# Quantum field theory in Lorentzian universes from nothing

John L. Friedman
*Department of Physics, University of Wisconsin, Milwaukee, Wisconsin 53201*

Atsushi Higuchi
*Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland*

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We examine quantum field theory in spacetimes that are time nonorientable but have no other causal pathology. These are Lorentzian universes from nothing, spacetimes with a single spacelike boundary that nevertheless have a smooth Lorentzian metric. A time-nonorientable, spacelike hypersurface serves as a generalized Cauchy surface, a surface on which freely specified initial data for wave equations have unique global time evolutions. A simple example is antipodally identified de Sitter space. Classically, such spacetimes are locally indistinguishable from their globally hyperbolic covering spaces. The construction of a quantum field theory is more problematic. Time nonorientability precludes the existence of a global algebra of observables, and hence of global states, regarded as positive linear functions on a global algebra. One can, however, define a family of local algebras on an atlas of globally hyperbolic subspacetimes, with overlap conditions on the intersections of neighborhoods. This family locally coincides with the family of algebras on a globally hyperbolic spacetime; and one can ask whether a sensible quantum field theory is obtained if one defines a state as an assignment of a positive linear function to every local algebra. We show, however, that the extension of a generic positive linear function from a single algebra to the collection of all local algebras violates positivity: one cannot find a collection of quantum states satisfying the physically appropriate overlap conditions. One can overcome this difficulty by artificially restricting the size of neighborhoods in a way that has no classical counterpart. Neighborhoods in the atlas must be small enough that the union of any pair is time orientable. Correlations between field operators at a pair of points are then defined only if a curve joining the points lies in a single neighborhood. Any state on one neighborhood of an atlas can be extended to a collection of states on the atlas, and the structure of local algebras and states is thus locally indistinguishable from quantum field theory on a globally hyperbolic spacetime. But the artificiality of the size restriction on neighborhoods means that the structure is not a satisfactory global field theory. The structure is not unique, because there is no unique maximal atlas. The resulting theory allows less information than quantum field theory in a globally hyperbolic spacetime, because there are always sets of points in the spacetime for which no correlation function is defined. Finally, in showing that one can extend a local state to a collection of states, we use an antipodally symmetric state on the covering space, a state that would not yield a sensible state on the spacetime if all correlations could be measured.

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## I. INTRODUCTION

In a Lorentzian path-integral approach to quantum gravity, one can, as in the Euclidean case, imagine constructing a wave function of the universe from a sum over all Lorentzian four-geometries with a single spacelike boundary. Spacetimes of this kind provide the only examples of topology change in which one can have a smooth, nondegenerate Lorentz metric without closed timelike curves; instead, the spacetimes are time nonorientable.

The simplest examples of such spacetimes have the topology of a finite timelike cylinder, $S^3 \times \mathbb{R}^+$, with diametrically opposite points of its past spherical boundary identified. This is the topology of antipodally identified de Sitter space. It is a four-dimensional analogue of the Möbius strip, which can be constructed from a finite two-dimensional timelike cylinder by identifying diametrically opposite points of its circular past boundary $\tilde{\Sigma}$ (see Fig. 1). A more familiar representation of the same strip is shown in Fig. 2, whose median circle $\Sigma$ was the one just constructed by identifying points of $\tilde{\Sigma}$. The orientable double-covering space of the strip is a cylinder $\tilde{M}$ of double the timelike length (Fig. 3), and $M$ is constructed from $\tilde{M}$ by identifying antipodal points. If the covering space has the metric of a flat, timelike cylinder, the Möbius strip $M$ will be time nonorientable with a locally flat metric and a timelike Killing vector (defined globally only up to sign) perpendicular to $\Sigma$. If the covering spacetime $\tilde{M}$ is given the two-dimensional de Sitter metric, the Möbius strip will acquire the metric of antipodally identified de Sitter space.

As recent authors have noted, the time nonorientability of these spacetimes prevents one from carrying through the standard construction of a Fock space or a Weyl algebra of observables [1–3]. Kay [2] requires existence of a globally defined * algebra and imposes what he terms the "$F$-locality condition," which demands in essence,

 

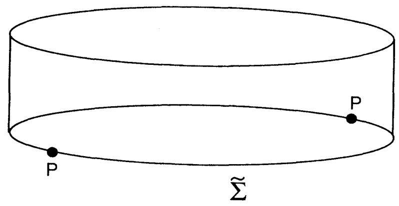


FIG. 1. Diametrically opposite points of the past boundary $\tilde{\Sigma}$ are identified to construct a smooth Lorentzian universe with no past boundary.

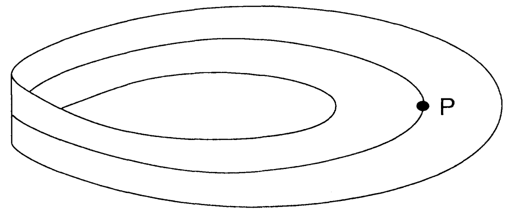


FIG. 2. For a two-dimensional cylinder, the resulting spacetime is a Möbius strip whose median circle $\Sigma$ is obtained by the identification of points in $\tilde{\Sigma}$.

that the * algebra satisfy the canonical commutation relations in a neighborhood of any point with respect to one time orientation. Under these conditions he proves that the spacetime must be time orientable. An independent study by Gibbons [1] concludes that one is forced to use a real (i.e., noncomplex) Hilbert space to describe quantum field theory in some time-nonorientable spacetimes including antipodally identified de Sitter space. This also suggests that one cannot construct a globally defined conventional quantum field theory in a time-nonorientable spacetime.

The conclusion is surprising, because an observer in an antipodally identified de Sitter space $M$ cannot classically distinguish the spacetime from de Sitter space $\tilde{M}$. The past of a timelike world line in $M$ (defined by a choice of orientation near the world line) is isometric to the past of either of the two corresponding world lines in $\tilde{M}$. In fact, the Cauchy problem is well defined on $M$ for fields with initial data on $\Sigma$,[1] and the solutions will be identical to those seen by an observer on $\tilde{\Sigma}$ who travels along the corresponding world line and sees the same data.

We are concerned in this paper with whether one can evade these global results by piecing together local quantum algebras and states. We find that one *can* evade Kay's no-go theorem if, instead of a globally defined * algebra one demands only a set of * algebras, each defined in a local neighborhood. Overlap conditions on the * algebras then ensure that the local algebraic structure coincides with that on a globally hyperbolic spacetime. One would like to use this structure of local algebras to define quantum states as collections of positive linear functions (PLF's) on the local algebras, again with consistency conditions on the intersection of globally hyperbolic neighborhoods. We find, however, that if one considers the algebras of observables on an atlas consisting of *all* globally hyperbolic subspacetimes that inherit their causal structure from the spacetime $M, g$, then one cannot consistently define states. In particular, if the union of a pair of neighborhoods is time nonorientable, one cannot consistently extend a generic PLF to the pair of neighborhoods without violating positivity.

One can define a collection of local states on *smaller* atlases, restricted so that the union of any two neighborhoods is time orientable. The collection of algebras and states is then locally indistinguishable from that on the globally hyperbolic covering space of any Lorentzian universe from nothing. In particular, any local state can be consistently extended to a collection of states on all algebras associated with the atlas. But the restriction on the size of neighborhoods amounts to a restriction on the size of regions over which one can define correlations between field operators, and this has unpleasant implications. The specification of a collection of states on the neighborhoods that cover and share an initial value surface does not *uniquely* determine a time evolution: The extension to a collection of states on the set of all algebras is not unique. In addition, in showing that one can extend a local state to a collection of states, we use an antipodally symmetric state on the covering space, and such a state would not yield a well-defined state on the spacetime if all correlations could be measured. Finally, the families of states and algebras depend on the choice of atlas, and there is no unique maximal atlas.

The ability to construct a family of states and algebras that agrees locally with that of a globally hyperbolic spacetime relies on the fact that the spacetimes we consider, although time nonorientable, have no closed timelike curves (CTC's). The simplest example above, the flat Möbius strip, has CTC's if one extends the strip to a timelike thickness greater than its circumference. [These are smooth timelike curves $c(\lambda)$ that intersect the same point twice; the tangent vectors at the point of intersection have opposite time orientation.] In the nonchronal region (the region with CTC's) nearby points that are spacelike separated with respect to the causal structure of a globally hyperbolic neighborhood are joined by timelike curves in the full spacetime. Thus events that are locally spacelike separated will influence one another, and one naturally expects that field operators at points whose local separation is spacelike will fail to commute or that there will be restrictions on algebraic states in the local neighborhoods. In Yurtsever's [4] generalization of the algebraic approach to quantum field theory to spacetimes with CTC's, massive scalar field theory will not in general have local algebras of observables that agree with

[1]This is because the lift to $\tilde{\Sigma}$ of data on $\Sigma$ will evolve to an antipodally invariant field on $\tilde{M}$. The field on $\tilde{M}$ will therefore be the lift of a solution to the field equation on $M$.

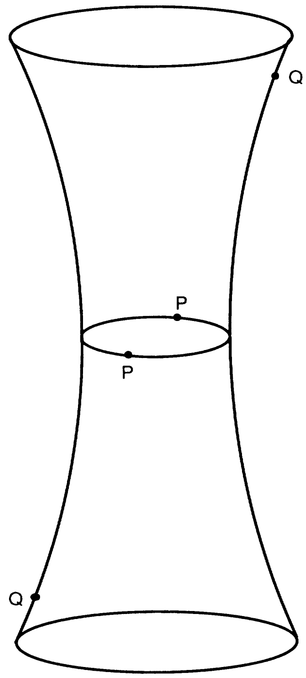


FIG. 3. The double cover $\tilde{M}$ is related to $M$ by the identification of antipodal points.

the ordinary local algebras of observables associated (by the usual construction) with sufficiently small globally hyperbolic neighborhoods of each spacetime point. On the other hand, in Kay's approach [2], one *requires* such an agreement ($F$-quantum compatibility). This requirement can be implemented both for massless and massive scalar field theories in some spacetimes with CTC's [2,5] though it is not clear how these works can be extended to more general spacetimes. Antipodally identified de Sitter space avoids CTC's by expanding rapidly enough that timelike curves that loop through $\Sigma$ cannot quite return to their starting point.

Lorentzian universes with no past boundary and no CTC's can be constructed in the way shown in Fig. 3 from any compact orientable three-manifold $\tilde{\Sigma}$ that admits an involution—a diffeomorphism $I$ that acts freely on $\tilde{\Sigma}$ and for which $I^2 = 1$. That is, if $T : \mathbb{R} \to \mathbb{R}$ is the map $t \mapsto -t$, the manifold is the quotient

$$M = (\mathbb{R} \times \tilde{\Sigma})/(T \times I). \tag{1.1}$$

Countably many three-manifolds admit free involutions, including all lens spaces and most other spherical spaces; each gives rise to a topologically distinct class of Lorentzian universes from nothing.

## II. QUANTUM FIELD THEORY WITHOUT A CHOICE OF TIME ORIENTATION

### A. Minkowski space

We begin with a Fock-space framework for concreteness and to make the subsequent algebraic treatment of oppositely oriented observers more transparent.

The quantum theory of a neutral scalar field on Minkowski space can be described in terms of the space $V$ of real solutions to the Klein-Gordon equation,

$$Kf := (-\nabla_a \nabla^a + m^2) f = 0\,, \tag{2.1}$$

which are finite in the norm,

$$\int_\Sigma dS\,(|n^a \nabla_a f|^2 + \kappa^2 f^2)\,, \tag{2.2}$$

where $\Sigma$ is $t$ =const surface, $n_a$ is a unit normal to $\Sigma$, and $\kappa^2$ is an arbitrary positive constant. One makes $V$ into a complex vector space by a choice of complex structure $J$, which in turn relies on choosing an orientation of time. If $\mathcal{O}$ is the orientation for which $n^a$ is future pointing, then

$$Jf = i(f^{(+)} - f^{(-)})\,, \tag{2.3}$$

where $f^{(+)}$ is the positive frequency part of $f$ with respect to $\mathcal{O}$. A reversal of time orientation reverses the assignment of positive and negative frequencies. By itself, however, the space of real solutions is independent of orientation.

Given a choice $\mathcal{O}$ of time orientation, one can define a symplectic product $\omega$ on $V$ by writing

$$\omega(f,g) = \int_\Sigma dS_a f \overleftrightarrow{\nabla}^a g\,, \tag{2.4}$$

where

$$dS_a = n_a dS \tag{2.5}$$

with $n^a$ the normal that is future directed with respect to $\mathcal{O}$. The corresponding inner product on $V$ has the form

$$\langle f|g\rangle = \tfrac{1}{2}\,\omega(f,Jg) + \frac{i}{2}\,\omega(f,g)\,. \tag{2.6}$$

The completion of $V$ in the inner product $\langle\ |\ \rangle$ with complex structure $J$ is the one-particle Hilbert space $\mathcal{H}$ of the free scalar field, and the corresponding Fock space is

$$\mathcal{F} = \mathbb{C} + \mathcal{H} + \mathcal{H} \otimes_s \mathcal{H} + \cdots\,. \tag{2.7}$$

An observer with time orientation $\check{\mathcal{O}}$ opposite to $\mathcal{O}$ will use normal

$$\check{n}_a = -n_a\,, \tag{2.8}$$

surface element

$$d\check{S}_a = \check{n}_a dS = -dS_a\,, \tag{2.9}$$

and symplectic structure

$$\check{\omega}(f,g) = \int_\Sigma d\check{S}_a f \overleftrightarrow{\nabla}^a g = -\omega(f,g)\,. \tag{2.10}$$

The complex structure on $V$ similarly changes sign, because the positive and negative frequency parts of $f \in V$ are interchanged:

$$\check{J} = -J\,. \tag{2.11}$$

Equation (2.6) then implies that oppositely oriented observers assign to the same pair of real solutions (and hence to the same one-particle state) complex conjugate inner products:

$$\begin{aligned}\langle \check{f}|\check{g}\rangle &= \frac{1}{2}\,\check{\omega}(f,\check{J}g) + \frac{i}{2}\,\check{\omega}(f,g)\\ &= \frac{1}{2}\,\omega(f,Jg) - \frac{i}{2}\,\omega(f,g)\\ &= \overline{\langle f|g\rangle}\,. \end{aligned}\qquad(2.12)$$

The map $|\rangle \mapsto |\check{\rangle}$ induces an antiunitary map $\mathcal{I}:\mathcal{F}\longrightarrow\check{\mathcal{F}}$, with

$$\mathcal{I}\alpha|f_1\rangle\otimes_s\cdots\otimes_s|f_k\rangle = \bar{\alpha}|f_1\check{\rangle}\otimes_s\cdots\otimes_s|f_k\check{\rangle}\,. \qquad(2.13)$$

A pure state $[\Psi]$ can be regarded as assigning to time orientations $\mathcal{O}$ and $\check{\mathcal{O}}$ vectors $\Psi\in\mathcal{F}$ and $\check{\Psi}\in\check{\mathcal{F}}$, with $\check{\Psi}=\mathcal{I}\check{\Psi}$; more generally, an (algebraic) state $[\rho]$ assigns states $\rho$ and $\check{\rho}$ to orientations $\mathcal{O}$ and $\check{\mathcal{O}}$.

The Heisenberg field operator $[\phi]$ similarly assigns to time orientations $\mathcal{O}$ and $\check{\mathcal{O}}$ operators $\hat{\phi}$ and $\check{\phi}$, acting on $\mathcal{F}$ and $\check{\mathcal{F}}$ respectively. For orientation $\mathcal{O}$, smeared field operators

$$\hat{\phi}(F) = \int \hat{\phi}(x)F(x)d^4x \qquad(2.14)$$

have commutation relations

$$[\hat{\phi}(E),\hat{\phi}(F)] = i\int E(x)(G_{\rm adv}-G_{\rm ret})(x,y)F(y)d^4x d^4y\,, \qquad(2.15)$$

where $G_{\rm adv}$ ($G_{\rm ret}$) is the advanced (retarded) Green function. An observer with opposite time orientation $\check{\mathcal{O}}$ will adopt the opposite sign for the commutator, because she will use the opposite definitions $\check{G}_{\rm adv}=G_{\rm ret}$ and $\check{G}_{\rm ret}=G_{\rm adv}$:

$$[\check{\phi}(E),\check{\phi}(F)] = i\int E(x)(\check{G}_{\rm adv}-\check{G}_{\rm ret})(x,y)F(y)d^4x d^4y\,. \qquad(2.16)$$

The structure of the algebra is clearer if one uses the fact that each smeared field operator can be written as the symplectic product of $\hat{\phi}$ with a real solution $f$ to the Klein-Gordon equation

$$\hat{\phi}(F) = \omega(\hat{\phi},f) = \int_\sigma dS\,[\hat{\phi}(x)n_a\nabla^a f(x) - \hat{\pi}(x)f(x)]\,, \qquad(2.17)$$

where

$$f(x) = \int (G_{\rm adv}-G_{\rm ret})(x,y)F(y)d^4x\,. \qquad(2.18)$$

The canonical commutation relations are simply

$$[\omega(\hat{\phi},f),\omega(\hat{\phi},g)] = i\omega(f,g)\,. \qquad(2.19)$$

Expectation values of elements in the algebra depend on orientation in the manner

$$\check{\langle}\check{\Psi}|\check{\phi}(F)\cdots\check{\pi}(G)|\check{\Psi}\check{\rangle} = \overline{\langle\Psi|\hat{\phi}(F)\cdots[-\hat{\pi}(G)]|\Psi\rangle}\,, \qquad(2.20)$$

or, for general algebraic state $[\rho]$,

$$\check{\rho}[\check{\phi}(F)\cdots\check{\pi}(G)] = \overline{\rho(\hat{\phi}(F)\cdots[-\hat{\pi}(G)])}\,. \qquad(2.21)$$

Note that Eq. (2.21) follows from the relation between $\rho$ and $\check{\rho}$ acting on a string of smeared $\check{\phi}$'s,

$$\check{\rho}[\check{\phi}(F)\cdots\check{\phi}(G)] = \overline{\rho[\hat{\phi}(F)\cdots\hat{\phi}(G)]}\,, \qquad(2.22)$$

because $\hat{\pi}(F)=-\hat{\phi}(\partial_t F)$ and $\check{\pi}(F)=-\check{\phi}(\partial_{\check{t}} F)$, where $\check{t}=-t$.

### B. Globally hyperbolic spacetimes

We now generalize the treatment of a scalar field to arbitrary globally hyperbolic spacetimes $M,g$. We use an algebraic approach for quantum fields (see Haag [6] for a review) developed for curved spacetimes by a number of earlier authors (Ashtekar and Magnon [7], Isham [8], Kay [9], Hájíček [10], Dimock [11], Fredenhagen and Haag [12], and Kay and Wald [13,14]).

Corresponding to orientations $\mathcal{O}$ and $\check{\mathcal{O}}$, one defines abstract algebras $\mathcal{A}$ and $\check{\mathcal{A}}$ as the free complex algebras generated by symbols of the form $\{\hat{\phi}(F), F\in C_0^\infty(M)\}$ and $\{\check{\phi}(F), F\in C_0^\infty(M)\}$, modulo the commutation relations:

$$\begin{aligned}[\hat{\phi}(E),\hat{\phi}(F)] &= i\int E(x)(G_{\rm adv}-G_{\rm ret})\\ &\qquad\times(x,y)F(y)d^4V_x d^4V_y,\\ [\check{\phi}(E),\check{\phi}(F)] &= i\int E(x)(\check{G}_{\rm adv}-\check{G}_{\rm ret})\\ &\qquad\times(x,y)F(y)d^4V_x d^4V_y. \end{aligned}\qquad(2.23)$$

To each globally hyperbolic subspacetime $U,g|_U$ of $M,g$ and each choice of orientation $\mathcal{O}$ on $U$ corresponds a local algebra $\mathcal{A}_{(U,\mathcal{O})}$ defined as above, with $M$ replaced by $U$. On each overlap, $U\bigcap U'$, there is a linear or antilinear isomorphism $\mathcal{I}$ between the restrictions of $\mathcal{A}_{(U,\mathcal{O})}$ and $\mathcal{A}_{(U',\mathcal{O}')}$ to $\phi$'s smeared with functions having support on $U\bigcap U'$. For $F\in C_0^\infty(U\cap U')$,

$$\begin{aligned}\mathcal{I}\hat{\phi}(F) &= \hat{\phi}'(F),\quad \mathcal{I}i = i \ \text{ if } \ \mathcal{O}=\mathcal{O}',\\ \mathcal{I}\hat{\phi}(F) &= \phi'(F),\quad \mathcal{I}i = -i \ \text{ if } \ \mathcal{O}\neq\mathcal{O}'\,. \end{aligned}\qquad(2.24)$$

In particular, for $U'=M$ with agreeing orientations, the map $\mathcal{I}$ embeds $\mathcal{A}_{(U,\mathcal{O})}$ in $\mathcal{A}_{(M,\mathcal{O})}$ as a subalgebra.

In the algebraic approach, with a fixed time orientation, a physical state is a positive linear function (PLF) $\rho$ on the algebra of observables. When the algebra is represented by a set of linear operators on a Hilbert space, a state $\rho$ is represented by a density matrix. Again, one can democratically define a state $[\rho]$ as an assignment of a PLF, $\rho$, or $\check{\rho}$, to each choice of orientation ($\mathcal{O}$ or $\check{\mathcal{O}}$), where $\check{\rho} \in \check{\mathcal{F}} \otimes \check{\mathcal{F}}^*$. Formally, a state $[\rho]$ is an equivalence class of pairs, $(\rho, \mathcal{O})$, satisfying

$$(\rho', \mathcal{O}') \equiv (\rho, \mathcal{O}) \Longleftrightarrow$$

$$\rho' = \rho \text{ if } \mathcal{O}' = \mathcal{O}, \text{ and } \rho' = \check{\rho} \text{ if } \mathcal{O}' \neq \check{\mathcal{O}}, \quad (2.25)$$

where, for an arbitrary string of operators,

$$\check{\rho}[\check{\phi}(F) \cdots \check{\phi}(G)] = \overline{\rho[\hat{\phi}(F) \cdots \hat{\phi}(G)]}, \quad (2.26)$$

implying

$$\check{\rho}[\check{\phi}(F) \cdots \check{\pi}(G)] = \overline{\rho(\hat{\phi}(F) \cdots [-\hat{\pi}(G)])}. \quad (2.27)$$

The restriction of a state $[\rho]$ to the pair of subalgebras associated with a globally hyperbolic neighborhood $U$ is a state $[\rho_U]$ on $U$. As in Eq. (2.21), states on overlapping neighborhoods $U$ and $U'$ are related by

$$\rho_{(U,\mathcal{O})}(A) = \rho_{(U',\mathcal{O}')}(A') \text{ if } \mathcal{O} = \mathcal{O}',$$

$$\rho_{(U,\mathcal{O})}(A) = \overline{\rho_{(U',\mathcal{O}')}(A')} = \rho_{(U',\mathcal{O}')}(A'^{\dagger}) \text{ if } \mathcal{O} \neq \mathcal{O}' \quad (2.28)$$

where $A' = \mathcal{I}A$, with $\mathcal{I}$ given by Eq. (2.24).

## III. QUANTUM FIELD THEORY ON TIME-NONORIENTABLE SPACETIMES

### A. Existence of a family of local algebras

As in the previous section, we will define local algebras of observables in a neighborhood of each point, show that the algebras on overlapping regions are isomorphic, and define global states as positive linear functions on each local algebra that respect the overlap isomorphisms. The construction uses the fact that there is a well-defined initial value problem on the spacetimes $M, g$ that we consider and that one can find an atlas of globally hyperbolic neighborhoods which inherit their causal structure from $M$.

*Definition.* A spacelike hypersurface $\Sigma$ of a spacetime $M, g$ is an *initial value surface* (for the Klein-Gordon equation) if, for any smooth choice of $\phi$ and its normal derivative on $\Sigma$, there is a unique $\phi$ on $M$ satisfying $K\phi = 0$.

We are concerned with time-nonorientable spacetimes $M, g$ which have no closed timelike curves and whose double-covering $\tilde{M}, \tilde{g}$ is globally hyperbolic; as noted in Sec. I, any hypersurface $\Sigma \subset M$ is an initial value surface if its lift to $M$ is a Cauchy surface.

A local algebra of observables can be defined on any neighborhood $U \subset M$ for which (i) $U, g|_U$, regarded as a spacetime, is globally hyperbolic, and (ii) $U$ is connected and *causally convex* [15].

An open set $U$ is causally convex if no causal curve in $M$ intersects $U$ in a disconnected set. If $U$ is not causally convex, then some points that are spacelike separated in the spacetime $U, g|_U$ are joined by a null or timelike curve in $M$, and the commutation relations for field operators cannot be deduced by the causal structure of $U$. A causally convex neighborhood inherits its causal structure from $M$.[2]

Let $C = \{U\}$ be an atlas for $M$, a collection of open sets that cover $M$, for which (i) and (ii) above hold for each set $U \in C$, and (iii) each $U \in C$ has a Cauchy surface that can be completed to an initial value surface of $M$.

One would like to define a collection of local algebras and states on all oriented subspacetimes $U, g|_U$, satisfying (i)–(iii), where algebras on overlapping neighborhoods are related by linear or antilinear isomorphisms $\mathcal{I}$ and states on overlapping neighborhoods are related by Eq. (2.28). Although a consistent definition of a collection of states will require an additional unwanted restriction on the size of neighborhoods, the structure of local algebras coincides with that of the globally hyperbolic spacetime.

Let $\mathcal{C} = \{(U, \mathcal{O}), (U, \check{\mathcal{O}}), \ldots\}$ be the collection of all pairs with $U \in C$ and $\mathcal{O}$ a choice of orientation for $U$. Given a neighborhood $U \in C$, we associate with orientations $\mathcal{O}$ and $\check{\mathcal{O}}$ algebras of observables, $\mathcal{A}_{(U,\check{\mathcal{O}})}$ and $\mathcal{A}_{(U,\mathcal{O})}$, using the fact that $U, g|_U$ is globally hyperbolic to construct the Green functions $G_{\text{ret}}$, $G_{\text{adv}}$, $\check{G}_{\text{ret}} = G_{\text{adv}}$, $\check{G}_{\text{adv}} = G_{\text{ret}}$.

*Definition.* The algebras $\mathcal{A}_{(U,\mathcal{O})}$ and $\mathcal{A}_{(U,\check{\mathcal{O}})}$ are the free complex algebras generated by $\{\hat{\phi}(F), F \in C_0^\infty(U)\}$ and $\{\check{\phi}(F), F \in C_0^\infty(U)\}$ modulo the canonical commutation relations (2.23).

The algebras are related by an antilinear isomorphism, $\mathcal{I} : \mathcal{A}_{(U,\mathcal{O})} \longrightarrow \mathcal{A}_{(U,\check{\mathcal{O}})}$, given by

$$\begin{aligned} \mathcal{I}\hat{\phi}(F) &= \check{\phi}(F), \\ \mathcal{I}i &= -i. \end{aligned} \quad (3.1)$$

Writing $\hat{\pi}(F) := -\hat{\phi}(\nabla_a(n^a F))$, with $n^a$ the future pointing normal with respect to orientation $\mathcal{O}$, we have

$$\mathcal{I}\hat{\pi}(F) = -\check{\pi}(F). \quad (3.2)$$

We thus have a collection of pairs $(\mathcal{A}_U, \mathcal{O})$, related on each overlap $U \bigcap U'$ by the linear or antilinear isomorphism given in Eq. (2.24). Thus one can consistently define a pair of oppositely oriented algebras for every globally hyperbolic neighborhood $U$ that inherits its causal structure from $M$. By allowing a pair of algebras at each point we evade Kay's "$F$-locality" condition [2].

[2]Although conditions (i) and (ii) above resemble what is called the local causality property [15], the latter is much more restrictive: the closure of a local causality neighborhood is required to lie in a geodesically convex normal neighborhood.

### B. Nonexistence of states on the family of all local algebras

Suppose now that one tries to define a state $\rho$ as an assignment of a PLF $\rho_{(U,\mathcal{O})}$ to the algebra $\mathcal{A}_{(U,\mathcal{O})}$ of each oriented neighborhood $(U,\mathcal{O})$, satisfying overlap conditions on intersections of neighborhoods. That is, in order that the state $\rho$ looks locally like a state on a globally hyperbolic spacetime, defined in Sec. II, it must obey the same conditions on intersections of neighborhoods: For any $A$ in the subalgebra generated by $\hat{\phi}(F) \in \mathcal{A}_U$ with $F \in C_0^\infty(U \bigcap V)$,

$$\rho_{(U,\mathcal{O})}(A) = \rho_{(U',\mathcal{O}')}(A') \quad \text{if} \quad \mathcal{O} = \mathcal{O}',$$
$$\rho_{(U,\mathcal{O})}(A) = \overline{\rho_{(U',\mathcal{O}')}(A')} = \rho_{(U',\mathcal{O}')}(A'^{\dagger}) \quad \text{if} \quad \mathcal{O} \neq \mathcal{O}', \tag{3.3}$$

where $A' = \mathcal{I}A$, with $\mathcal{I}$ given by Eq. (3.1). We show that one cannot extend a generic state $\rho_{(U,\mathcal{O})}$ to a collection of states on all neighborhoods satisfying (i)–(iii).[3]

The difficulty is associated with pairs of neighborhoods whose union is time nonorientable. Consider such a pair $(V,\mathcal{O}),(V',\mathcal{O}') \in \mathcal{C}$. Because $V \bigcup V'$ is time nonorientable, the intersection $V \bigcap V'$ includes disjoint regions $\hat{U}$ and $\check{U}$, such that the inherited time orientations agree on $\hat{U}$ and disagree on $\check{U}$. The restrictions of the states to $\hat{U} \sqcup \check{U}$ are required to yield pairs of physically equivalent states, seen by observers whose orientations agree on $\hat{U}$ and disagree on $U$ [see (3.8) and (3.9) below for the precise definition]. Without loss of generality, we may (by choosing open subsets of $\hat{U}$ and $\check{U}$, if necessary) assume that $\hat{U}$ and $\check{U}$ are globally hyperbolic and causally convex.

Each choice of orientation $\mathcal{O}$ and $\mathcal{O}'$ gives a well-defined quantum field theory on the *globally hyperbolic* spacetime $U := \hat{U} \sqcup \check{U}, g|_U$. The difficulty arises from the relation between the two field theories, the requirement that for each state $\rho$, there exists a physically equivalent state $\rho'$. That is, corresponding to each orientation of $U$ is an algebra, $\mathcal{A}$ or $\mathcal{A}'$, generated by commuting subalgebras on $\hat{U}$ and $\check{U}$; and for each PLF $\rho$ on the algebra $\mathcal{A}'$ there must be a PLF $\rho'$ on $\mathcal{A}'$ whose values on observable quantities agree with those of $\rho$ on the observables in $\mathcal{A}$ having the same physical meaning.

We assume that any symmetric element of $\mathcal{A}$ (or $\mathcal{A}'$) is an observable. Let $\hat{\mathcal{A}} := \mathcal{A}_{(\hat{U},\mathcal{O}|\hat{U})}$, $\hat{\mathcal{A}}' := \mathcal{A}'_{(\hat{U},\mathcal{O}'|\hat{U})}$, and denote by $\hat{\phi}(F)$ and $\hat{\phi}'(F)$ the smeared field operators that generate $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}'$. Then, because the orientations agree, physically equivalent observables are related by the isomorphism $\hat{\mathcal{I}}$ given by

$$\hat{\phi}'(F) = \hat{\mathcal{I}}\hat{\phi}(F), \qquad i = \hat{\mathcal{I}}i. \tag{3.4}$$

Similarly, denote by $\check{\phi}(F)$ and $\check{\phi}'(F)$ the smeared fields generating $\check{\mathcal{A}} := \mathcal{A}_{(\check{U},\mathcal{O}|\check{U})}$ and $\check{\mathcal{A}}' := \mathcal{A}_{(\check{U},\mathcal{O}'|\check{U})}$. Because the orientations $\mathcal{O}$ and $\mathcal{O}'$ disagree on $\check{U}$, physically equivalent observables in $\check{\mathcal{A}}$ and $\check{\mathcal{A}}'$ are related by the antilinear map $\check{\mathcal{I}}$, with

$$\check{\phi}'(F) = \check{\mathcal{I}}\check{\phi}(F), \qquad -i = \check{\mathcal{I}}i. \tag{3.5}$$

For any PLF $\rho$ on $\mathcal{A}$, the corresponding PLF $\rho'$ on $\mathcal{A}'$ must satisfy

$$\rho'(\hat{\mathcal{I}}\hat{A}) = \rho(\hat{A}), \qquad \rho'(\check{\mathcal{I}}\check{A}) = \rho(\check{A}), \tag{3.6}$$

for all symmetric $\hat{A} \in \hat{\mathcal{A}}$, $\check{A} \in \check{\mathcal{A}}$. Any element of $\hat{\mathcal{A}}$ (or of $\check{\mathcal{A}}$) can be written as a linear combination, $\hat{A} = \hat{A}_1 + i\hat{A}_2$, of symmetric elements, $A_1 = \frac{1}{2}(\hat{A}+\hat{A}^\dagger)$, $A_2 = \frac{1}{2i}(\hat{A}-\hat{A}^\dagger)$. Linearity of $\rho$ and $\rho'$ and Eq. (3.6) then imply

$$\rho'(\hat{\mathcal{I}}\hat{A}) = \rho(A), \forall \hat{A} \in \hat{\mathcal{A}},$$
$$\rho'[(\check{\mathcal{I}}\check{A})^\dagger] = \rho(A), \forall \check{A} \in \check{\mathcal{A}}. \tag{3.7}$$

Now the product $\hat{A}\check{A}$ of two symmetric elements $\hat{A} \in \hat{\mathcal{A}}$, $\check{A} \in \check{\mathcal{A}}$ is again symmetric and corresponds to the observable $\hat{\mathcal{I}}\hat{A}\ \check{\mathcal{I}}\check{A} \in \mathcal{A}'$. Thus, we require

$$\rho'(\hat{\mathcal{I}}\hat{A}\ \check{\mathcal{I}}\check{A}) = \rho(\hat{A}\check{A}). \tag{3.8}$$

This final requirement uniquely determines $\rho'$, because any element of $\hat{\mathcal{A}}$ can be written as a linear combination of terms of the form $\hat{A}\check{A}$ with $\hat{A}$ and $\check{A}$ symmetric. The resulting $\rho'$ satisfies

$$\rho'[\hat{\mathcal{I}}\hat{A}\ (\check{\mathcal{I}}\check{A})^\dagger] = \rho(\hat{A}\check{A}), \forall \hat{A} \in \hat{\mathcal{A}},\ \check{A} \in \check{\mathcal{A}}. \tag{3.9}$$

[This equation makes sense, because the factorization is unique up to a complex number $c$—one can represent the same algebra element as the product $(c\hat{A})(c^{-1}\check{A})$—and the maps $\hat{A} \mapsto \hat{\mathcal{I}}\hat{A}$ and $\check{A} \mapsto (\check{\mathcal{I}}\check{A})^\dagger$ are both linear.]

We claim that the linear function $\rho'$ defined by Eqs. (3.7) and (3.8) is not in general positive. Let $\hat{A} \in \hat{\mathcal{A}}$, $B \in \check{\mathcal{A}}$, and a state $\rho_0$ satisfy

$$[\hat{A}, \hat{A}^\dagger] = 1, \quad [\check{B}, \check{B}^\dagger] = 1\,, \tag{3.10}$$

and

$$\rho_0(\hat{A}^\dagger\hat{A}) = \rho_0(\check{B}^\dagger\check{B}) = 0\,. \tag{3.11}$$

Using the Schwarz inequality, $|\rho_0(X\hat{A})|^2 \leq \rho_0(XX^\dagger)\rho_0(\hat{A}^\dagger\hat{A})$, for any operator $X$, one finds $\rho_0(X\hat{A}) = 0$, and similarly, $\rho_0(X\check{B}) = 0$. For the special case where $\hat{A} = \hat{\phi}(F)$ and $\check{B} = \check{\phi}(G)$ for some (complex) functions $F$ and $G$ with support in $\hat{U}$ and $\check{U}$, respectively, and where $\rho_0 = |\psi\rangle\langle\psi|$ for a state $|\psi\rangle$ in a Hilbert space, assumption (3.11) implies $\hat{A}|\psi\rangle = \check{B}|\psi\rangle = 0$.

Now define a PLF $\rho_1$ by

$$\rho_1(X) := \frac{1}{1+c^2}\rho_0[(1+c\hat{A}\check{B})X(1+c\hat{A}^\dagger\check{B}^\dagger)]\,, \tag{3.12}$$

where $c > 0$. We will show that the corresponding state $\rho_1'$ satisfying (3.9) is not positive. Consider the positive operator $\mathcal{I}O \in \mathcal{A}'$ defined by

$$\mathcal{I}O := [(\hat{\mathcal{I}}\hat{A})^\dagger - (\check{\mathcal{I}}\check{B})^\dagger](\hat{\mathcal{I}}\hat{A} - \check{\mathcal{I}}\check{B})\,. \tag{3.13}$$

[3]We will need to generalize (3.3) for this.

According to (3.9), we have

$$\rho_1'(\mathcal{I}\mathcal{O}) = \rho_1(\hat{A}^\dagger \hat{A}) + \rho_1(B^\dagger \check{B}) - 2\mathrm{Re}[\rho_1(\hat{A}\check{B})] = \frac{2c(c-1)}{1+c^2}\,. \quad (3.14)$$

Thus, $\rho_1'$ is nonpositive if $c < 1$.

The state $\rho_0$ satisfying (3.11) is not likely to be physically realistic, considering the fact that no annihilation operator for the vacuum state in Minkowski space can be localized. However, by choosing $\hat{A}$ and $\check{B}$ to be approximate annihilation operators for high-frequency modes, one should be able to satisfy condition (3.11) approximately for a physically realistic state. Moreover, one can show a similar nonpositivity with a weaker and physically realistic condition

$$\rho[(\hat{A}^\dagger \hat{A})^3],\ \rho[(B^\dagger \check{B})^3] < \frac{1}{\sqrt{24}}\,, \quad (3.15)$$

as demonstrated in the Appendix.

In fact it is likely that a linear function in a large globally hyperbolic neighborhood $U_L$ which contains an initial surface $\Sigma$ except for a measure-zero boundary cannot be positive under the assumption of reasonable short-distance behavior. The argument is as follows. Consider a small neighborhood $U_S$ which contains part of the above-mentioned boundary. The set $U_S \bigcap U_L$ can be approximated near the measure-zero boundary by the left and right Rindler wedges in Minkowski space. Now, any physically realistic state on $\mathcal{A}_{(U_S,\mathcal{O}|U_S)}$ should behave like the Minkowski vacuum for high-frequency modes. The Minkowski vacuum can be expressed as a linear combination of products of left and right Rindler states [16]. On the other hand, in $U_L$ the approximate left and right Rindler wedges have opposite time directions. Then, a construction similar to that given above, with the left and right Rindler wedges playing the role of $\hat{U}$ and $\check{U}$, is likely to show that there is a nonpositive linear function on $\mathcal{A}_{(U_L,\mathcal{O}|U_L)}$ whose restriction to $U_S \bigcap U_L$ corresponds to a PLF on $\mathcal{A}_{(U_S,\mathcal{O}|U_S)}$.

### C. Collections of local states associated with restricted atlases

We have argued that one cannot consistently define a collection of local states $\rho_{(U,\mathcal{O})}$ on an atlas that includes pairs of neighborhoods whose union is time nonorientable. We now consider restricted atlases of neighborhoods satisfying (i)–(iii) together with the following additional condition: (iv) The closure of $U \bigcup V$ is time orientable for each $U, V \in C$. One maximizes the information available in such a collection of states by considering a maximal collection $C$ of neighborhoods (i)–(iv) and covering $M$. Again we denote by $\mathcal{C}$ the corresponding collection of all oriented neighborhoods, $\mathcal{C} = \{(U,\mathcal{O})|U \in C\}$.

Condition (iv) is necessary to extend a local state to a collection of states on neighborhoods of the atlas $\mathcal{C}$. We now show that it is sufficient. We first consider the problem in the globally hyperbolic context and then return to our time-nonorientable spacetimes.

In a globally hyperbolic spacetime, one is free to choose a state $\rho_U$ on the algebra of any globally hyperbolic subspacetime $U, g|_U$ that satisfies conditions (i)–(iii). One can then construct a global state of which $\rho_U$ is the restriction to $U$. Elements of the * algebra $\mathcal{A}_{(U,\mathcal{O}_U)}$ are linear combinations of products of Klein-Gordon inner products $\omega_U(\hat{\phi}, f)$ that involve only data for $f$ on a Cauchy surface $\Sigma_U$. A state is specified by the expectation values, $\rho_U[\omega_U(\hat{\phi}, f)\cdots\omega_U(\hat{\phi}, g)]$, or, equivalently, by $n$-point distributions, $\rho[\phi(x)\cdots\phi(y)]$, $\rho[\phi(x)\cdots\phi(y)\pi(z)], \ldots, \rho[\pi(x)\cdots\pi(y)]$, where $x, y, \ldots, z \in \Sigma_U$. A state $\rho_U$ can be extended to a state on the larger spacetime by adjoining values of $\rho[\omega(\hat{\phi}, f)\cdots\omega_U(\hat{\phi}, g)]$, where at least one of $f, \ldots, g$ has support on the part of the full Cauchy surface $\Sigma$ that lies outside of $U$.

The simplest way to extend $\rho_U$ to a global state is as follows. First, one specifies a state in the interior of the domain of dependence of $\Sigma\backslash\Sigma_U$. Call this state $\rho_{\bar{U}}$. Then, we define the global state $\rho := \rho_U \otimes \rho_{\bar{U}}$. That is

$$\begin{aligned}\rho&\left[\omega(\hat{\phi}, f_1)\cdots\omega(\hat{\phi}, f_n)\omega(\hat{\phi}, g_1)\cdots\omega(\hat{\phi}, g_m)\right]\\ :&= \rho_U\left[\omega_U(\hat{\phi}, f_1)\cdots\omega_U(\hat{\phi}, f_n)\right]\\ &\quad\times\rho_{\bar{U}}\left[\omega_{\bar{U}}(\hat{\phi}, g_1)\cdots\omega_{\bar{U}}(\hat{\phi}, g_m)\right]\,,\end{aligned} \quad (3.16)$$

where $f_1, \ldots, f_n$ involve only data on $\Sigma_U$, whereas $g_1, \ldots, g_m$ involve only data on $\Sigma\backslash\Sigma_U$. The function $\rho$ is positive if $\rho_U$ and $\rho_{\bar{U}}$ are.

The global state $\rho$ is rather unphysical in the sense that there is no correlation between the field operators on $\Sigma_U$ and those on $\Sigma\backslash\Sigma_U$. It is also likely that the (renormalized) stress-energy tensor will be singular on the light cone of the boundary of $\Sigma_U$, since the state $\rho$ is analogous to the direct product of the left and right Rindler vacua in Minkowski space, which is known to possess such singularities [17]. It will be interesting to establish "extendibility" of states under some physical requirements, such as the absence of singularity in the stress-energy tensor. However, not much is known about these issues, as far as we are aware. We suspect, but have not verified, that if one restricts consideration to Hadamard states, then one can extend such a state on $U$ with suitable behavior at $\partial U$ to a Hadamard state on the full spacetime.

Now, consider a Lorentzian universe from nothing, a time-nonorientable spacetime $M, g$ of the form (1.1),

$$M = (\mathbb{R} \times \tilde{\Sigma})/Q\,, \quad (3.17)$$

with

$$Q = T \times I\,, \quad (3.18)$$

where $T(t) = -t$, for $t \in \mathbb{R}$, and $I$ is an involution of $\tilde{\Sigma}$ with no fixed points. We will denote the orientable double cover by $p : \tilde{M} \to M$, where $\tilde{M} = \mathbb{R} \times \tilde{\Sigma}$. We will write $\Sigma = \tilde{\Sigma}/I$, and, for simplicity, we will denote by $\tilde{\Sigma}$ and $\Sigma$ the particular copies $\tilde{\Sigma} \times \{0\}$ and $p(\tilde{\Sigma} \times \{0\})$ of these three-manifolds in $\tilde{M}$ and $M$. The metric $g$ is chosen to make $\tilde{M}$, $\tilde{g} = p_* g$ globally hyperbolic, with Cauchy surface $\tilde{\Sigma}$. Then $\Sigma$ is an initial value surface of $M, g$.

Let $\mathcal{C}$ be an atlas for $M$ satisfying conditions (i)–(iv). On any oriented neighborhood $(U, \mathcal{O})$ in the atlas, one can freely specify a state $\rho_{(U,\mathcal{O})}$. We are to extend the state to a collection of states on the algebras associated with $\mathcal{C}$, satisfying Eq. (3.3). To do this, we first lift the local state to the globally hyperbolic covering spacetime $\tilde{M}, \tilde{g}$, extend that lifted state to an antipodally symmetric state on $\tilde{M}$, and then use the antipodally symmetric state on $\tilde{M}$ to provide a collection of states on the atlas $\mathcal{C}$.

Given an orientation $\tilde{\mathcal{O}}_{\tilde{M}}$ on $\tilde{M}$, there is a 1-1 correspondence between oriented neighborhoods $(U, \mathcal{O}_U)$ and neighborhoods $\tilde{U}$ of $\tilde{M}$. [Because the two lifts of $(U, \mathcal{O}_U)$ to $\tilde{M}$ have opposite time orientation, there is a unique lift of $(U, \mathcal{O})$ to an oriented neighborhood $(\tilde{U}, \tilde{\mathcal{O}}_U)$ for which the orientations induced by $\mathcal{O}_U$ and $\tilde{\mathcal{O}}_{\tilde{M}}$ agree.] The atlas $\mathcal{C}$ on $M$ is thus mapped to an atlas $\tilde{\mathcal{C}}$ of oriented neighborhoods that cover $\tilde{M}$, all with orientation $\tilde{\mathcal{O}}_{\tilde{M}}$.

The isomorphisms $p|_{\tilde{U}} : \tilde{U} \to U$ induce algebra isomorphisms $\tilde{\mathcal{A}}_{(\tilde{U},\tilde{\mathcal{O}})} \to \mathcal{A}_{(U,\mathcal{O})}$, given by

$$\phi(F) \to \tilde{\phi}(F \circ p|_{\tilde{U}}) . \tag{3.19}$$

Thus the family of algebras (including both orientations) associated with the atlas $\mathcal{C}$ of $M$ is identical to the family of algebras (all with the same orientation) associated with the oriented atlas $\tilde{\mathcal{C}}$ of $\tilde{M}$. Because $\tilde{M}$ is globally hyperbolic, one can regard all local algebras as subalgebras of a global algebra $\tilde{\mathcal{A}}_{\tilde{M}}$ associated with the orientation $\tilde{\mathcal{O}}_M$.

The family of algebras on $M$, however, has an additional structure that plays no role in quantum field theory on the covering space itself, namely, the collection of antilinear isomorphisms between oppositely oriented neighborhoods $(U, \mathcal{O})$ and $(U, \check{\mathcal{O}})$. On $\tilde{M}$, these can be regarded as antilinear isomorphisms between the algebras associated with antipodally related neighborhoods, or, equivalently, as an antilinear isomorphism $\mathcal{Q}$ of the global algebra $\tilde{\mathcal{A}}_{\tilde{M}}$ given by

$$\mathcal{Q}\tilde{\phi}(F) = \tilde{\phi}(F \circ Q), \qquad \mathcal{Q}i = -i . \tag{3.20}$$

A collection of states associated with an atlas $\mathcal{C}$ of $M, g$ that satisfies the overlap conditions (3.3) can then be regarded as a collection of states associated with the atlas $\tilde{\mathcal{C}}$ on $\tilde{M}$; it must satisfy the usual overlap condition for a collection of states on the oriented atlas $\tilde{\mathcal{C}}$ and the additional condition

$$\rho_{\tilde{U}}(\tilde{A}) = \rho_{Q(\tilde{U})}(\mathcal{Q}\tilde{A}) . \tag{3.21}$$

If the collection of states corresponds to a single state on $\tilde{\mathcal{A}}_{\tilde{M}}$, the additional condition is simply the statement that it is *antipodally symmetric*.

Given a PLF $\rho_U$ on the algebra $\mathcal{A}_{(U,\mathcal{O}_U)}$ associated with an oriented neighborhood $(U, \mathcal{O}_U) \in \mathcal{C}$, one can extend it as follows to a collection of states on all algebras associated with $\mathcal{C}$. Again denote by $\Sigma$ an initial value surface of $M$ shared by $U$—for which $\Sigma_U$ is a Cauchy surface for $U, g|_U$—and denote by $\tilde{\Sigma}$ the corresponding Cauchy surface of $\tilde{M}$. Then $\rho_U$ can be regarded as a PLF $\rho_{\tilde{U}}$ on $\tilde{\mathcal{A}}_{\tilde{U}}$; and $\rho_{Q(\tilde{U})}$ given by Eq. (3.21) can be regarded as a PLF on the disjoint neighborhood $Q(\tilde{U})$. We can now use the construction given above for globally hyperbolic spacetimes to extend it to a PLF $\tilde{\rho}_0$ on $\tilde{\mathcal{A}}_{\tilde{M}}$. The PLF $\tilde{\rho}_0$ will not in general be antipodally symmetric, but we can obtain a PLF that is both antipodally symmetric and positive by writing

$$\tilde{\rho} = \tfrac{1}{2}(\tilde{\rho}_0 + \tilde{\rho}_0 \circ \mathcal{Q}) . \tag{3.22}$$

Then the restrictions of $\tilde{\rho}$ to the subalgebras $\hat{\mathcal{A}}_{\tilde{U}}$ (and the identification of the subalgebras with algebras associated with $\mathcal{C}$) yield a collection of states $\rho_{(U,\mathcal{O})}$ satisfying the overlap conditions (3.3) as required.

What are the implications of condition (iv), restricting possible atlases to neighborhoods small enough that no two neighborhoods contain an orientation-reversing curve? If our Universe has the topology of antipodally identified de Sitter space and a volume larger than the currently visible universe, one can choose an atlas that includes open sets with spatial extent as large as the visible universe. This is enough to allow one mechanically to replicate the observable part of quantum field theory with a collection of states and algebras associated with an atlas restricted by condition (iv).

From a more fundamental point of view, however, the theory is not satisfactory. Let us reiterate, in hindsight, the objections mentioned earlier. Because the correlations that are allowed depend on the atlas, one obtains a different theory for every choice of atlas. There is no unique way to pick a largest atlas satisfying conditions (i)–(iv), and thus no unique theory. The missing correlations mean that the information contained in a collection of states associated with neighborhoods that cover an initial value surface of $M$ is incomplete.[4] One is not enti-

[4]One can extend a collection of states given on an atlas $\mathcal{C}$ in more than one way to an antipodally symmetric state on $\tilde{M}$, because there are sets of points (and small neighborhoods about them) among which no correlations are defined. Here is an example of two antipodally symmetric algebraic states on de Sitter space that are extensions of the same collection of states on an atlas $\mathcal{C}$. Let $f$ and $g$ be complex solutions to the scalar wave equation on $M$ (antipodally identified de Sitter), whose initial data on an initial value surface $\Sigma$ have support on disjoint neighborhoods $U$ and $V$ of $\Sigma$ that do not both belong to a single neighborhood in $\mathcal{C}$. Then no correlations between points of $U$ and $V$ are defined by the collection of states on $M$. Let also $i\omega(\bar{f}, f)$, $i\omega(\bar{g}, g) > 0$ with respect to a chosen time direction. Then let $\tilde{f}$ and $\tilde{g}$ be the (normalized) antipodally symmetric lifts of $f$ and $g$ to $\tilde{M}$. (Thus, $\tilde{f}$ and $\tilde{g}$ are mapped to their complex conjugates under the antipodal map.) Pick as a Hilbert space $\mathcal{H}$ on $\tilde{M}$ a Fock space associated with a state $|0\rangle$ that is annihilated by the annihilation operators $\tilde{\omega}(\tilde{\phi}, \tilde{f})$ and $\tilde{\omega}(\tilde{\phi}, \tilde{g})$ corresponding to $f$ and $g$. Define an antipodally symmetric state by $|\psi\rangle = \tilde{\omega}(\tilde{\phi}, \tilde{f})^\dagger \tilde{\omega}(\tilde{\phi}, \tilde{g})^\dagger |0\rangle$. Let $\rho = 1/2(|0\rangle\langle 0| + |\psi\rangle\langle\psi|)$. Then, for small $e$, $\rho' = 1/2(|0\rangle\langle 0| + |\psi\rangle\langle\psi|) + e(|0\rangle\langle\psi| + |\psi\rangle\langle 0|)$ is positive, and $\rho$ and $\rho'$ give the same family of states on $M$.

tled to regard a state as an assignment of local states to a collection of local algebras restricted by condition (iv), because a "state" so defined has no unique time evolution. Finally, in order to extend a local state to a family of states on an atlas $\mathcal{C}$, we used an antipodally symmetric state on $\tilde{M}$. This suggests that by artificially restricting the collection of neighborhoods, we are simply providing a way to interpret an antipodally symmetric state: one can make such a state consistent if one chooses neighborhoods small enough. But if one includes all globally hyperbolic neighborhoods that inherit their causal structure from the spacetime $M, g$, an antipodally symmetric state on the covering space does not yield a consistent collection of local states on $M, g$.

A "theory" in which the correlations that one can measure are limited and depend on the choice of atlas might be more acceptable if one regards, say, a path-integral formulation as fundamental and relegates the usual quantum field theory to a subsidiary position. In a path-integral interpretation where the measuring instrument is included in the system, the measurements that can be made depend on the state of the instrument. Different states of the instrument will pick out different observables. Measuring a correlation between field operators at two spacetime points plausibly enforces a choice of time orientation at each of the two points in such a theory. This suggests that each state might carry with it an implicit atlas (or partial atlas) of oriented neighborhoods, covering at least regions of spacetime where measurements are effectively made. But this is a much weaker structure than the one we have considered, and it suggests that, if there is to be a sensible quantum field theory on time-nonorientable spacetimes, condition (3.9), which presupposes a physical meaning of correlations irrespective of time orientations, may be too strong.

## IV. A NOTE ON FERMION FIELDS

It is common in the general-relativity literature to regard two-component spinors as fields built from the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$ [18,19,21,20]. Chiral fermions, however, are really acted on by a larger group that includes time reversal. That is, a Weyl spinor belongs to an action of a covering group of that subgroup $L_+$ of the full Lorentz group comprising the component of the identity and the component of time-reversing, space-preserving Lorentz transformations. Readers of earlier work [21–23] may have been left with the misimpression that one cannot define two-component spinors on time-nonorientable spacetimes, and the present section summarizing work from [24] (see also the sequel [25] by Chamblin and Gibbons) is intended as a clarification.

The group $L_+$ has two double covers, $\mathrm{Sin}^+$ and $\mathrm{Sin}^-$, depending on the sign of $\mathcal{T}^2$, where $\mathcal{T}$ is either of the two elements of the covering group that correspond to the choice $T$ of time reversal. Only $\mathrm{Sin}^-$ acts on the usual two-component spinors associated with Weyl neutrinos in Minkowski space. In an orientable spacetime, the difference between an $\mathrm{SL}(2,\mathbb{C})$-spinor structure and a $\mathrm{Sin}^-$-spinor structure is unimportant, unless one wishes explicitly to discuss time reversal. In a time-nonorientable spacetime, however, the difference is essential. Because one cannot pick a bundle of time-oriented frames, time-nonorientable spacetimes have no $\mathrm{SL}(2,\mathbb{C})$-spinor structure, and two-component spinors rely for their definition on an action of the covering group $\mathrm{Sin}^-$ of $L_+$. Other authors have considered generalized spinor structures on generic-nonorientable spacetimes [26–30]. For these generic spacetimes, the situation is somewhat different, because one must consider actions of the full Lorentz group; and the usual action of parity requires four-component spinors.

Every Lorentzian universe from nothing, every spacetime of the form (1.1), has a $\mathrm{Sin}^+$-spinor structure, but only a subclass has a $\mathrm{Sin}^-$-spinor structure. Inequivalent $\mathrm{Sin}^+$- and $\mathrm{Sin}^-$-spinor structures correspond to members of two classes of homomorphisms from $\pi_1(\tilde{M})$ to $\mathbb{Z}_2$, where $\tilde{M}$ is the orientable double cover of the spacetime manifold $M$.

A precise statement is as follows.

*Proposition.* Let $M, g$ be a spacetime of the form (1.1) and let $\tilde{M}$ be its orientable double cover. Then the inequivalent $\mathrm{Sin}^+$-spinor structures ($\mathrm{Sin}^-$-spinor structures) are in 1-1 correspondence with homomorphisms $h \in H^1(\tilde{M})$ that respect the antipodal map $A$ and for which $h[\widetilde{c^2}] = +1$ ($h[\widetilde{c^2}] = -1$), for every time-reversing curve $c$ in $M$. In particular, every such spacetime has a $\mathrm{Sin}^+$-spinor structure.

Here $\widetilde{c^2}$ is a lift of $c^2$ to $\tilde{M}$. A field of two-component spinors is then a cross section of a bundle associated to a $\mathrm{Sin}^-$-spinor structure. Lorentzian universes from nothing for which $\Sigma$ is a three-torus have both $\mathrm{Sin}^+$- and $\mathrm{Sin}^-$-spinor structures, while antipodally identified de Sitter space has only a $\mathrm{Sin}^+$-spinor structure and so does not admit global fields of the usual kind of chiral fermions. Even in a time-nonorientable spacetime that allows the usual two-component spinors, however, one cannot construct a global Lagrangian density that violates time-reversal invariance.

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## APPENDIX: NONPOSITIVITY OF $\rho_1'$ WITH CONDITION (3.15)

In this appendix we prove that $\rho_1'$ defined in Sec. III, with a slight technical modification, is nonpositive under condition (3.15). Let us first prove some general inequalities. By noting

$$\rho_0[(\hat{A}^\dagger \hat{A})^2] = \rho_0[\hat{A}^\dagger \hat{A}] + \rho_0[(\hat{A}^\dagger)^2(\hat{A})^2] \qquad \text{(A1)}$$

and

$$\rho_0[(\hat{A}^\dagger\hat{A})^3] = \rho_0[(\hat{A}^\dagger\hat{A})^2] + \rho_0[(\hat{A}^\dagger)^2\hat{A}^2] + \rho_0[(\hat{A}^\dagger)^2\hat{A}\hat{A}^\dagger\hat{A}^2], \quad \text{(A2)}$$

we obtain

$$\rho_0[\hat{A}^\dagger\hat{A}] \le \rho_0[(\hat{A}^\dagger\hat{A})^2] \le \rho_0[(\hat{A}^\dagger\hat{A})^3] < \epsilon, \quad \text{(A3)}$$

where $0 < \epsilon < 1/\sqrt{24}$. Using these inequalities, we find

$$\rho_0[(\hat{A}\hat{A}^\dagger)^3] = \rho_0[(\hat{A}^\dagger\hat{A}+1)^3] \le 1 + 7\epsilon. \quad \text{(A4)}$$

Then, using the Schwarz inequality, we have

$$|\rho_0[\hat{A}^\dagger\hat{A}\check{B}^\dagger\check{B}]|^2 \le \rho_0[(\hat{A}^\dagger\hat{A})^2]\rho_0[(B^\dagger\check{B})^2]. \quad \text{(A5)}$$

Hence

$$\rho_0[\hat{A}^\dagger\hat{A}B^\dagger\check{B}] \le \epsilon. \quad \text{(A6)}$$

Next we will prove

$$\rho_0[(\hat{A}\hat{A}^\dagger)^2\check{B}\check{B}^\dagger] + \rho_0[\hat{A}\hat{A}^\dagger(\check{B}\check{B}^\dagger)^2] \le 2(1+7\epsilon). \quad \text{(A7)}$$

Define

$$s_1 := \rho_0[(\hat{A}\hat{A}^\dagger)^2\check{B}\check{B}^\dagger] \quad \text{(A8)}$$

and

$$s_2 := \rho_0[\hat{A}\hat{A}^\dagger(\check{B}\check{B}^\dagger)^2]. \quad \text{(A9)}$$

By using the Schwarz inequality $|\rho_0(X^\dagger Y)|^2 \le \rho_0(X^\dagger X)\rho_0(Y^\dagger Y)$ with $X = \hat{A}^\dagger\hat{A}\hat{A}^\dagger$ and $Y = \hat{A}^\dagger\check{B}\check{B}^\dagger$, we find $s_1^2 \le s_0 s_2$, where $s_0 = 1 + 7\epsilon$. In a similar manner we find $s_2^2 \le s_0 s_1$. Then, from these two inequalities we have $s_1 s_2 \le s_0^2$. Hence

$$(s_1 + s_2)^2 \le s_0(s_1+s_2) + 2s_0^2. \quad \text{(A10)}$$

From this we immediately obtain (A7), i.e., $s_1+s_2-2s_0 \le 0$.

Now, given a PLF $\rho_0$ satisfying (3.15), we define a new PLF $\rho_1$ by

$$\rho_1(X) := k\rho_0[(1 + ce^{i\alpha}\hat{A}\check{B})X(1 + ce^{-i\alpha}\hat{A}^\dagger\check{B}^\dagger)], \quad \text{(A11)}$$

where $k$ is the normalization factor given by

$$\begin{aligned} k^{-1} &= \rho_0[(1 + ce^{i\alpha}\hat{A}\check{B})(1 + ce^{-i\alpha}\hat{A}^\dagger\check{B}^\dagger)] \\ &= \rho_0[(1 + ce^{-i\alpha}\hat{A}^\dagger\check{B}^\dagger)(1 + ce^{i\alpha}\hat{A}\check{B})] \\ &\quad + c^2\rho_0([\hat{A}\check{B}, \hat{A}^\dagger\check{B}^\dagger]). \end{aligned} \quad \text{(A12)}$$

The last term equals $c^2(\hat{A}^\dagger\hat{A} + \check{B}^\dagger\check{B} + 1)$. Hence $k^{-1} > 0$ and $\rho_1$ is indeed a PLF. Next we consider a positive operator $\mathcal{IO}$ in the other neighborhood $V'$ defined by

$$\mathcal{IO} := [(\hat{\mathcal{I}}\hat{A})^\dagger - e^{i\alpha}(\check{\mathcal{I}}\check{B})^\dagger](\hat{\mathcal{I}}\hat{A} - e^{-i\alpha}\check{\mathcal{I}}\check{B}). \quad \text{(A13)}$$

Then, according to (3.8), the operator $\mathcal{IO}$ takes the value for the state $\rho_1'$

$$\rho_1'(\mathcal{IO}) = \rho_1(\hat{A}^\dagger\hat{A}) + \rho_1(\check{B}^\dagger\check{B}) - 2\mathrm{Re}[e^{i\alpha}\rho_1(\hat{A}\check{B})]. \quad \text{(A14)}$$

Hence, we have

$$k^{-1}\rho_1'(\mathcal{IO}) = \rho_0[(1 + ce^{i\alpha}\hat{A}\check{B})(\hat{A}^\dagger\hat{A} + \check{B}^\dagger\check{B})(1 + ce^{-i\alpha}\hat{A}^\dagger\check{B}^\dagger)] - 2\mathrm{Re}\{e^{i\alpha}\rho_0[(1 + ce^{i\alpha}\hat{A}\check{B})\hat{A}\check{B}(1 + ce^{-i\alpha}\hat{A}^\dagger\check{B}^\dagger)]\}. \quad \text{(A15)}$$

When we expand this expression, the sum of the terms proportional to $e^{\pm i\alpha}$ or $e^{\pm 2i\alpha}$ takes the form

$$A\cos(\alpha + \delta_1) + B\cos(2\alpha + \delta_2).$$

This can be made nonpositive by choosing $\alpha$ appropriately. Hence, we may drop these terms and obtain

$$k^{-1}\rho_1'(\mathcal{IO}) \le \rho_0(\hat{A}^\dagger\hat{A} + \check{B}^\dagger\check{B}) - 2c\rho_0(\hat{A}\hat{A}^\dagger\check{B}\check{B}^\dagger) + c^2\{\rho_0[(\hat{A}\hat{A}^\dagger)^2\check{B}\check{B}^\dagger] + \rho_0[\hat{A}\hat{A}^\dagger(\check{B}\check{B}^\dagger)^2]\}. \quad \text{(A16)}$$

By using

$$\rho_0(\hat{A}\hat{A}^\dagger\check{B}\check{B}^\dagger) = \rho_0[(\hat{A}^\dagger\hat{A} + 1)(\check{B}^\dagger\check{B} + 1)], \quad \text{(A17)}$$

we find

$$k^{-1}\rho_1(\mathcal{IO}) \le (1-2c)[\rho_0(\hat{A}^\dagger\hat{A}) + \rho_0(\check{B}^\dagger\check{B})] - 2c[1 + \rho_0\{\hat{A}^\dagger\hat{A}\check{B}^\dagger\check{B})] + c^2(\rho_0[(\hat{A}\hat{A}^\dagger)^2\check{B}\check{B}^\dagger] + \rho_0[\hat{A}\hat{A}^\dagger(\check{B}\check{B}^\dagger)^2]\}. \quad \text{(A18)}$$

Using inequalities (A3), (A6), and (A7) and assuming $c < 1/2$, we obtain

$$k^{-1}\rho_1'(\mathcal{IO}) \le 2c^2(1+7\epsilon) - 2c(1+2\epsilon) + 2\epsilon. \quad \text{(A19)}$$

For the right-hand side to have a negative value for some $c$, it is sufficient to have

$$(1+2\epsilon)^2 - 4(1+7\epsilon)\epsilon > 0. \quad \text{(A20)}$$

Hence, if

$$\epsilon < \frac{1}{\sqrt{24}} \quad \text{(A21)}$$

and

$$\frac{1+2\epsilon-\sqrt{1-24\epsilon^2}}{2(1+7\epsilon)} < c < \frac{1}{2}, \quad \text{(A22)}$$

then $\rho_1'(\mathcal{I}O) < 0$ as claimed.


[1] G. W. Gibbons, Nucl. Phys. **B271**, 986 (1986).
[2] B. S. Kay, Math. Phys. (special issue), p. 167 (1992).
[3] N. Sanchez and B. F. Whiting, Nucl. Phys. **B283**, 605 (1987).
[4] U. Yurtsever, Class. Quantum Grav. **11**, 999 (1994).
[5] C. J. Fewster and A. Higuchi, "Quantum field theory on certain non-globally hyperbolic spacetimes," University of Bern report (unpublished).
[6] R. Haag, *Local Quantum Physics* (Springer-Verlag, Berlin, 1992).
[7] A. Ashtekar and A. Magnon, Proc. R. Soc. London **A346**, 375 (1975).
[8] C. J. Isham, in *Differential Methods in Mathematical Physics II*, edited by K. Bleuler, H. Petry, and A. Reetz (Springer, New York, 1978).
[9] B. S. Kay, Commun. Math. Phys. **62**, 55 (1978); **71**, 29 (1980).
[10] P. Hájíček, in *Differential Methods in Mathematical Physics II* [8].
[11] J. Dimock, Commun. Math. Phys. **77**, 219 (1980).
[12] K. Fredenhagen and R. Haag, Commun. Math. Phys. **127**, 273 (1990).
[13] B. S. Kay and R. M. Wald, Phys. Rep. **207**, 49 (1991).
[14] R. M. Wald, in *Gravitation and Quantization*, Proceedings of the Les Houches Summer School, Les Houches, France, 1992, edited by B. Julia and J. Zinn-Justin, Les Houches Summer School Proceedings Vol. LVII (North-Holland, Amsterdam, 1993).
[15] R. Penrose, *Techniques of Differential Topology in Relativity* (Society for Industrial and Applied Mathematics, Philadelphia, 1972).
[16] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
[17] P. Candelas and D. Deutsch, Proc. R. Soc. London **A354**, 79 (1977); **A362**, 251 (1978).
[18] K. Bichteler, J. Math. Phys. **9**, 813 (1968).
[19] R. Geroch, J. Math. Phys. **9**, 1729 (1968).
[20] R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[21] R. Penrose and W. Rindler, *Spinors and Space-Time, Vol. I* (Cambridge University Press, Cambridge, England, 1984); *Spinors and Space-Time* (Cambridge University Press, Cambridge, England, 1986), Vol. II.
[22] G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. **148**, 345 (1992).
[23] G. W. Gibbons and S. W. Hawking, Phys. Rev. Lett. **69**, 1719 (1993).
[24] J. L. Friedman, Class. Quantum Grav. **12**, 2231 (1995).
[25] A. Chamblin and G. W. Gibbons, Class. Quantum Grav. **12**, 2243 (1995).
[26] M. Karoubi, Ann. Sci. Éc. Norm. Sup. Series 4, **1**, 161 (1968).
[27] G. S. Whiston, Gen. Relativ. Gravit. **6**, 463 (1975).
[28] L. Dabrowski, *Group Actions on Spinors*, Monographs and Textbooks in Physical Science (Bibliopolis, Naples, 1988).
[29] A. Chamblin, Commun. Math. Phys. **164**, 65 (1994).
[30] G. W. Gibbons, Int. J. Mod. Phys. **3**, 61 (1994).