Probability for primordial black holes

R. Bousso* and S. W. Hawking†
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom
(Received 28 June 1995)

We consider two quantum cosmological models with a massive scalar field: an ordinary Friedmann universe and a universe containing primordial black holes. For both models we discuss the complex solutions to the Euclidean Einstein equations. Using the probability measure obtained from the Hartle-Hawking no-boundary proposal we find that the only unsuppressed black holes start at the Planck size but can grow with the horizon scale during the roll down of the scalar field to the minimum.

PACS number(s): 98.80.Hw, 04.60.Kz, 04.70.Dy, 98.80.Bp

I. INTRODUCTION

In this paper we ask how likely it is for the universe to have contained primordial black holes. We investigate universes which undergo a period of inflation in their earliest stage, driven by a scalar field ϕ with a potential $V(\phi)$ with a minimum $V(0) = 0$. The results do not depend qualitatively on the exact form of the potential, so for simplicity we consider a massive minimally coupled scalar $V(\phi) = \frac{1}{2}m^2\phi^2$. The scalar field starts out at a large initial value ϕ_0 and acts as a cosmological constant for some time until it reaches the minimum of its potential and inflation ends. We consider two different types of spacetimes: in the first, the spacelike sections are simply three-spheres and no black holes are present; in the second, they have the topology $S^1 \times S^2$, which is the topology of the spatial section of the Schwarzschild–de Sitter solution. Thus these spaces can be interpreted as inflationary universes with a pair of black holes. In the inflationary period, the first type will be similar to a de Sitter universe, the second to a Nariai universe [1]. To find the likelihood for primordial black holes, we assign probabilities to both types of spacetimes using the Hartle-Hawking no-boundary proposal (NBP) [2]. This is the only proposal for the boundary conditions of the universe that seems to give a well-defined answer in this situation. It is not clear how to apply the so-called "tunneling proposal" in the $S^1 \times S^2$ case. If one takes the action to appear with the opposite sign as is done in the S^3 case, one would reach the conclusion that a universe with a pair of black holes was more likely than a universe without, and that the probability would increase with the size of the black holes. This is clearly absurd.

The NBP framework is summarized in Sec. II. In Secs. III and IV we review its implementation for cases with a fixed cosmological constant. In Sec. V we introduce a massive scalar field and discuss the solutions of the Euclidean Einstein equations for the S^3 case. They will be slightly complex due to the time dependence of the effective cosmological constant $(m\phi)^{-2}$. We obtain the Euclidean action for those solutions. In Sec. VI we go through a similar procedure for the $S^1 \times S^2$ case. We find that the black hole grows during the inflationary period, a noteworthy difference to the Nariai case with a fixed cosmological constant. In Sec. VII we use the action to estimate the relative probability of the two types of universes. We find that black holes are suppressed for all but very large initial values of ϕ_0.

*Electronic address: R.Bousso@damtp.cam.ac.uk
†Electronic address: S.W.Hawking@damtp.cam.ac.uk

II. THE WAVE FUNCTION OF THE UNIVERSE

The Hartle-Hawking no-boundary proposal states that the wave function of the universe is given by

$$\Psi_0[h_{ij}, \Phi_{\partial M}] = \int D(g_{\mu\nu}, \Phi) \exp\left[-I(g_{\mu\nu}, \Phi)\right], \quad (2.1)$$

where $(h_{ij}, \Phi_{\partial M})$ are the three-metric and matter field on a spacelike boundary ∂M and the path integral is taken over all compact Euclidean four geometries $g_{\mu\nu}$ that have ∂M as their only boundary and matter field configurations Φ that are regular on them; $I(g_{\mu\nu}, \Phi)$ is their action.

The gravitational part of the action is given by

$$I_E = -\frac{1}{16\pi}\int_M d^4x\, g^{1/2}(R - 2\Lambda) - \frac{1}{8\pi}\int_{\partial M} d^3x\, h^{1/2} K, \quad (2.2)$$

where R is the Ricci-scalar, Λ is the cosmological constant, and K is the trace of K_{ij}, the second fundamental form of the boundary ∂M in the metric g. For the origin of the boundary term, see, e.g., Ref. [3].

In the standard 3+1 decomposition [4], the metric is written as

$$ds^2 = N^2 d\tau^2 + h_{ij}(dx^i + N^i d\tau)(dx^j + N^j d\tau). \quad (2.3)$$

Assuming that the NBP is satisfied at $\tau = 0$, the Euclidean action then takes the form

$$I_E = -\frac{1}{16\pi}\int_{\tau=0}^{\tau_{\partial M}} N d\tau \int d^3x\, h^{1/2} \times(-K_{ij}K^{ij} + K^2 + {}^3R - 2\Lambda) + \frac{1}{8\pi}\int_{\tau=0} d^3x\, h^{1/2}K. \quad (2.4)$$

Here ${}^3R$ is the scalar curvature of the surface, and tensor operations are carried out with respect to the surface metric h_{ij}. In the first term the boundary terms are implicitly subtracted out at $\tau = 0$ and $\tau = \tau_{\partial M}$. But it is an essential prescription of the NBP that there *is no* boundary at $\tau = 0$. So the second term explicitly adds the contribution from $\tau = 0$ back in. It vanishes for universes with spacelike sections of topology S^3, but can be nonzero for the topology $S^1 \times S^2$.

There are unresolved questions on how to choose the integration contour and make the integral converge [5], but we shall not discuss them here. Instead, we will use the semiclassical approximation

$$\Psi_0[h_{ij}, \Phi_{\partial M}] \approx \sum_n A_n e^{-I_n}, \quad (2.5)$$

where the sum is over the saddle points of the path integral, i.e., the solutions of the Euclidean Einstein equations. In this paper, we neglect the prefactors A_n and take only one saddle point into account for a given argument of the wave function. So the probability measure will be

$$|\Psi_0[h_{ij}, \Phi_{\partial M}]|^2 = \left|e^{-I}\right|^2 = e^{-2I^{\mathrm{Re}}}, \quad (2.6)$$

where I^{Re} is the real part of the Euclidean saddle-point action.

By considering only spaces of high symmetry (homogeneous S^3 or $S^1 \times S^2$ spacelike sections) we restrict the degrees of freedom in the metric to a finite number q^α. The Euclidean action for such a minisuperspace model with bosonic matter will typically have the form

$$I = -\int N d\tau \left[\frac{1}{2} f_{\alpha\beta}\frac{dq^\alpha}{d\tau}\frac{dq^\beta}{d\tau} + U(q^\alpha)\right]. \quad (2.7)$$

The saddle points will in general be complex solutions $q^\alpha(\tau)$ in the τ plane. In the semiclassical approximation the following relations for the real and imaginary parts of the saddle-point actions hold:

$$-\frac{1}{2}\left(\nabla I^{\mathrm{Re}}\right)^2 + \frac{1}{2}\left(\nabla I^{\mathrm{Im}}\right)^2 + U(q^\alpha) = 0, \quad (2.8)$$

$$\nabla I^{\mathrm{Re}} \cdot \nabla I^{\mathrm{Im}} = 0, \quad (2.9)$$

where the gradient and the dot product are both with respect to $f^{\alpha\beta}$. Therefore I^{Im} will be a solution of the Lorentzian Hamilton-Jacobi equation in regions of minisuperspace where Ψ has the property that

$$\left(\nabla I^{\mathrm{Re}}\right)^2 \ll \left(\nabla I^{\mathrm{Im}}\right)^2. \quad (2.10)$$

This allows us to reintroduce a concept of Lorentzian time in such regions: We find the integral curves of ∇I^{Im} in minisuperspace and define the Lorentzian time t as the parameter naturally associated with them. Reversely, if we demand that the NBP should predict classical Lorentzian universes at sufficiently late Lorentzian time, condition (2.10) must be satisfied. This means that there must be saddle-point solutions for which the path in the τ plane can be deformed such that it is eventually almost parallel to the imaginary τ axis and that all the q^α should be virtually real at late Lorentzian times. In summary, the following conditions must be met.

(i) The NBP must be satisfied at $\tau = 0$.

(ii) At the end point $\tau_{\partial M}$ of the path, the q^α must take on the real values $q^\alpha_{\partial M}$ of the arguments of the wave function:

$$q^\alpha(\tau_{\partial M}) = q^\alpha_{\partial M}. \quad (2.11)$$

(iii) The q^α must remain nearly real in the Lorentzian vicinity of the end point:

$$\mathrm{Re}\left(\left.\frac{dq^\alpha}{d\tau}\right|_{\tau_{\partial M}}\right) \approx 0. \quad (2.12)$$

III. THE de SITTER SPACETIME

In this and the next section we review vacuum solutions of the Euclidean Einstein equations with a cosmological constant Λ. First we look for a solution with spacelike sections S^3. Therefore we choose the metric ansatz

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_3^2. \quad (3.1)$$

The Euclidean action is

$$I = -\frac{3\pi}{4}\int N d\tau a\left(\frac{\dot{a}^2}{N^2} + 1 - \frac{\Lambda}{3}a^2\right), \quad (3.2)$$

An overdot denotes differentiation with respect to τ. We define

$$H = \sqrt{\frac{\Lambda}{3}}. \quad (3.3)$$

Variation of a and N yields the equation of motion

$$\frac{\ddot{a}}{a} + H^2 = 0 \quad (3.4)$$

and the Hamiltonian constraint

$$\frac{\dot{a}^2}{a^2} - \frac{1}{a^2} + H^2 = 0 \quad (3.5)$$

in the gauge $N = 1$. A solution of Eqs. (3.4) and (3.5) is given by

$$a(\tau) = H^{-1}\sin H\tau. \quad (3.6)$$

It is called the de Sitter spacetime. The NBP is satisfied at $\tau = 0$, where $a = 0$ and $\frac{da}{d\tau} = 1$. If we choose a path along the $\tau^{\rm Re}$ axis to $\tau = \frac{\pi}{2H}$, the solution will describe half of the Euclidean de Sitter instanton S^4. Choosing the path to continue parallel to the $\tau^{\rm Im}$ axis, $a(\tau)$ remains real and the conditions (i) to (iii) of the previous section will be satisfied:

$$a(\tau^{\rm Im})\big|_{\tau^{\rm Re}=\frac{\pi}{2H}} = H^{-1}\cosh H\tau^{\rm Im}. \tag{3.7}$$

This describes half of an ordinary Lorentzian de Sitter universe.

So with the above choice of path, Eq. (3.6) corresponds to half of a real Euclidean 4-sphere joined to a real Lorentzian hyperboloid of topology $R^1 \times S^3$. It can be matched to any $a_{\partial M} > 0$ by choosing the end point appropriately, and for $a_{\partial M} > H^{-1}$ the wave function oscillates and a classical Lorentzian universe is predicted.

The real part of the action for this saddle point is

$$I^{\rm Re}_{\rm de\ Sitter} = \frac{3\pi}{2}\int_0^{\frac{\pi}{2H}} d\tau^{\rm Re} a\left(H^2a^2 - 1\right) = -\frac{3\pi}{2\Lambda}. \tag{3.8}$$

The Lorentzian segment of the path only contributes to $I^{\rm Im}$.

IV. THE NARIAI SPACETIME

We still consider vacuum solutions of the Euclidean Einstein equations with a cosmological constant, but we now look for solutions with spacelike sections $S^1 \times S^2$. The corresponding ansatz is the Kantowski-Sachs metric

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 dx^2 + b(\tau)^2 d\Omega_2^2. \tag{4.1}$$

The Euclidean action is

$$I = -\pi\int N d\tau\, a\left(\frac{\dot b^2}{N^2} + 2\frac{b}{a}\frac{\dot a \dot b}{N^2} + 1 - \Lambda b^2\right) + \pi\left[-\dot a b^2 - 2ab\dot b\right]_{\tau=0}, \tag{4.2}$$

where the second term is the surface term of equation (2.4). We define

$$H = \sqrt{\Lambda}. \tag{4.3}$$

Variation of a, b, and N gives the equations of motion and the Hamiltonian constraint:

$$\frac{\ddot b}{b} - \frac{\dot a \dot b}{ab} = 0, \tag{4.4}$$

$$\frac{\ddot b}{b} + \frac{\dot a \dot b}{ab} + \frac{\ddot a}{a} + H^2 = 0, \tag{4.5}$$

$$2\frac{\dot a \dot b}{ab} + \frac{\dot b^2}{b^2} - \frac{1}{b^2} + H^2 = 0. \tag{4.6}$$

A solution is given by

$$a(\tau) = H^{-1}\sin H\tau,\ b(\tau) = H^{-1} = \text{const}. \tag{4.7}$$

It is called the Nariai spacetime. The NBP is satisfied at $\tau = 0$, where

$$a = 0,\ \dot a = 1,\ b = b_0, \ \text{ and } \dot b = 0. \tag{4.8}$$

(There is a second way of satisfying the NBP for the Kantowski-Sachs metric [6], but it will not lead to a universe containing black holes.) The path along the $\tau^{\rm Re}$ axis describes half of the Euclidean Nariai instanton $S^2 \times S^2$. Both two-spheres have the radius H^{-1}. Continuing parallel to the $\tau^{\rm Im}$ axis, the solution remains real:

$$a(\tau^{\rm Im})\big|_{\tau^{\rm Re}=\frac{\pi}{2H}} = H^{-1}\cosh H\tau^{\rm Im}, \tag{4.9}$$

$$b(\tau^{\rm Im})\big|_{\tau^{\rm Re}=\frac{\pi}{2H}} = H^{-1}. \tag{4.10}$$

This describes half of a Lorentzian Nariai universe. Its spacelike sections can be visualized as three-spheres of radius a with a "hole" of radius b punched through the North and South pole. This gives them the topology of $S^1 \times S^2$. Their physical interpretation is that of three-spheres containing two black holes at opposite ends. The black holes have the radius b and accelerate away from each other as a grows. The Nariai universe is a degenerate case of the Schwarzschild–de Sitter spacetime, with the black hole horizon and the cosmological horizon having equal radius [7].

The above path corresponds to half of a two-sphere joined to a two-dimensional hyperboloid at its minimum radius H^{-1}, cross a two-sphere of constant radius H^{-1}. It can be matched to any $a_{\partial M} > 0$ but only to $b_{\partial M} = H^{-1}$ so the wave function will be highly peaked around that value of b.

The first term of Eq. (4.2) vanishes and so the real part of the action for the Nariai solution comes entirely from the second term:

$$I^{\rm Re}_{\rm Nariai} = -\pi b_0^2 = -\frac{\pi}{\Lambda}. \tag{4.11}$$

Now we compare the probability measures corresponding to the de Sitter and Nariai solutions. We find that in these models with a fixed cosmological constant primordial black holes are strongly suppressed, unless Λ is at least of order 1 in Planck units:

$$\exp\left(-2I^{\rm Re}_{\rm Nariai}\right) = \exp\left(\frac{2\pi}{\Lambda}\right) \ll \exp\left(\frac{3\pi}{\Lambda}\right) = \exp\left(-2I^{\rm Re}_{\rm de\ Sitter}\right). \tag{4.12}$$

V. AN INFLATIONARY MODEL WITHOUT BLACK HOLES

Of course, we know that $\Lambda \approx 0$, and therefore the models of the previous section are rather unrealistic. However, in inflationary cosmology it is assumed that the very early universe underwent a period of exponential expansion. It has proven very successful to model this behavior by introducing a massive scalar field Φ with a potential $\frac{1}{2}m^2\Phi^2$. If this field is sufficiently far from equilibrium at the beginning of the universe, the corresponding energy density acts like a cosmological constant until the field has reached its minimum and starts oscillating. During this time the universe behaves much like the Lorentzian de Sitter or Nariai universes described above.

But there are two important differences due to the time dependence of the effective cosmological constant Λ_{eff}: Firstly, for the solutions of the Euclidean Einstein equations in the complex τ plane one can no longer find a path on which the minisuperspace variables are always real. However, we shall see that it is possible to satisfy conditions (i) to (iii) of Sec. II by choosing appropriate complex initial values. Secondly, it will be found in the next section that the black hole radius b is no longer constant during inflation.

In this section, we introduce the massive scalar field for the model corresponding to de Sitter spacetime, where the spacelike slices are three-spheres containing no black holes. This model was first put forward by Hawking [8]. From the fluctuations in the cosmic microwave background as measured by the Cosmic Background Explorer (COBE) [9] it follows that m is small compared to the Planck mass [10]:

$$m \sim 10^{-5}. \tag{5.1}$$

We will find complex solutions and the complex initial value of the scalar field, and we calculate the real part of the action. This has been done before by Lyons [11], but his paper contains a logical error to which we will come back later.

The ansatz for the Euclidean metric is again

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_3^2. \tag{5.2}$$

Using the rescaled field

$$\phi^2 = 4\pi\Phi^2, \tag{5.3}$$

we obtain the Euclidean action

$$I = -\frac{3\pi}{4}\int N d\tau a\left(\frac{\dot{a}^2}{N^2} + 1 - \frac{1}{3}a^2\frac{\dot{\phi}^2}{N^2} - \frac{1}{3}a^2m^2\phi^2\right), \tag{5.4}$$

so that the effective cosmological constant is

$$\Lambda_{\text{eff}}(\tau) = m^2\phi(\tau)^2. \tag{5.5}$$

In analogy to Eq. (3.3) we define

$$H(\tau) = \sqrt{\frac{\Lambda_{\text{eff}}(\tau)}{3}} = \frac{m\phi(\tau)}{\sqrt{3}}. \tag{5.6}$$

Variation with respect to a, ϕ, and N gives the Euclidean equations of motion and the Hamiltonian constraint:

$$\frac{\ddot{a}}{a} + \frac{2}{3}\dot{\phi}^2 + \frac{1}{3}m^2\phi^2 = 0, \tag{5.7}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - m^2\phi = 0, \tag{5.8}$$

$$\frac{\dot{a}^2}{a^2} - \frac{1}{a^2} - \frac{1}{3}\dot{\phi}^2 + \frac{1}{3}m^2\phi^2 = 0. \tag{5.9}$$

To evaluate $\Psi_0(a_{\partial M}, \phi_{\partial M})$ using a semiclassical approximation we must find solutions in the complex τ plane that meet conditions (i) to (iii) of Sec. II. In particular, the NBP must be satisfied:

$$a = 0,\ \dot{a} = 1,\ \phi = \phi_0, \text{ and } \dot{\phi} = 0 \text{ for } \tau = 0. \tag{5.10}$$

Assume that the initial value of the scalar field is large and nearly real:

$$\phi_0^{\text{Re}} \gg 1 \gg \phi_0^{\text{Im}}. \tag{5.11}$$

An approximate solution near the origin is given by

$$a_{\mathcal{I}}(\tau) = \frac{1}{H_0^{\text{Re}}}\sin H_0^{\text{Re}}\tau, \tag{5.12}$$

$$\phi_{\mathcal{I}}(\tau) = \phi_0 + \sum_{n=1}^{\infty}\frac{1}{n!}\gamma_n\tau^n \tag{5.13}$$

for $|\tau| < O\left(1/H_0^{\text{Re}}\right)$, where the Taylor series is obtained by solving Eq. (5.8) iteratively for $\ddot{\phi}$, using the NBP conditions (5.10) and the approximation (5.12) for a. It has the property that

$$\gamma_{2n+1} = 0 \text{ for all } n. \tag{5.14}$$

We call Eqs. (5.12) and (5.13) the *inner approximation.* Writing down the Taylor expansion explicitly to lowest nontrivial order,

$$\phi(\tau) = \phi_0\left[1 + \frac{3}{8\phi_0^2}(H_0\tau)^2\right] + O(\tau^4), \tag{5.15}$$

shows that ϕ is almost constant near the origin.

As an *outer approximation* we use

$$\phi_{\mathcal{O}}(\tau) = \psi_0 + \frac{im}{\sqrt{3}}\tau + \chi_0\exp(3iH_0\tau), \tag{5.16}$$

$$a_{\mathcal{O}}(\tau) = a_0\exp\left[-\frac{im}{\sqrt{3}}\int_0^\tau \phi(\tau')d\tau'\right] + c_0\exp\left[\frac{im}{\sqrt{3}}\int_0^\tau \phi(\tau')d\tau'\right] \tag{5.17}$$

for $0 < \tau^{\text{Im}} \ll \sqrt{3}\phi_0^{\text{Re}}/m$. While this solution does not satisfy the NBP, it will be good outside the validity of the inner approximation. Both the χ_0 term and the c_0 term can be neglected for $\tau^{\text{Im}} \gg 1/H_0^{\text{Re}}$, but they are useful for matching $a_{\mathcal{O}}$ and $\phi_{\mathcal{O}}$ to $a_{\mathcal{I}}$ and $\phi_{\mathcal{I}}$ at some $|\tau| \sim 1/H_0^{\text{Re}}$. Comparison with Eq. (5.15) shows that

$$\chi_0 \sim \frac{\sqrt{3}}{\phi_0^{\text{Re}}},\ \text{Im}(\chi_0) \approx 0. \tag{5.18}$$

In the region of the inner approximation, a will be nearly real on the Lorentzian line $\tau^{\text{Re}} = \frac{\pi}{2H_0^{\text{Re}}}$. Matching $a_{\mathcal{O}}$ to $a_{\mathcal{I}}$ fixes

$$a_0 \approx \frac{i}{2H_0^{\text{Re}}},\ c_0 \approx \frac{-i}{2H_0^{\text{Re}}} \tag{5.19}$$

and ensures that a will remain nearly real on this line. To make $\phi(\tau)$ roughly real on the same line, by Eqs. (5.16) and (5.18) we have to choose

$$\psi_0^{\rm Im} = -\frac{\pi}{2\phi_0^{\rm Re}} \tag{5.20}$$

in the outer approximation.

$\phi_0^{\rm Im}$ in turn is fixed by matching $\phi_{\mathcal{I}}$ to $\phi_{\mathcal{O}}$. Since it is very small, this requires evaluation of Eq. (5.13) to a very high order n. However, we need not calculate any coefficients since, by Eq. (5.14), $\phi_{\mathcal{I}}^{\rm Im}$ is constant along the imaginary axis to any order n:

$$\phi_{\mathcal{I}}^{\rm Im}(\tau^{\rm Im})\big|_{\tau^{\rm Re}=0} = \phi_0^{\rm Im}. \tag{5.21}$$

Therefore it is convenient to choose a matching point τ_M on the imaginary axis:

$$\tau_M^{\rm Re} = 0,\ \tau_M^{\rm Im} = O\left(1/H_0^{\rm Re}\right). \tag{5.22}$$

By Eqs. (5.16) and (5.18) $\phi_{\mathcal{O}}^{\rm Im}$ is also constant along this axis,

$$\phi_{\mathcal{O}}^{\rm Im}(\tau^{\rm Im})\big|_{\tau^{\rm Re}=0} = \psi_0^{\rm Im}, \tag{5.23}$$

so the result of the matching analysis will be independent of the precise choice of τ_M on the axis, as it should be. The matching condition is

$$\phi_{\mathcal{I}}^{\rm Im}(\tau_M) = \phi_{\mathcal{O}}^{\rm Im}(\tau_M) \tag{5.24}$$

and by Eqs. (5.20), (5.21), and (5.23) we obtain

$$\phi_0^{\rm Im} = \psi_0^{\rm Im} = -\frac{\pi}{2\phi_0^{\rm Re}}. \tag{5.25}$$

This result is nontrivial (e.g., $\phi_0^{\rm Re} \neq \psi_0^{\rm Re}$). We now see why the correct value for $\phi_0^{\rm Im}$ is obtained in Ref. [11], although actually only $\psi_0^{\rm Im}$ is calculated there.

We have thus satisfied condition (ii) of Sec. II. By the continuity of the outer approximation, condition (iii) can be satisfied by fine-tuning $\phi_0^{\rm Im}$. Condition (i) is satisfied by the construction of the inner approximation. The only freedom left is the choice of $\phi_0^{\rm Re}$. This variable parametrizes the set of solutions.

To calculate the Euclidean action for the solutions given above, we consider a path going along the real τ axis from the origin to $\tau^{\rm Re} = \frac{\pi}{2H_0^{\rm Re}}$ and then parallel to the imaginary τ axis to $\tau_{\partial M}$. Both a and ϕ are nearly real on the Lorentzian segment of this path, so the real part of the action can be approximated by an integral only over the first segment, using the inner approximation [11]:

$$\begin{aligned} I_{S^3}^{\rm Re} &\approx \frac{3\pi}{2}\int_0^{\pi/2H_0^{\rm Re}} d\tau^{\rm Re}\, a_{\mathcal{I}}\left(\frac{1}{3}a_{\mathcal{I}}^2 m^2\phi_{\mathcal{I}}^2 - 1\right) \\ &\approx -\frac{3\pi}{2m^2(\phi_0^{\rm Re})^2}. \end{aligned} \tag{5.26}$$

The outer approximation is not valid after inflation ends, when $\phi \approx 0$. However, at this point we are already well inside the classical regime. A dust phase will ensue where ϕ oscillates; a and ϕ will both remain real. Approximate solutions for this regime have been given by Hawking and Page [12].

VI. AN INFLATIONARY MODEL WITH BLACK HOLES

We now introduce a massive scalar field on a universe with spacelike sections $S^1 \times S^2$. Thus we will obtain a cosmological model similar to the Nariai universe of Sec. IV. We find the complex solutions, initial conditions and the action in analogy to the previous section, but point out a few differences.

Again we use the Kantowski-Sachs metric

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 dx^2 + b(\tau)^2 d\Omega_2^2 \tag{6.1}$$

and the rescaled field

$$\phi^2 = 4\pi\Phi^2. \tag{6.2}$$

The Euclidean action is

$$\begin{aligned} I = -\pi \int N d\tau\, a\left(\frac{\dot b^2}{N^2} + 2\frac{b}{a}\frac{\dot a \dot b}{N^2} + 1 - b^2\frac{\dot\phi^2}{N^2} - b^2 m^2\phi^2\right) \\ +\pi\left[-\dot a b^2 - 2ab\dot b\right]_{\tau=0}, \end{aligned} \tag{6.3}$$

and like in the previous section the effective cosmological constant is given by

$$\Lambda_{\rm eff}(\tau) = m^2\phi(\tau)^2. \tag{6.4}$$

In analogy to Eq. (4.3) we define

$$H(\tau) = \sqrt{\Lambda_{\rm eff}(\tau)} = m\phi(\tau). \tag{6.5}$$

Variation with respect to a, b, ϕ, and N gives the Euclidean equations of motion and the Hamiltonian constraint:

$$\frac{\ddot b}{b} - \frac{\dot a \dot b}{ab} + \dot\phi^2 = 0, \tag{6.6}$$

$$\frac{\ddot b}{b} + \frac{\dot a \dot b}{ab} + \frac{\ddot a}{a} + \dot\phi^2 + m^2\phi^2 = 0, \tag{6.7}$$

$$\ddot\phi + \left(\frac{\dot a}{a} + 2\frac{\dot b}{b}\right)\dot\phi - m^2\phi = 0, \tag{6.8}$$

$$2\frac{\dot a \dot b}{ab} + \frac{\dot b^2}{b^2} - \frac{1}{b^2} - \dot\phi^2 + m^2\phi^2 = 0. \tag{6.9}$$

The NBP conditions corresponding to an instanton of topology $S^2 \times S^2$ are

$$\begin{aligned} &a = 0,\ \dot a = 1,\ b = b_0,\ \dot b = 0, \\ &\phi = \phi_0 \ \text{ and } \dot\phi = 0 \ \text{ for } \tau = 0. \end{aligned} \tag{6.10}$$

With the new definition (6.5) of H the *inner approximation* is given by

$$a_{\mathcal{I}}(\tau) = \frac{1}{H_0^{\rm Re}}\sin H_0^{\rm Re}\tau, \tag{6.11}$$

$$\phi_{\mathcal{I}}(\tau) = \phi_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\gamma_n \tau^n, \tag{6.12}$$

$$b_{\mathcal{I}}(\tau) = \frac{1}{m\phi_{\mathcal{I}}(\tau)} \tag{6.13}$$

for $|\tau| < O\left(1/H_0^{\rm Re}\right)$. The *outer approximation* is

$$\phi_{\mathcal{O}}(\tau) = \psi_0 + im\tau + \chi_0 \exp(iH_0\tau), \tag{6.14}$$

$$a_{\mathcal{O}}(\tau) = a_0 \exp\left[-im\int_0^\tau \phi(\tau')d\tau'\right] + c_0 \exp\left[im\int_0^\tau \phi(\tau')d\tau'\right], \tag{6.15}$$

$$b_{\mathcal{O}}(\tau) = \frac{1}{m\phi_{\mathcal{O}}(\tau)} \tag{6.16}$$

for $0 < \tau^{\rm Im} \ll \frac{\phi_0^{\rm Re}}{m}$.

A matching analysis completely analogous to that of the previous section shows that a, b, and ϕ will be nearly real on the Lorentzian line $\tau^{\rm Re} = \frac{\pi}{2H_0^{\rm Re}}$, if we choose the initial values

$$\phi_0^{\rm Im} = -\frac{\pi}{2\phi_0^{\rm Re}},\ b_0 = \frac{1}{m\phi_0}; \tag{6.17}$$

$\phi_0^{\rm Re}$ is a free parameter.

An interesting feature of the outer approximation is that the black hole radius grows with the horizon scale during inflation. On the Lorentzian line $\tau^{\rm Re} = \frac{\pi}{2H_0^{\rm Re}}$ the field decreases linearly with time until it reaches zero and inflation ends. By Eqs. (6.14) and (6.16) b becomes very large on the time scale

$$\Delta\tau_{\rm growth} = \frac{\phi_0^{\rm Re}}{m}. \tag{6.18}$$

Again the inner approximation is used to calculate the real part of the Euclidean action. As in Sec. IV it comes entirely from the $\tau = 0$ term:

$$I_{S^1\times S^2}^{\rm Re} \approx -\pi\left(b_0^{\rm Re}\right)^2 \approx -\frac{\pi}{m^2\left(\phi_0^{\rm Re}\right)^2}. \tag{6.19}$$

VII. THE PROBABILITY FOR PRIMORDIAL BLACK HOLES

In the previous two sections we have calculated the action for two inflationary universes. We now compare the corresponding probability measures

$$P_{S^3}(\phi_0^{\rm Re}) = \exp\left(\frac{3\pi}{m^2\left(\phi_0^{\rm Re}\right)^2}\right) \tag{7.1}$$

and

$$P_{S^1\times S^2}(\phi_0^{\rm Re}) = \exp\left(\frac{2\pi}{m^2\left(\phi_0^{\rm Re}\right)^2}\right). \tag{7.2}$$

The universe containing black holes is heavily suppressed, if $\phi_0^{\rm Re}$ is not large enough to make the initial effective cosmological constant equal to the Planck value. Thus the formation of black holes with initial sizes significantly larger than the Planck scale is very unlikely. The semiclassical approximation should be good in these situations, so one can have confidence in this conclusion.

The semiclassical approximation will break down for solutions with initial cosmological constants of the Planck value in a region where the curvature is on the Planck scale. However, this region contributes an action less than one in Planck units and one would not expect quantum effects to change this. Thus it seems clear that the only primordial black holes with any significant probability start with no more than the Planck size:

$$b_0^{\rm Re} \lesssim 1. \tag{7.3}$$

This corresponds to a large initial value of the scalar field

$$\phi_0^{\rm Re} \gtrsim 10^5. \tag{7.4}$$

The Nariai solution is unstable to quantum fluctuations [7]. At the beginning of inflation it becomes a nondegenerate Schwarzschild–de Sitter spacetime. Once the black hole horizon is inside the cosmological horizon the black hole will start to lose mass due to Hawking radiation. If the black hole horizon is somewhat smaller than the cosmological horizon, the black hole will evaporate and disappear. However, there is a significant probability that the areas of the two horizons will be nearly enough equal for them to increase together. The consequences of this result for the global structure of the universe will be presented in a forthcoming paper.

[1] H. Nariai, Sci. Rep. Tohoku Univ. **35**, 62 (1951).
[2] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[3] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[4] A. Arnowitt, S. Deser, and C. Misner, in *Gravitation*, edited by M. J. Bowick and F. Gürsey (World Scientific, Singapore, 1986).
[5] J. J. Halliwell and J. Louko, Phys. Rev. D **42**, 3997 (1990).
[6] R. LaFlamme, Ph.D. thesis, 1988.
[7] P. Ginsparg and M. J. Perry, Nucl. Phys. **B222**, 245 (1983).
[8] S. W. Hawking, Nucl. Phys. **B239**, 257 (1984).
[9] G. Smoot *et al.*, Astrophys. J. Lett. **396**, L1 (1992).
[10] J. J. Halliwell and S. W. Hawking, Phys. Rev. D **31**, 1777 (1985).
[11] G. W. Lyons, Phys. Rev. D **46**, 1546 (1992).
[12] S. W. Hawking and D. N. Page, Nucl. Phys. **B298**, 789 (1988).