Quantum probes of spacetime singularities

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It is shown that there are static spacetimes with timelike curvature singularities which appear completely nonsingular when probed with quantum test particles. Examples include extreme dilatonic black holes and the fundamental string solution. In these spacetimes, the dynamics of quantum particles is well defined and uniquely determined.

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I. INTRODUCTION

In general relativity, a spacetime is considered singular if it is geodesically incomplete. This is intuitively reasonable since geodesics describe the motion of test particles. Thus, if a spacetime is (timelike) geodesically incomplete, then the evolution of some test particle is not defined after a finite proper time. The use of geodesic incompleteness is not only intuitively appealing, it has also been quite useful in establishing that large classes of solutions to Einstein's equations are singular.

There has been extensive debate over whether these singularities in general relativity will be "smoothed out' in quantum gravity. Various model systems have been quantized with inconclusive results (see [1,2] for some classic treatments). As a first step toward understanding the relation between quantum theory and singularities, we consider the motion of a quantum test particle in a classical singular spacetime. We will see that there are static spacetimes with timelike singularities in which a quantum test particle is completely well behaved for all time. Even more significantly, these singularities do not introduce any new ambiguities or require additional boundary conditions in the definition of the quantum particle. The dynamics is uniquely defined *by the spacetime*, just as on a nonsingular background.

Thus, even though these spacetimes appear singular when probed with classical test particles, they are nonsingular when the test particles are treated quantum mechanically. Roughly speaking, the reason for the difference is that these spacetimes produce an effective repulsive barrier which shields their classical singularity, and quantum wave packets simply bounce off this barrier. From this viewpoint, geodesics correspond to the geometric optics limit of infinite frequency waves. Only in this unphysical limit is the singularity reached.

Another motivation for studying the motion of quantum test particles in a classical spacetime comes from string theory. Classical solutions to string theory are associated with two-dimensional conformal field theories. These theories describe the motion of quantum test strings in a background classical geometry. A solution to string theory is singular if there does not exist a well defined evolution for these quantum test strings. Since a string consists of an infinite number of modes which represent particles of increasing mass and spin, studying the behavior of a single quantum test particle will give a preliminary indication of the behavior of a test string. (Unfortunately, these results will not be conclusive since even if the quantum particle is singular, there may exist an equivalent "dual" description of the solution in string theory which is nonsingular [3].)

If one wants to investigate quantum probes of singularities, one needs a condition in the quantum theory of the test particle which determines whether or not it is singular. The general definition of a singularity in a quantum theory is still controversial. Some people have suggested looking at the expectation value of certain "physical operators" to see whether they diverge. The notion of a singularity that we will study is somewhat different. We will be interested in particular in the analogue of a timelike singularity. As such, we will say that a system is *non*singular when the evolution of any state is *uniquely* defined for all time. If this is not the case, then there is some loss of predictability and we will say that the system is singular.

To illustrate this, consider nonrelativistic quantum mechanics on a bounded interval. Note that this system is classically singular as the associated 'spacetime' is geodesically incomplete. One can initially define the Hamiltonian H to be the Laplacian acting on wave functions that vanish smoothly at the boundary. This operator is symmetric, but not yet self-adjoint. There are in fact many so-called extensions of this operator (given by defining H to act as the Laplacian on a slightly larger domain) which are self-adjoint and which correspond to the different boundary conditions which might be imposed at the edges. One of these extensions must be chosen in order to evolve quantum states. This is directly analogous to a typical timelike singularity in classical general relativity, as in this case one must make a choice of boundary conditions at the singularity. In both cases, the evolution

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is not unique until extra information is specified. One can imagine another type of singularity in nonrelativistic quantum mechanics which is more analogous to a spacelike singularity than a timelike one. This occurs if H is time dependent and is self-adjoint for $t < t_0$ but fails to be self-adjoint (or even fails to exist) at $t = t_0$. Here, however, we will concentrate on the case of timelike singularities.

The hydrogen atom is the prime example of a singular classical theory which is nonsingular quantum mechanically. There is a direct analog of this for singular geometries. Consider the (nonrelativistic) quantum mechanics of a free particle moving on an $(n+1)$-dimensional Reimannian manifold M, g. The Hilbert space consists of square integrable functions on M with the measure given by the proper volume element. The Hamiltonian is proportional to the Laplacian on the manifold. It is known that if the metric g is geodesically complete, then the Laplacian has a unique self-adjoint extension [4] (operators for which this is the case are called essentially self-adjoint). This means that if the space is classically nonsingular, then it is nonsingular quantum mechanically as well. We are interested in determining whether the metric can be geodesically incomplete and still have a unique self-adjoint Laplacian.

It is easy to see that the answer is yes. Consider a spherically symmetric metric

$$ds^2 = dr^2 + R^2(r)d\Omega_n \tag{1.1}$$

where $d\Omega_n$ is the standard metric on the n sphere. We first take the domain of the Laplacian to consist of smooth functions with compact support away from the origin. The key question is whether the resulting operator is essentially self-adjoint. A sufficient condition for this to be the case is to consider solutions to $D^2\psi \pm i\psi = 0$ and show that such solutions are not square integrable [5]. Using separation of variables, $\psi = f(r)Y(\text{angles})$, we obtain the radial equation

$$f'' + \frac{nR'}{R}f' - \frac{c}{R^2}f \pm if = 0 \tag{1.2}$$

where a prime denotes the derivative with respect to r and $c \geq 0$ is an eigenvalue of (minus) the Laplacian on the n sphere. Essential self-adjointness is in fact equivalent [5] to the statement that, for each c and each choice of $\pm i$, there is one solution to (1.2) which fails to be square integrable near the origin. It suffices to consider the case $c = 0$ since increasing c increases the divergence of one solution at $r = 0$. Near the origin, if $R = r^p$, then the two solutions are $f = r^\alpha$ where $\alpha = 0$ or $\alpha = 1 - np$ (note that the $\pm if$ term is negligible near $r = 0$). If $p \geq 3/n$, the latter solution fails to be square integrable with respect to the proper volume element $r^{pn}drd\mu$, where $d\mu$ is the volume element on the unit n sphere. We conclude that any metric of the form (1.1) which behaves like $R(r) = r^p$ with $p \geq 3/n$ near the origin is nonsingular in quantum mechanics. Of course, the metric (1.1) is geodesically incomplete unless[1] $p = 1$. So there is a large class of geometries which are singular classically, but not quantum mechanically. They are geometric analogues of the hydrogen atom.

II. STATIC SPACETIMES

A. General condition for quantum regularity

For a static, globally hyperbolic spacetime, there is a well defined quantum theory for a single relativistic particle (see, for example, [6]). We will show that for certain static spacetimes with timelike singularities, this is still the case. We will consider a relativistic particle with mass $m \geq 0$, which is described quantum mechanically by a positive frequency solution to the wave equation of mass m. Some time ago, Wald discussed solutions to the (massless) wave equation in the presence of singularities [7]. Our discussion will be based on his approach.

Consider a static spacetime with timelike Killing field ξ^μ. Let t denote the Killing parameter, and Σ denote a static slice. The wave equation $(\nabla^\mu\nabla_\mu - m^2)\psi = 0$ can be rewritten in the form

$$\frac{\partial^2\psi}{\partial t^2} = VD^i(VD_i\psi) - V^2m^2\psi \ , \tag{2.1}$$

where $V^2 = -\xi^\mu\xi_\mu$, and D_a is the spatial covariant derivative on Σ. Let A denote (minus) the operator on the right-hand side:

$$A \equiv -VD^i(VD_i) + V^2m^2. \tag{2.2}$$

Consider the Hilbert space $\mathcal{H}$ of square integrable functions on Σ with the inner product V^{-1} times the proper volume element. If we initially define the domain of A to be smooth functions of compact support on Σ, then since $V^2m^2 \geq 0$, A is a positive symmetric operator. (Recall that in general relativity, the "singular points" are not included as part of Σ.) We note that A of 2.2 is also a *real* differential operator so that its deficiency indices [5] are always equal and self-adjoint extensions always exist. The key question is whether such an extension is unique. If it is, then this extension A_E is always positive definite and we may define its positive self-adjoint square root. Then the wave function for a free relativistic particle satisfies

$$i\frac{d\psi}{dt} = (A_E)^{1/2}\psi \tag{2.3}$$

[1]This argument seems to imply that the Laplacian in three-dimensional Euclidean space is not essentially self-adjoint. This is indeed the case if one initially defines the operator only away from the origin and is consistent with the fact that $\mathbf{R}^3 - \{0\}$ is geodesically incomplete. However when the space is regular at the origin (and only in this case) one can require $D^2\psi \pm i\psi = 0$ at the origin as well. This removes the ambiguity.

with solution

$$\psi(t) = \exp\left[-it(A_E)^{1/2}\right]\psi(0). \qquad (2.4)$$

The right-hand side is well defined using standard properties of self-adjoint operators. If there is more than one self-adjoint extension of A, then (2.3) and (2.4) are ambiguous. This is our criterion for calling the quantum theory singular.

Wald [7] considered the second order wave equation. He did not require the operator A to be essentially self-adjoint, but instead picked an arbitrary positive definite self-adjoint extension and studied the resulting solution. He showed that it agreed with the usual Cauchy evolution inside the domain of dependence of the initial surface.

B. Examples

In this section we will consider some examples of static solutions that have recently been discussed in the literature. All of these solutions are geodesically incomplete, and we wish to determine whether they are singular when probed by quantum test particles. We first consider a general static, spherically symmetric metric in $n+2$ dimensions,

$$ds^2 = -V^2dt^2 + V^{-2}dr^2 + R^2 d\Omega_n \ , \qquad (2.5)$$

where V and R are functions of r only. As discussed above, the crucial question is whether the spatial operator A (2.2) is essentially self-adjoint. Consider the equation $A\psi \pm i\psi = 0$. Separating variables $\psi = f(r)Y(\text{angles})$ leads to the following radial equation for f:

$$f'' + \frac{(V^2R^n)'}{V^2R^n}f' - \frac{c}{V^2R^2}f - \frac{m^2}{V^2}f \pm i\frac{f}{V^4} = 0. \qquad (2.6)$$

The operator A will be essentially self-adjoint if one of the two solutions to this equation (for each c and each sign of the imaginary term) fails to be square integrable with respect to the measure R^nV^{-2} near $r=0$.

Suppose that, for $m=0$, one solution of (2.6) fails to be square integrable near the origin. Then, because $m^2 \geq 0$ and $V^2 \geq 0$, the addition of the term $-\frac{m^2 f}{V^2}$ acts like a repulsive potential in quantum mechanics. That is, it will increase the rate at which the larger solution diverges at the origin while driving the other more quickly to zero. It follows that if A is essentially self-adjoint for $m=0$, it is essentially self-adjoint for all $m \geq 0$ as well. Thus, we need only consider the massless case below.

The metric (2.5) can have a null singularity instead of a timelike one. The difference is seen as follows. Define a new radial coordinate $dr_* = dr/V^2$, so that radial null geodesics follow curves of constant $t \pm r_*$. If the singularity is at a finite value of r_*, then it is timelike. But if it is at $r_* = -\infty$, then it is null. (A Penrose diagram of the resulting spacetime would resemble the region $r > 2M$ of the Schwarzschild solution, with a singularity along the horizon $r = 2M$.) If the singularity is null, the spacetime is globally hyperbolic. It follows immediately that the operator A must be essentially self-adjoint. This is because, if there were more than one self-adjoint extension, there would be two distinct evolutions of initial data to the wave equation, of the form described by Wald [7]. But these solutions must agree with ordinary Cauchy evolution (which is unique), so all self-adjoint extensions of A must agree. We will therefore consider only spacetimes with timelike singularities.

Consider first the (four-dimensional) negative mass Schwarzschild solution. It is easy to verify that both solutions to (2.6) are locally normalizable near $r=0$. Thus, it remains singular even when probed with quantum test particles. This is fortunate, since if the negative mass Schwarzschild solution was nonsingular in some theory, then that theory would probably not have a stable ground state [8]. One can also verify that the Reissner-Nordström solution remains singular for all values of the charge to mass ratio Q/M. It is interesting to note that the $M<0$ Schwarzschild solution is timelike geodesically complete. As a result, a massive relativistic classical particle in this spacetime is nonsingular while the corresponding quantum theory is singular. We thus have a counterexample to Wheeler's "rule of unanimity" [9].[2]

We now consider four-dimensional, charged dilatonic black holes. They are extrema of the action

$$S = \int d^4x\sqrt{-g}\left[R - 2(\nabla\phi)^2 - e^{-2a\phi}F^2\right] \ , \qquad (2.7)$$

where ϕ is the dilaton, F is the Maxwell field, and a is a constant which governs the strength of the dilaton coupling. For $a=\sqrt{3}$, this action is equivalent to Kaluza-Klein theory. In other words, given an extremum of (2.7) with this value of a, one can reconstruct a solution of the five-dimensional vacuum Einstein equation. The charged black hole solution to this theory (for general a) is given by a metric of the form (2.5) with [11]

$$V^2 = \left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{(1-a^2)}{(1+a^2)}} \ ,$$

$$R^2 = r^2\left(1-\frac{r_-}{r}\right)^{\frac{2a^2}{(1+a^2)}} \ . \qquad (2.8)$$

Notice the product of these two quantities is independent of a and is simply

$$V^2R^2 = (r-r_+)(r-r_-). \qquad (2.9)$$

For $r_+ > r_-$ and $a \neq 0$, this metric describes a black hole with an event horizon at $r = r_+$ and a singularity at $r = r_-$. (For the special case $a=0$, $r=r_-$ denotes the inner Cauchy horizon of the Reissner-Nordström solution which is nonsingular.) The extremal limit $r_+ = r_-$

[2] A similar, but simpler, counterexample is given by the Hamiltonian $p^2 + 1/x$ for a nonrelativistic particle on the half line $x > 0$.

describes a globally static spacetime with a curvature singularity at $r = r_+$. This singularity is null for $a \leq 1$, but is timelike for $a > 1$.

The operator A (2.2) must be essentially self-adjoint for $r_+ = r_-$ and $a \leq 1$ since the singularity is null. We wish to investigate whether this continues to be the case for $a > 1$. To begin, let $\rho = r - r_+$, so $V^2R^2 = \rho^2$. Since $a > 1$, $V^2 > \rho$ so that the imaginary term in (2.6) may be ignored. Then one solution to (2.6) behaves like $f = \rho^\alpha$ with $\alpha \leq -1$ near the singularity $\rho = 0$. The least divergent solution, $\alpha = -1$, corresponds to the S wave, $c = 0$. From (2.8) we see that this solution has norm

$$\langle f|f\rangle = \int \rho^{-4/(1+a^2)} d\rho \tag{2.10}$$

which diverges near $\rho = 0$ for $a^2 \leq 3$. Thus, extremal dilaton black holes with $1 < a^2 \leq 3$ are examples of static spacetimes with timelike singularities for which quantum test particles are well behaved. The fact that the solutions (2.8) have infinite repulsive barriers when $a^2 > 1$ was noticed earlier by Holzhey and Wilczek [10]. However, their analysis did not distinguish between a^2 greater than three and less than three. We now see that for $a^2 > 3$ quantum mechanics does not exclude the solution that grows near $\rho = 0$.

Notice that extreme Kaluza-Klein black holes are included in the class of solutions which are quantum mechanically nonsingular. One might wonder if this is related to the fact that the *five*-dimensional metric for an extreme magnetically charged black hole does not have a curvature singularity. The answer is clearly no. First, our analysis applies to both electric and magnetically charged solutions [since the metric (2.8) is the same] and the electrically charged solution remains singular in five dimensions. Second, if one dimensionally reduces the $(4+m)$-dimensional Einstein action to four dimensions one can obtain the action (2.7) with $a = \sqrt{(m+2)/m}$ [11]. So all of these extremal Kaluza-Klein black holes are nonsingular quantum mechanically, even though most have curvature singularities in $4+m$ (as well as four) dimensions.

As another example, we consider the fundamental string solution discovered by Dabholkar *et al.* [12]. This was originally found as a solution to the low energy string action

$$S = \int d^Dx\sqrt{-g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2\right) \tag{2.11}$$

(where D is the spacetime dimension and H is the three-form) but was later shown to be an exact solution to string theory [13]. The metric is given by

$$ds^2 = V^2(-dt^2 + dz^2) + dx_i dx^i \ , \tag{2.12}$$

$$V^{-2} = 1 + \frac{M}{r^{D-4}} \ , \tag{2.13}$$

where $r^2 = x_i x^i$. This solution describes the field outside of a straight fundamental string located at $r = 0$, which is a curvature singularity. This singularity is null for $D \geq 6$ but is timelike for $D = 5$. Thus, A must be essentially self-adjoint for $D \geq 6$. By performing an analysis similar to that above, one can show that A remains essentially self-adjoint when $D = 5$. So this provides another example of a classically singular spacetime which is nonsingular quantum mechanically.

However, this result is not directly applicable to singularities in string theory since we have not included the effect of the dilaton on the test particle. Recall that the lowest mode of a (bosonic) string is the tachyon which is coupled to the dilaton via

$$S = \int d^Dx\sqrt{-g}e^{-2\phi}[(\nabla\psi)^2 + m^2\psi^2]. \tag{2.14}$$

For a static spacetime, the wave functions of the tachyon modes satisfy the equation of motion

$$\frac{\partial^2\psi}{\partial t^2} = -\tilde{A}\psi \tag{2.15}$$

where

$$\tilde{A} = -Ve^{2\phi}D_i[Ve^{-2\phi}D^i\psi] + m^2V^2 \tag{2.16}$$

and the notation is the same as in (2.2). Since $m^2 < 0$ for the tachyon, we must keep the mass term for now. This operator is symmetric with respect to an L^2 inner product with measure equal to the proper volume element divided by $Ve^{2\phi}$.

We now show that $\tilde{A}$ is essentially self-adjoint for the $D = 5$ fundamental string (2.12). The dilaton for this solution is given by $e^\phi = V$. After separating variables $\psi = f(r)e^{ikz}Y(\text{angles})$, the equation $\tilde{A}\psi \pm i\psi = 0$ yields the following radial equation for f:

$$f'' + \frac{2}{r}f' - \left(\frac{k^2}{V^2} + \frac{c}{r^2}\right)f - m^2 f \pm i\frac{f}{V^2} = 0 \ , \tag{2.17}$$

where $c \geq 0$ is again an eigenvalue of (minus) the Laplacian on the sphere. Since $V^2 = r/M$ near $r = 0$, we see that the k^2 term, the mass term, and the imaginary term are all negligible near the origin. Thus, one solution in this region is $f = r^\alpha$ where $\alpha \leq -1$. This solution always has infinite norm near $r = 0$ since the appropriate inner product is

$$\langle f|f\rangle = \int \frac{|f|^2Vr^2dr}{Ve^{2\phi}} = \int |f|^2 Mr dr. \tag{2.18}$$

Therefore, even when the coupling to the dilaton is included, the singularity in the fundamental string does not prevent unique evolution of the tachyon. This suggests that other modes of the string will similarly have unique evolution, but the effect of spin needs to be investigated.

Another exact solution to string theory is an orbifold, which is constructed by starting with flat Euclidean space and identifying points under the action of a discrete group. If the group has fixed points, then the quotient is geodesically incomplete. Nevertheless, it is believed that string theory is well behaved on these backgrounds

[14]. From our discussion in the introduction, it is clear that for a two-dimensional orbifold (which is a cone), the operator governing evolution of a scalar test particle (or the tachyon) is *not* essentially self-adjoint. This suggests that the propagation of test strings is also not well defined without further specification of boundary conditions at the singularity. This is not a problem in dimensions greater than three, so the most commonly discussed case of a six-dimensional orbifold is nonsingular for quantum test particles.

C. Scattering

The evolution defined by $A_E^{1/2}$ has all of the nice properties of familiar quantum mechanical systems. By construction, the evolution is unitary and the energy $A_E^{1/2}$ is conserved. However, we have not yet ruled out the possibility that an incoming wave packet might remain localized near the singularity, resulting in a nonunitary S matrix. Indeed, we expect this will happen whenever the singularity is null, since the wave then takes an infinite (coordinate) time to reach the singularity. However, we now show that, at least for highly symmetric cases, this cannot occur for timelike singularities. For such cases, the S matrix is unitary.

Consider a spherically symmetric metric of the form (2.5) with a timelike singularity at the origin. As usual, spherical symmetry and time independence imply that energy and angular momentum are conserved in the scattering so that we can confine our attention to the radial eigenfunction equation. Since any eigenstate of $A_E^{1/2}$ is also an eigenstate of A_E, it is in fact sufficient to study wave functions f that solve

$$f'' + \frac{(V^2R^n)'}{V^2R^n}f' - \frac{c}{V^2R^2}f - \frac{m^2}{V^2}f + \frac{Ef}{V^4} = 0. \quad (2.19)$$

Let $R = r^p$ and $V = r^q$ near the origin, and consider first the case $c = m = 0$. Since the singularity is timelike, $q < 1/2$. Thus, the term Ef/V^4 is negligible near $r = 0$, and the two solutions to (2.19) take the form $f = r^\alpha$ with $\alpha = 0, 1 - 2q - np$. By our previous discussion, the condition that the classical singularity not affect quantum test particles is that the solution $r^{1-2q-np}$ must not be square integrable near $r = 0$ with respect to the measure $R^nV^{-2}dr$. Since this measure is r^{np-2q} near the origin, the condition that the singularity be timelike ($q < 1/2$) guarantees that the other solution r^0 is always square integrable. If c and m^2 are nonzero, the above equation is modified by the addition of a repulsive potential (assuming nontachyonic particles) which increases the divergence of the more singular solution and forces the less singular solution to vanish more quickly. Thus, for any $c \geq 0$ and $m^2 \geq 0$, there is exactly one allowable solution of (2.19). It is real, with equal incoming and outgoing flux. Thus, the S matrix is unitary. A similar argument establishes unitarity for nontachyonic particles in any cylindrically symmetric spacetime. For the special case of the $D = 5$ fundamental string solution (2.12), one can verify that tachyon scattering is also unitary.

III. EXTENSIONS

In the previous section, we considered only the propagation of quantum test particles on a static (time-independent) background. Any extension to more general cases will clearly require a change of outlook, if not of techniques. Indeed, for a general time-dependent background there is no consistent quantum theory of a single free particle in the usual sense and the only appropriate description is in terms of quantum field theory. Since linear quantum field theory is defined by the solutions of classical field theory, the essential step is to study the evolution of classical test fields on a singular background.

This may not be as difficult as it sounds. As described in [7], techniques similar to those applied here can be used to define classical field evolution in static singular spacetimes. Given a scalar field ϕ satisfying a wave equation of the form

$$\frac{\partial^2}{\partial t^2}\phi = -A\phi \quad (3.1)$$

with A a symmetric operator on the Hilbert space $\mathcal{H}$ of Sec. II A and any self-adjoint extension A_E of A, the field

$$\phi(t) = \cos[A_E^{1/2}t]\phi(0) + A_E^{-1/2}\sin[A_E^{1/2}t]\dot{\phi}(0) \quad (3.2)$$

is the unique solution of $\frac{\partial^2}{\partial t^2}\phi = -A_E\phi$ [which takes the value $\phi(0)$ at $t = 0$ and has time derivative $\dot{\phi}(0)$ at $t = 0$] and also satisfies (3.1) in any hyperbolic domain. Thus, when A is essentially self-adjoint, there is a unique solution of this form and no boundary conditions need be imposed.

What about the general nonstatic case? It is not difficult to make the first steps. By reformulating the general wave equation in the first order form

$$\frac{\partial}{\partial t}\begin{bmatrix}\phi(t)\\ \dot{\phi}(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -A(t) & iB(t)\end{bmatrix}\begin{bmatrix}\phi(t)\\ \dot{\phi}(t)\end{bmatrix}, \quad (3.3)$$

it is clear that our task is to define the path ordered exponential

$$\begin{bmatrix}\phi(t)\\ \dot{\phi}(t)\end{bmatrix} = \mathcal{P}\exp\left(\int_0^t \begin{bmatrix}0 & 1\\ -A & iB\end{bmatrix}\right)\begin{bmatrix}\phi(0)\\ \dot{\phi}(0)\end{bmatrix}. \quad (3.4)$$

As before, we will need to work in certain Hilbert spaces, and the choices

$$\left(\begin{bmatrix}\phi_1\\ \dot{\phi}_1\end{bmatrix}, \begin{bmatrix}\phi_2\\ \dot{\phi}_2\end{bmatrix}\right)_{\mathcal{H}_t} = \int_{\Sigma_t}(\phi_1^*\phi_2 + \dot{\phi}_1^*\dot{\phi}_2)\sqrt{-g}g^{tt}d^{n-1}\Sigma_t \quad (3.5)$$

are natural. Note that in the static case $\sqrt{-g}g^{tt}d^{n-1}\Sigma_t$ is V^{-1} times the proper volume element on Σ_t, so that this is a straightforward generalization of $\mathcal{H}$ from Sec. II A. In the special case of a static spacetime, the path ordered exponential is well defined and gives the solution (3.2). The case in which the spacetime is stationary (so that $\mathcal{H}$, A, and B are time independent) and $[A, B] = 0$ is also straightforward to exponentiate and yields

$$\phi(t) = e^{itB/2} \cos\left(\frac{t}{2}\sqrt{B^2 + 4A}\right)\phi(0) + e^{itB/2}\frac{2}{\sqrt{B^2+4A}} \sin\left(\frac{t}{2}\sqrt{B^2+4A}\right) \times \left(\dot{\phi}(0) - i\frac{B}{2}\phi(0)\right). \quad (3.6)$$

While the general time-dependent case remains to be investigated, we mention that the following two results can be derived by elementary methods. First, by using an 'interaction picture,' it is readily shown that if $A(t)$ and $B(t)$ differ from the operators associated with either of the solutions (3.2) or (3.6) by an appropriately bounded perturbation,[3] then there is a unique (and well defined) solution of the form (3.4). Also, by assuming that a solution of the form (3.4) is well defined, it is readily shown (using much the same method as [7]) to agree with the solution of the usual wave equation in any hyperbolic domain. Such a solution also conserves the Klein-Gordon inner product over the entire spacetime.

Whether such ideas can be developed further is an interesting question for future research. Also of interest would be a search for corresponding results for higher spin fields. This would be an important step toward extending these results from test particles to test strings. Since the (four-dimensional) Maxwell equations are conformally invariant, one can construct examples of singular spacetimes in which Maxwell fields are well behaved but scalar fields are not. If it is found that a large class of fields have nonsingular evolution on some singular background, then such a spacetime need not be seen as a threat to cosmic censorship. Instead of being shielded by a horizon, the timelike singularity would be shielded by the effective repulsive barrier that it presents to wave propagation.

[3]Unfortunately, since perturbations of wave operators are polynomial differential operators, they are not bounded, so that this case is not of direct physical interest.

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