{rs}\bar{\rho}^r\bar{z}^s$ leads to a variation in $M_l + iS_l$ such that

$$a_{0l} = M_l + iS_l + \text{LOM} \tag{42}$$

"LOM" is an abbreviation for lower order moments: some combination of M_j and S_k with $j < l$ and $k < l$. Equivalently by virtue of Eq. (24),

$$a_{l0} = (-1)^{l/2}\frac{(l-1)!!}{l!!}M_l + \text{LOM}, \tag{43a}$$

$$a_{l-1,1} = i(-1)^{(l-1)/2}\frac{l!!}{(l-1)!!}S_l + \text{LOM} \tag{43b}$$

Given an integer m, for even m, knowing the a_{j0} up to a_{m0} and the a_{j1} up to $a_{m-2,1}$ is equivalent to knowing $M, S_1, M_2, S_3, M_4, \ldots, S_{m-1}, M_m$; for odd m, knowing the a_{j0} up to $a_{m-1,0}$ and the a_{j1} up to $a_{m-1,1}$ is equivalent to knowing $M, S_1, M_2, S_3, M_4, \ldots, M_{m-1}, S_m$. Thus there is a unique term in the power series expansion of any one of the functions $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω where each multipole moment appears to leading order, and there is a prescribed algorithm for obtaining the moments.

E. Deriving expansions for $\Delta E/\mu$, Ω_ρ/Ω, and Ω_z/Ω

Finally, Eqs. (17)–(19) can be derived as follows. First, use the method of Sec. III D above to compute M_l as a function of $a_{00}, a_{20}, \ldots, a_{l0}$, and $a_{01}, a_{21}, \ldots, a_{l-2,1}$ (or S_l as a function of $a_{00}, a_{20}, \ldots, a_{l-1,0}$, and $a_{01}, a_{21}, \ldots, a_{l-1,1}$). Then, by inverting the series, obtain a_{l0} as a function of $M_0, S_1, M_2, \ldots, S_{l-1}, M_l$, (or $a_{l-1,1}$ as a function of $M_0, S_1, M_2, \ldots, M_{l-1}, S_l$). Inverting is trivial as long as the problem is solved for the $l-1$ case before trying to solve for the l case. The metric functions and from these, $\Delta E/\mu$, Ω_ρ/Ω, or Ω_z/Ω, can then be expressed as functions of the a_{j0} and a_{j1} using the equations in Sec. III A. Then inserting the values of these a_{j0} and a_{j1} in terms of the multipole moments, we obtain Eqs. (17)–(19).

Alternatively, we can derive the expansions by simply figuring out how the different combinations of the multipole moments appear in the expansions. First of all, each term has as many powers of M as are required to produce the correct dimensions. Then, for example, to find the S_1S_3 dependence in the Ω_ρ/Ω function, an Ω_ρ/Ω can be chosen (by varying the function order by order as needed) such that when the above algorithm to compute the multipole moments is performed on this chosen Ω_ρ/Ω, all the multipole moments except S_1 and S_3 are zero, while S_1 and S_3 take on different nonzero values. Then, looking at the $(\Omega_\rho/\Omega)_n$ function as computed in Step 2 of the above iterative process, the dependencies of Ω_ρ/Ω on S_1S_3, $S_1^2S_3$, $S_1S_3^2$, etc., can be inferred by examining how $(\Omega_\rho/\Omega)_n$ changes as S_1 and S_3 change values. For brevity, shown in Eqs. (17)–(19) are the first few terms only, but additional ones are not hard to compute. The calculation was verified by checking that when the moments take on their Kerr values, Eqs. (17)–(19) give the correct expressions that can be computed independently, directly from the Kerr metric.

IV. DETERMINATION OF THE MASS FOR Ω_z/Ω

With $\Delta E/\mu$ or Ω_ρ/Ω known as a function of Ω, it is easy to determine the mass M since it appears in the first term in either expansion, Eqs. (17) or (18). For Ω_z/Ω, it will be shown in this section that M can be determined in the case that there is some precession (Ω_z/Ω is not zero for all Ω). This is possible because up to any order in the Ω expansion of $\Omega_z/\Omega = \sum_\alpha F_\alpha\Omega^{\alpha/3}$, there are roughly twice as many terms as multipole moment variables, and information of the mass is contained in the redundant terms.

If the coefficient of the Ω term in the expansion of Ω_z/Ω is nonzero ($F_3 \neq 0$), then a method to determine M can be derived by examining Eq. (19). If $F_4 \neq 0$, then the mass is

$$M = \left(\frac{4F_6 - 7F_3^2}{8F_4}\right)^{3/2}, \tag{44a}$$

while if $F_4 = 0$, then the mass is

$$M = \left(\frac{2F_8}{3F_3^2} + \sqrt{\frac{4F_8^2}{9F_3^4} + \frac{41F_7}{36F_3} - \frac{F_{10}}{3F_3^2}}\right)^{3/2}. \tag{44b}$$

In the case that the coefficient of the Ω term in the Ω_z/Ω expansion is zero ($F_3 = 0$), there is a general procedure that can be followed to obtain the mass. With the equations of Sec. III, specifically, those leading up to expressions (33) but carrying the process out to one more order, the next-to-leading order effects of the $a_{2n,1}$ (for $n \geq 1$) and $a_{2n+2,0}$ (for $n \geq 0$) on $(\Omega_z/\Omega)_n$ are

$$-2in(2n+3)a_{2n,1}M^{-(2n+1)/3}\Omega^{(4n+5)/3}, \tag{45a}$$

$$-2(n+1)^2(2n+3)a_{2n+2,0}M^{-(2n+3)/3}\Omega^{(4n+6)/3}. \tag{45b}$$

Comparing these with Expressions (33), the mass can be determined by looking at the first nonzero term in the Ω_z/Ω expansion. If the first nonzero term is an $\Omega^{(4n+3)/3}$ term for integer $n \geq 1$, then the mass is

$$M = \left(\frac{(n+1)F_{4n+5}}{n(2n+3)F_{4n+3}}\right)^{3/2} . \tag{46a}$$

If the first nonzero term is an $\Omega^{(4n+4)/3}$ term for integer $n \geq 0$, then the mass is

$$M = \left(\frac{F_{4n+6}}{(2n+2)F_{4n+4}}\right)^{3/2} . \tag{46b}$$

After M is determined, then the multipole moments can be determined as described in Sec. III, where it is assumed that M is known.

V. LEADING-ORDER EFFECT OF THE MULTIPOLE MOMENTS ON THE GRAVITATIONAL-WAVE PHASE EVOLUTION

Another interesting but much more accurately measurable function of Ω is the gravitational-wave phase evolution for circular orbits in the equatorial plane, expressed as ΔN as a function of Ω, as defined in Eq. (4).

Unfortunately, a similar analysis cannot be conducted for ΔN as was done for the other functions, because the dE_{wave}/dt that appears in ΔN cannot be computed from the Ernst formalism. Rather $dE_{\text{wave}}/dt = -dE/dt$ can only be computed by solving wave equations to compute the wave generation: equations which (apparently) will not decouple from each other nor allow a separation-of-variables solution. These hindrances make the calculation much more difficult than solving perturbations of the Kerr metric, for which decoupling and separation-of-variables do in fact occur and simplify the problem. To make the situation in the general case even more difficult, ΔN depends also on the inner boundary conditions for the gravitational-wave equations and on the amount of energy absorbed by the central body through, for example, a horizon or tidal heating of matter. These inner conditions are not, in general, determined from just the multipole moments. However, at least in the case of a Schwarzshild black hole, the effects of the horizon do not appear until a very high order [14]. It is perhaps possible that just as we made the idealizing assumption (iv) of energy balance when computing ΔE, we can also make some type of simplicity assumption (such as regularity of the wave functions at the origin), and get an accurate enough answer, but this is not clear. Despite this uncertainty, if in the future the task were undertaken to determine ΔN as a function of at least the lowest few multipole moments, the potential to experimentally test the "no-hair" theorem for black holes would be very promising [4]. It will be shown below that if we once again make our four idealizing assumptions, then ΔN contains full information of all the multipole moments. While we cannot yet construct a general algorithm to actually extract all the multipole moments from ΔN, we can, it turns out, extract M, S_1, and M_2 (enough, in principle, to test the no-hair theorem). The following is just a limited discussion of how each multipole moment appears to leading order in ΔN, which in turn depends on how each multipole moment appears to leading order in ΔE and in the gravitational-wave luminosity.

We will divide the discussion in three parts. In Sec. V A we will show a simple way, based on the mass quadrupolar radiation formalism, to compute how central-body multipole moments with $l \geq 2$ (M_2, S_3, M_4, ...) show up to leading order in the gravitational-wave luminosity, $-dE/dt$. For example, we will see how M_2 first shows up at v^4 order (beyond where M first appears) in the luminosity. However, while we can compute this M_2v^4 term, we cannot compute, for example, M_2v^6 or M_2v^8 terms. In Sec. V B we will show that there is another effect which must be taken into account when calculating the leading order influence of S_1 (at v^3 order) on the luminosity. Moreover, we will calculate the leading order occurrence of not only S_1 but also S_1^2 (which shows up at v^4 order) in the series expansion for the luminosity. The $S_1^2v^4$ term is calculated for its usefulness in Sec. V C, where the leading-order effect of the multipole moments M_l, S_l, and S_1^2 on ΔN are computed. From these leading order effects, and some well-known terms derived elsewhere, we also infer the entire series for ΔN up through v^4 order (including S_1v^3, $S_1^2v^4$, and M_2v^4). From this fully known part of the series we get a simple way of testing the no-hair theorem. Incidentally, we could also, for example, calculate the leading-order effect of M_2^2 on the luminosity, which is an $M_2^2v^8$ term, but this would be of little practical value since we cannot calculate M_2v^8 terms at present anyway. Therefore, we will limit this discussion to just the leading-order effects of M_l, S_l, and S_1^2 on ΔN, and save the more general discussion of higher order terms and combinations of multipole moments [such as an expression similar to Eqs. (17)–(19)] for future work.

A. The dominant contribution to $-dE/dt$

The luminosity $dE_{\text{wave}}/dt = -dE/dt$ can be determined by computing the symmetric trace-free radiative multipole moments [15] that determine the gravitational field of the source. The mass multipole moments I_L and current multipole moments J_L are those of the entire source (including the orbiting object of small mass μ), as opposed to the M_l and S_l moments which are the moments of the central body alone. The L subscript is shorthand for l indices: L means $a_1a_2 \cdots a_l$. Because the entire source is not axisymmetric, these I_L and J_L are not reducible to scalar moments, as the M_l and S_l moments are. For nearly Newtonian sources, in terms of an integral over the mass density $\tilde{\rho}$ of the source and Cartesian coordinates y_k, these moments are given by (Ref. [15], Eqs. (5.28))

$$I_L(t) = \left[\int d^3\mathbf{y}\,\tilde{\rho}(\mathbf{y},t)y_L\right]^{\text{STF}} , \tag{47a}$$

$$J_L(t) = \left[\int d^3\mathbf{y}\,\tilde{\rho}(\mathbf{y},t)y_{L-1}\epsilon_{a_lkm}y_ku_m,\right]^{\text{STF}} . \tag{47b}$$

The STF superscript means that the expression is to be symmetrized and made trace-free on its l free indices.

Repeated indices are summed. The u_m is the material's velocity, so that $\tilde{\rho} u_m$ is the mass-current density. The expression y_L means $y_{a_1} y_{a_2} \cdots y_{a_l}$ and y_{L-1} means $y_{a_1} y_{a_2} \cdots y_{a_{l-1}}$.

In terms of these radiative multipole moments, the gravitational-wave luminosity is (Ref. [15], Eq. (4.16′))

$$-\frac{dE}{dt} = \sum_{l=2}^{\infty} \frac{(l+1)(l+2)}{(l-1)l} \frac{1}{l!(2l+1)!!} \langle I_L^{(l+1)} I_L^{(l+1)} \rangle + \sum_{l=2}^{\infty} \frac{4l(l+2)}{(l-1)} \frac{1}{(l+1)!(2l+1)!!} \langle J_L^{(l+1)} J_L^{(l+1)} \rangle. \tag{48}$$

The angular brackets indicate averaging over time. A number in parentheses to the above right of a moment indicates taking that many time derivatives of that radiative moment.

The leading-order contribution comes from the mass quadrupole radiative moment I_{ij}. This quadrupolar contribution to the energy loss for a mass μ moving in a circle of radius ρ at angular frequency Ω is (see, for example, Ref. [16], Eq. (3.6))

$$-\left.\frac{dE}{dt}\right|_{I_{ij}} = \frac{32}{5} \mu^2 \rho^4 \Omega^6. \tag{49}$$

It turns out that for all central-body moments except S_1, the leading-order correction to Eq. (49) arises from a modification of the orbital radius ρ as a function of Ω. Each mass moment M_l ($l > 0$) or current moment S_l changes ρ by the following [where $v = (M\Omega)^{1/3}$]:

$$\rho = M v^{-2} \left(1 + \frac{(-1)^{l/2}\,(l+1)!!\,M_l\,v^{2l}}{3\,l!!\,M^{l+1}}\right), \tag{50a}$$

$$\rho = M v^{-2} \left(1 - \frac{2(-1)^{(l-1)/2}\,l!!\,S_l\,v^{2l+1}}{3\,(l-1)!!\,M^{l+1}}\right). \tag{50b}$$

Equations (50) are derived by using (43), (20)–(23), (13), and (9). Inserting Eqs. (50) into (49) gives the following leading-order effects of the central-body moments on the energy loss, due to the mass quadrupole radiation contribution:

$$-\left.\frac{dE}{dt}\right|_{I_{ij}} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10} \times\left[1 + \sum_{l=2,4,\ldots} \frac{4(-1)^{l/2}\,(l+1)!!\,M_l\,v^{2l}}{3\,l!!\,M^{l+1}} - \sum_{l=1,3,\ldots} \frac{8(-1)^{(l-1)/2}\,l!!\,S_l\,v^{2l+1}}{3\,(l-1)!!\,M^{l+1}}\right]. \tag{51}$$

B. Additional contributions from S_1 and S_1^2

In this section we discuss another contribution to the radiated power, dE/dt, which arises for all central body moments, but which is negligible compared to the ρ-change contribution (51) in all cases except for S_1. For S_1, the second effect together with (51), comprises the full leading order $-dE/dt$.

In computing this second contribution, it will be sufficient to treat each radiative moment in its Newtonian sense: the gravitational field is the sum of the field due to the small mass and the field due to the large mass. The contribution from the small mass comes directly from using Eqs. (47). The contribution of the large mass can be computed as follows.

If the orbiting object of mass μ were absent, then the radiative moments would be determined from just the moments of the central body: $I_L \propto M_l$ and $J_L \propto S_l$. These moments are stationary and therefore do not radiate. However, in the presence of the orbiting object, the large mass moves along a path $-(\mu/M)x_k$, where x_k is the path of the small mass [$x_1 = \rho\cos(\Omega t)$, $x_2 = \rho\sin(\Omega t)$, and $x_3 = 0$]. Therefore, the multipole moments due to the large mass are what the stationary moments would be in a Cartesian coordinate system displaced by $(\mu/M)x_k$. The changes in the $l+1$ radiative multipole moments, due to this displacement, are

$$\delta I_{L+1} = \left[-(l+1) I_L (\mu/M) x_{a_{l+1}}\right]^{\mathrm{STF}}, \tag{52a}$$

$$\delta J_{L+1} = \left[-\frac{(l+2)l}{l+1} J_L (\mu/M) x_{a_{l+1}}\right]^{\mathrm{STF}}. \tag{52b}$$

These can be derived from simply applying a coordinate displacement to the metric of Eqs. (11.1) of Ref. [15].

For example, the current quadrupole radiative moment J_{ij} picks up a contribution from the $J_j = S_1 \delta_{j3}$ moment of the large mass, and when added to the direct contribution from the orbiting object, it produces for the total radiative current quadrupole moment

$$J_{ij} = \left[\mu x_i \epsilon_{jkm} x_k \frac{dx_m}{dt} - \frac{3}{2}\frac{\mu}{M} x_i S_1 \delta_{j3}\right]^{\mathrm{STF}}. \tag{53}$$

This result for J_{ij} is given in Kidder, Will, and Wiseman [17].

Equation (53) inserted into (48) leads to a contribution to the luminosity of

$$-\left.\frac{dE}{dt}\right|_{J_{ij}} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10} \times\left[\frac{1}{36}v^2 - \frac{1}{12}\frac{S_1}{M^2}v^3 + \frac{1}{16}\frac{S_1^2}{M^4}v^4\right]. \tag{54}$$

The $S_1 v^3$ term is of the same magnitude as the leading-order S_1 term in Eq. (51). However, it is easy to verify that no other central-body moment, $M_2, S_3, M_4, S_5, \ldots$, contributes to dE/dt by this means, through Eqs. (48) and (52), at the same leading order as in Eq. (51). This is because the time derivatives in Eq. (48) each contribute a factor of Ω, enough factors that the contributions of the I_L ($l > 2$) and J_L moments end up being suppressed sufficiently that they do not appear in the luminosity at leading order.

Now, to finish computing the gravitational-wave luminosity $-dE/dt$, we want the leading order occurrence of each multipole moment, but in addition, to facilitate a discussion below of testing the no-hair theorem, we also want the entire series through order v^4. Equation (54) can be added to (51), since both are contributions to $-dE/dt$, and this gives us the first appearances of the multipole moments. But to get the series through order v^4, we also need to add in additional contributions to the luminosity: these terms, which do not involve any multipole moments except for M_0, are derived elsewhere (see, for example, Ref. [16], Eq. (3.13)). Adding all these terms up, we get

$$-\frac{dE}{dt} = \frac{32}{5}\left(\frac{\mu}{M}\right)^2 v^{10}\left[1 - \frac{1247}{336}v^2 + 4\pi|v|^3 - \frac{44711}{9072}v^4 - \frac{11}{4}\frac{S_1}{M^2}v^3 + \frac{1}{16}\frac{S_1^2}{M^4}v^4 - 2\frac{M_2}{M^3}v^4 + \sum_{l=4,6,\ldots}\frac{4(-1)^{l/2}\,(l+1)!!\,M_l\,v^{2l}}{3\,l!!\,M^{l+1}} - \sum_{l=3,5,\ldots}\frac{8(-1)^{(l-1)/2}\,l!!\,S_l\,v^{2l+1}}{3\,(l-1)!!\,M^{l+1}}\right]. \quad (55)$$

Above, the first line has the terms that were derived elsewhere. The second line shows the remaining terms that appear through v^4 order (the M_2 term is explicitly written out, rather than including it in the summation of the third line, which could have also been done). The third and fourth lines show the leading order occurrences of the higher ($l > 3$) moments. The $-\frac{11}{4}S_1M^{-2}v^3$ term from Eq. (55) is well known [17,18]. The $(\frac{1}{16}S_1^2M^{-4} - 2M_2M^{-3})v^4$ term agrees with previous work (Ref. [16], Eq. (3.13), the $\frac{33}{16}q^2v^4$ term) for the Kerr metric.

C. Computation of ΔN

Finally, we want to compute, from Eq. (4), the leading-order effects of the central-body multipole moments on ΔN. This computation requires, in addition to the leading-order effects on $-dE/dt = dE_{\rm wave}/dt$, also the leading-order effects of the moments on ΔE. By combining Eq. (26b) with (43a), as well as combining Eq. (26a) with (43b), and using Eq. (17) to get the contributions through v^4 order, we get

$$\frac{\Delta E}{\mu} = \frac{1}{3}v^2\left[1 - \frac{3}{2}v^2 - \frac{81}{8}v^4 - \sum_{l=2,4,\ldots}\frac{(-1)^{l/2}\,(4l-2)\,(l+1)!!\,M_l\,v^{2l}}{3\,l!!\,M^{l+1}} + \sum_{l=1,3,\ldots}\frac{(-1)^{(l-1)/2}\,(8l+12)\,l!!\,S_l\,v^{2l+1}}{3\,(l-1)!!\,M^{l+1}}\right]. \quad (56)$$

(There is no S_1^2 contribution at v^4 order.)

Combining Eqs. (4), (55), and (56), we get all the terms in ΔN through v^4 order, as well as the leading-order effects of the higher moments M_l, S_l ($l \geq 3$):

$$\Delta N = \frac{5}{96\pi}\left(\frac{M}{\mu}\right)v^{-5}\left[1 + \frac{743}{336}v^2 - 4\pi|v|^3 + \frac{113}{12}\frac{S_1}{M^2}v^3 + \left(\frac{3058673}{1016064} - \frac{1}{16}\frac{S_1^2}{M^4} + 5\frac{M_2}{M^3}\right)v^4 - \sum_{l=4,6,\ldots}\frac{(-1)^{l/2}\,(4l+2)\,(l+1)!!\,M_l\,v^{2l}}{3\,l!!\,M^{l+1}} + \sum_{l=3,5,\ldots}\frac{(-1)^{(l-1)/2}(8l+20)l!!\,S_l\,v^{2l+1}}{3\,(l-1)!!\,M^{l+1}}\right]. \quad (57)$$

Since each multipole moment makes its first appearance at a different order, then ΔN does contain full information of the multipole moments.

It should be stressed that Eqs. (55), (56), and (57) ignore many higher order terms—only the first appearance of each multipole moment is shown.

If ΔN can be measured and written as a series expansion in powers of $\Omega^{1/3}$, and the coefficients of the $\Omega^{-5/3}$, Ω^{-1}, $\Omega^{-2/3}$, and $\Omega^{-1/3}$ terms [i.e., the terms on the first two lines of Eq. (57)] can be determined, then from the four coefficient values, it would be possible to solve for the four unknowns: μ, M, S_1, and M_2. Then by checking to see whether $M_2 = -S_1^2/M$ or not, we could see whether the moments of the larger object correspond to those of a Kerr black hole satisfying the no-hair theorem or not [19]. In reality, as the orbiting object nears its last stable circular orbit, the v parameter in Eq. (57) becomes close to unity, so that many more terms in the series would need to be known for a high accuracy test of the no-hair theorem, as well as to look at higher order moments such as S_3.

There is still much work that is required even after a complete series for ΔN is developed. Some of our four idealizing assumptions made at the beginning of this paper need to be removed: We have considered only circular and equatorial orbits, but should generalize to all orbits. There is also the issue of how ΔN depends on the inner boundary conditions, that is, how it depends on whether or not the central body absorbs energy through a horizon, through tidal heating, etc. Traveling through an accretion disk would also change the orbiting object's energy and angular momentum, thereby affecting ΔN.

However, seeing that the information of the multipole moments is contained in many ways in the gravitational waves is encouragement that even after the problem is solved for the general case, we most likely will still have the ability to determine the central body's spacetime geometry from future gravitational-wave measurements.

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