Perturbations of an anisotropic spacetime: Ideal fluid

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In a previous paper we presented a general formulation that can treat the evolution of perturbations based on a Bianchi type-I background model universe. In this paper we apply the formulation to an ideal fluid system. The background model shows a smooth transition from the shear-dominated anisotropic stage to the ideal-fluid-dominated isotropic stage. In the shear-dominated stage the background anisotropy causes the couplings of the density perturbation with the gravitational wave and the rotation. The asymptotic solutions in the shear-dominated stage are derived. We investigate the solutions in the comoving gauge and the uniform-curvature gauge. Equations are presented using dimensionless variables; thus, they are suitable for numerical investigation.

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I. INTRODUCTION

In a homogeneous and isotropic universe [we call it the Friedmann-Lemaitre-Robertson-Walker (FLRW) model], inhomogeneities in the linear regime can be decomposed into three different types which are completely decoupled from each other: these are the scalar-type density fluctuation mode, the vector-type rotation mode, and the tensor-type gravitational wave mode. The FLRW model is consistent with current observations in the cosmological scale, and also with diverse theoretical and observational evidence extrapolated back into the early universe. However, as we go back in time, thus, as the universe contracts, a small anisotropy can grow faster than the contribution from the ordinary matter. Thus, if there exists a small global anisotropy at the present time, it could become important in the earlier epoch in the expanding medium. Investigating the evolution of perturbations in a homogeneous and anisotropic universe has some interesting aspects. In general, the background anisotropy causes the couplings among the scalar, vector, and tensor modes. Because of algebraic difficulties and complexity, there has been not as much work compared with the FLRW case.

The previous studies on the perturbation of Bianchi type-I spacetime include the following. The authors of [1] presented the evolution in the Newtonian context. In a relativistic context the authors of [2] considered a system of dust assuming that the perturbations propagate along one of the principal axes of the background anisotropy. The authors of [3] considered an ideal fluid system in which the propagation axis is on the plane of two principal axes. These studies are based on specific gauge conditions. The authors of [2] chose the synchronous gauge condition which coincides with the comoving gauge in a dust medium. The authors of [3] used some gauge-invariant variables which correspond to the uniform-curvature gauge and the comoving gauge variables.

One of the important results obtained by previous studies is that in the anisotropic universe there exist couplings of three different types of modes. In an isotropic background these three modes are decoupled and evolve independently. In an anisotropic background the existence of the background shear causes the coupling between the background shear and all three types of perturbations to the linear order. Thus these terms cause the appearance of intrinsic couplings of different modes in the evolution equations. For example, the gravitational wave perturbation can affect the density perturbation, and vice versa.

In [4], we presented a systematic formulation which can be used for investigating the evolution of perturbations in the Bianchi type-I spacetime. The formulation is designed so that it can be applied to very general situations. It includes full imperfect fluid contributions, and also can treat the system of a multicomponent medium. The fundamental equations are presented in a "gauge-ready form" so that it provides us with the freedom to choose the gauge conveniently.

In this paper we apply the formulation in [4] to the following situation. We consider a Bianchi type-I metric whose energy-momentum content is supplied by an *ideal fluid*, i.e., $p = p(\mu)$. Mainly, we will *consider* an ideal fluid with $w \equiv p/\mu = $ const. Such a system includes the radiation-dominated medium with $w = \frac{1}{3}$, and the dust-dominated medium with $w = 0$. The system shows the transition of the model from the Bianchi type-I universe into a spatially flat FLRW one. Consider a shear-dominated anisotropic universe filled with the fluid medium with constant w. For $w < 1$ the contribution to the background expansion from the anisotropy declines more rapidly than that from the matter contribution; see Eqs. (4) and (5). Thus in a later epoch the evolution becomes the matter-dominated FLRW one. In particular, for the radiation- and dust-dominated medium we can

find analytic solutions describing the background evolution; see Eqs. (A5) and (A6). Using such models we can investigate the evolution of perturbations as they experience the transition.

We will present the asymptotic solutions of the perturbations in the limit of shear domination, assuming that the perturbation scales are larger than the relevant horizon scale appearing in the perturbed equations. In the matter-dominated stage both the equations and the solutions become the ones found in the perturbed flat FLRW model; a thorough investigation in the FLRW case was made in [5]. In general, there exist couplings of all three types of modes, and the evolution of the system will be described by a differential equation higher than second order. In general, we may need a numerical treatment. A detailed numerical investigation of the cases where the ideal fluid is radiation or dust will be considered in a subsequent work.

In an ideal fluid case, we will consider mainly the comoving gauge and the uniform-curvature gauge. The spatial gauge freedoms are uniquely fixed by taking the C-gauge condition. In [4] we found that six different temporal gauge choices are available; see below Eq. (26). Since the synchronous gauge cannot fix the gauge mode completely we will not consider this gauge. Since each of the other five gauge conditions completely fixes the temporal gauge transformation property, each variable in any of these gauges has a unique corresponding gauge-invariant combination of variables. Thus, without losing generality, we can identify the variables in such gauge choices with the gauge-invariant ones. In an ideal fluid medium, since $p = p(\mu)$, the uniform-pressure gauge becomes identical to the uniform-density gauge. When we are interested in the evolution of the density perturbation the uniform-density gauge choice is not a particularly illuminating one. Thus we exclude these two gauge choices from consideration. In the FLRW case, it is known that the variables in the uniform-expansion gauge show very different behavior depending on whether the scale lies inside or outside the visual horizon; see [5]. Thus the general behavior is usually complicated in this gauge. We also exclude this gauge choice from our consideration in this paper. In the FLRW case, the density perturbation variable in the comoving gauge shows apparently similar behavior to the Newtonian one [5]. Such a similarity with a well known system is important for appropriate interpretations. We expect that such a similarity also appears in the anisotropic case compared with other gauge cases. In the FLRW case, the behavior of the perturbed potential and the perturbed velocity variables in the zero-shear gauge are most similar to the Newtonian ones. Although there exists shear in the background anisotropic universe, we can construct a uniform-shear (vanishing scalar part of the perturbed shear) condition as a proper temporal gauge-fixing condition; see Sec. III B 1 of [4]. However, it is interesting to note that the perturbed potential in the comoving gauge shows simpler behavior compared with the one in the zero-shear gauge. In the comoving gauge the perturbed potential is simply conserved independently of the changes in the background equation of state, and this is not true in the zero-shear gauge; see Ta-

ble 8 of [5]. In a scalar-field-dominated medium we know that the uniform-curvature gauge conveniently describes the perturbed scalar field. Compared with the ones in other gauges, the equation for the perturbed scalar field in the uniform-curvature gauge shows the most similar structure to the equation for the background scalar field [6]. The evolution of the perturbed scalar field in the anisotropic background will be considered in subsequent work.

In Sec. II A, we present the background evolution. In Sec. II B the variables are expanded in plane waves with the wave vector which is in a plane of two principal axes of the background anisotropy. In Sec. II C the variables are presented in dimensionless forms. In Sec. II D we present the complete set of perturbation equations without fixing the temporal gauge freedom; i.e., in a gauge-ready form. Comments on the parameter space which characterizes the background and perturbation configurations are made in Sec. II E. In Sec. III, we present the asymptotic solutions in the shear-dominated stage in the comoving gauge and the uniform-curvature gauge conditions. In the large scale limit, the solutions cover the parameter space of all background configurations. In Sec. IV, a discussion is given. As a unit we set $c \equiv 1$.

II. EQUATIONS

A. Ideal fluid background

For the background model, we consider the Bianchi type-I metric. It is spatially flat, homogeneous, and anisotropic. The equations are derived in Eqs. (7)–(10) of [4]. We *assume* that this background is supported by an ideal fluid, and ignore the cosmological constant:

$$\Pi_{\alpha\beta} = 0, \quad p = p(\mu), \quad \Lambda = 0, \tag{1}$$

where $\Pi_{\alpha\beta}(t)$, $p(t)$, $\mu(t)$, and Λ are the background anisotropic pressure, pressure, energy density, and cosmological constant, respectively. The background equations become

$$\dot{\mu} + 3\dot{s}\left(1 + w\right)\mu = 0, \tag{2}$$

$$\ddot{s}_\alpha + 3\dot{s}\dot{s}_\alpha = 0, \tag{3}$$

$$\dot{s}^2 = \frac{8\pi G\mu}{3} + \frac{1}{6}\sum_\alpha \dot{s}_\alpha^2, \tag{4}$$

where $w \equiv p/\mu$, $s \equiv s(t)$, $s_\alpha \equiv s_\alpha(t)$, and $\alpha = 1, 2, 3$. An overdot denotes the derivative with respect to the background proper time t. Assuming a constant ratio between the pressure and the energy density, $w \equiv p/\mu = $ const, from Eqs. (2) and (3) we have

$$\dot{s}_\alpha \propto e^{-3s}, \quad \mu \propto e^{-3(1+w)s}. \tag{5}$$

Solutions of Eqs. (2)–(4) are presented in the Appendix.

B. Notation

Our set of perturbation equations is obtained by using equations based on the Arnowitt-Deser-Misner (ADM)

formulations. We summarize the definitions of variables. For more details, see Sec. II of [4]. The metric is introduced as

$$g_{00} \equiv -e^{2s}(1 + 2A), \quad g_{0\alpha} \equiv e^{2s}B_\alpha,$$
$$g_{\alpha\beta} \equiv e^{2s}(\gamma_{\alpha\beta} + C_{\alpha\beta}), \tag{6}$$

where $\gamma_{\alpha\beta} \equiv e^{2s\alpha}\delta_{\alpha\beta}$. $A(\mathbf{x},t)$, $B_\alpha(\mathbf{x},t)$, and $C_{\alpha\beta}(\mathbf{x},t)$ are the perturbed order metric variables. The vector- and the tensor-type perturbation variables are decomposed into the scalar-, vector-, and tensor-type modes as

$$B_\alpha \equiv B_{,\alpha} + B_\alpha^{(v)},$$
$$C_{\alpha\beta} \equiv C\gamma_{\alpha\beta} + \bar{C}_{,\alpha\beta} + 2C_{(\alpha,\beta)}^{(v)} + C_{\alpha\beta}^{(t)},$$
$$Q_\alpha \equiv Q_{,\alpha} + Q_\alpha^{(v)}. \tag{7}$$

The tensor indices of the perturbed variables are based on $\gamma_{\alpha\beta}$. Q_α is the perturbed energy flux. In an ideal fluid, the perturbed anisotropic pressure vanishes; thus $\delta\Pi_{\alpha\beta} = 0$. The superscripts (v) and (t) denote the vector- and the tensor-type modes, respectively, where

$$B_\alpha^{(v)|\alpha} \equiv 0, \quad C_{\alpha\beta}^{(t)|\alpha} \equiv 0 \equiv C^{(t)\alpha}{}_\alpha, \quad \text{etc.} \tag{8}$$

We impose these conditions for decomposed variables *at all times*. Because of the transverse (divergenceless) condition, the vector mode variable has two independent components. Because of the transverse and traceless conditions, the symmetric tensor mode based on the spatial metric has two independent components.

In general, the perturbations of all three types are coupled in an anisotropic background. Thus, our decomposition of variables into the scalar-, vector-, and tensor-type modes becomes just a mathematical way of classifying them. Since these modes are coupled through equations describing their evolutions together, they lose their intrinsic meanings as the scalar, vector, and tensor modes; one mode can generate the other and vice versa. From the evolution equations we can identify that the coupling of the perturbed variables is induced through the nonvanishing background shear term. Thus these couplings will disappear in the isotropic background limit.

Without losing any generality or any mathematical convenience, we will fix the spatial (including the spatial scalar and the vector types) gauge freedom using the C-gauge condition (see Sec. III B of [4])

$$\bar{C} \equiv 0 \equiv C_\alpha^{(v)}. \tag{9}$$

We expand the variables in plane waves; $A(\mathbf{x},t) \propto \int e^{i\mathbf{k}\cdot\mathbf{x}}A_\mathbf{k}(t)d^3k$, etc. This is possible because the Bianchi type-I model is spatially flat. Since the $\mathbf{k}$-space variables satisfy the same form of equations as the variables in $\mathbf{x}$ space, we ignore the $\mathbf{k}$ subindices in $A_\mathbf{k}(t)$, etc. We *consider* the situation where the wave propagation vector of the perturbation k_α lies on the plane of two principal axes, say $\hat{x}_2$ and $\hat{x}_3$. Thus $k_\alpha \equiv (0, k_2, k_3)$. From Eq. (8) we can show (see Appendix F of [4])

$$Q_3^{(v)} = -\frac{k^2}{k^3}Q_2^{(v)}, \quad Q^{(v)3} = -\frac{k^2}{k_3}Q_2^{(v)}, \tag{10}$$

$$C^{(t)2}{}_2 = -\frac{k^3 k_3}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}{}_1, \quad C^{(t)3}{}_3 = -\frac{k^2 k_2}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}{}_1,$$

$$C^{(t)3}{}_2 = \frac{k^3 k_2}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}{}_1, \quad C^{(t)2}{}_3 = \frac{k^2 k_3}{\mathbf{k}\cdot\mathbf{k}}C^{(t)1}{}_1,$$

$$C^{(t)1}{}_3 = -\frac{k^2}{k^3}C^{(t)1}{}_2, \quad C^{(t)3}{}_1 = -\gamma_{11}\frac{k^2}{k_3}C^{(t)1}{}_2,$$

$$C^{(t)2}{}_1 = \gamma_{11}\gamma^{22}C^{(t)1}{}_2, \quad \text{etc.} \tag{11}$$

We will take $B_1^{(v)}$ and $B_2^{(v)}$ as two independent components for the vector mode, and $C^{(t)1}{}_1$ and $C^{(t)1}{}_2$ as two independent components for the tensor mode.

C. Nondimensionalization

It is convenient (for numerical investigation, in particular) to express the equations using dimensionless variables. We introduce the dimensionless variables

$$\bar{B} \equiv a\dot{s}B, \quad \delta\bar{K} \equiv \frac{1}{\dot{s}}\delta K, \quad \bar{Q} \equiv \frac{a\dot{s}}{\mu}Q,$$

$$B_v \equiv \frac{ik^2}{a\dot{s}}B_2^{(v)}, \quad Q_v \equiv \frac{ik^2}{a\dot{s}}\frac{1}{\mu}Q_2^{(v)}, \quad G \equiv \frac{1}{2}C^{(t)1}{}_1,$$

$$\bar{B}_v \equiv \frac{ik_2}{a\dot{s}}B^{(v)1}, \quad \bar{Q}_v \equiv \frac{ik_2}{a\dot{s}}\frac{1}{\mu}Q^{(v)1}, \quad \bar{G} \equiv \frac{1}{2}C^{(t)1}{}_2, \tag{12}$$

where $a \equiv e^s$. The variables A, C, and δ are dimensionless. We also introduce

$$\Delta \equiv -(k^2 k_2 + k^3 k_3), \quad \bar{\Delta} \equiv \frac{\Delta}{(a\dot{s})^2}. \tag{13}$$

We use $\bar{\Delta}\bar{B}$ as a variable. We can show that

$$\frac{\bar{\Delta}'}{\bar{\Delta}} = \frac{\Delta'}{\Delta} - 2\left(1 + \frac{\ddot{s}}{\dot{s}^2}\right), \tag{14}$$

where a prime denotes the time derivative using s as the time variable, $' \equiv \frac{\partial}{\partial s}$.

Under the gauge transformation, G, $\bar{G}$, B_v, and $\bar{B}_v$ transform as (see Appendix C of [4])

$$\hat{G} = G - \left(\dot{s}_1 - \frac{\dot{\Delta}}{4\Delta}\right)\xi^t,$$

$$\hat{\bar{G}} = \bar{G},$$

$$\hat{B}_v = B_v - ik_2\xi^{(v)2\prime} + i\left(2s_2' + \frac{\Delta'}{\Delta}\right)\xi_{,2}k^2,$$

$$\hat{\bar{B}}_v = \bar{B}_v - ik_2\xi^{(v)1\prime}, \tag{15}$$

where ξ^α is based on g_{ab}. For its definition, see Sec. III A of [4]. Under the C-gauge condition for spatial gauge fixing, from Eq. (40) of [4] we have

$$\xi = \frac{3}{4}\frac{\dot{\Delta}}{\Delta^2}\xi^t,$$

$$\xi_1^{(v)} = 0, \quad \xi_2^{(v)} = -\frac{1}{\Delta}\left(2\dot{s}_2 + \frac{\dot{\Delta}}{\Delta}\right)\xi^t{}_{,2}. \tag{16}$$

Thus under the C-gauge condition $\bar{B}_v$ becomes gauge free and B_v is dependent on ξ^t. With vanishing background

anisotropic pressure $Q_\alpha^{(v)}$, and thus Q_v and $\bar{Q}_v$, are gauge invariant; see Eq. (C13) of [4].

D. Equations in a gauge-ready form

In the following we present two sets of complete perturbation equations using dimensionless variables without fixing the temporal gauge. We assume an ideal fluid, thus $\delta\Pi_{\alpha\beta} = 0$ and $\pi = c_s^2\varepsilon$, where $c_s^2 \equiv \dot{p}/\dot{\mu}$; since $\dot{w} = -3\dot{s}(1+w)(c_s^2 - w)$, for $w = $ const we have $c_s^2 = w$.

In an ideal fluid situation together with the wave propagation in a plane of two principal axes, we find that there exist two sets of perturbations. In one set the scalar mode is coupled with one polarization state of the tensor mode, and one vector mode works as a source. In the other set we find a decoupled polarization state of the tensor mode with the other vector mode as a source. In both cases the vector modes work as source terms. The equations can be derived from Eqs. (D1)–(D12) of [4] using the notation presented in Secs. II B and II C.

For the set involving the scalar mode we have

$$\delta\bar{K} = 3A + \bar{\Delta}\bar{B} - \frac{3}{2}C', \tag{17}$$

$$\delta' + 3\left(c_s^2 - w\right)\delta + (1+w)\left(3A - \delta\bar{K}\right) + \bar{\Delta}\bar{Q} = 0, \tag{18}$$

$$\bar{Q}' - \left(3w + \frac{\ddot{s}}{\dot{s}^2}\right)\bar{Q} + c_s^2\delta + (1+w)A = -2\left(s_2' - s_3'\right)\frac{1}{\bar{\Delta}}Q_v, \tag{19}$$

$$\delta\bar{K}' + \left(2 + \frac{\ddot{s}}{\dot{s}^2}\right)\delta\bar{K} + \left[\bar{\Delta} + 3\frac{\ddot{s}}{\dot{s}^2} + 2\sum_\alpha s_\alpha'^2\right]A - \frac{\Delta'}{\Delta}\bar{\Delta}\bar{B} - \left(1 + 3c_s^2\right)\frac{4\pi G\mu}{\dot{s}^2}\delta$$

$$= -2\left(s_2' - s_3'\right)B_v + 4\left(s_1' - \frac{\Delta'}{4\Delta}\right)G' - 4\left(s_2' - s_3'\right)^2\frac{k^2k_2k^3k_3}{\Delta^2}G, \tag{20}$$

$$\frac{16\pi G\mu}{\dot{s}^2}\delta + 4\delta\bar{K} - 2\sum_\alpha s_\alpha'^2 A + \frac{\Delta'}{\Delta}\bar{\Delta}\bar{B} + 2\bar{\Delta}C$$

$$= 2\left(s_2' - s_3'\right)B_v - 4\left(s_1' - \frac{\Delta'}{4\Delta}\right)G' + 4\left(s_2' - s_3'\right)^2\frac{k^2k_2k^3k_3}{\Delta^2}G, \tag{21}$$

$$\frac{8\pi G\mu}{\dot{s}^2}\bar{Q} + \frac{2}{3}\left(\delta\bar{K} - \bar{\Delta}\bar{B}\right) + \frac{\Delta'}{2\Delta}\left(A - \frac{3}{2}C\right) = 2\left(s_1' - \frac{\Delta'}{4\Delta}\right)G, \tag{22}$$

$$\left(\bar{\Delta}\bar{B}\right)' + \left(3 + \frac{\ddot{s}}{\dot{s}^2} - \frac{3}{4}\frac{\Delta'}{\Delta}\right)\bar{\Delta}\bar{B} - \frac{3}{4}\frac{\Delta'}{\Delta}\left(A' - \frac{3}{2}C'\right) + \bar{\Delta}\left(A + \frac{1}{2}C\right)$$

$$= -3\left(s_2' - s_3'\right)B_v - 6\left(s_2' - s_3'\right)^2\frac{k^2k_2k^3k_3}{\Delta^2}G, \tag{23}$$

$$G'' + \left(3 + \frac{\ddot{s}}{\dot{s}^2}\right)G' - \left[\bar{\Delta} + 2\left(s_2' - s_3'\right)^2\frac{k^2k_2k^3k_3}{\Delta^2}\right]G = \left(s_1' - \frac{\Delta'}{4\Delta}\right)\left(A' - 3A + \delta\bar{K}\right) + \left(s_2' - s_3'\right)B_v, \tag{24}$$

$$Q_v' + \left(2 - 3w + \frac{\ddot{s}}{\dot{s}^2} - 2\frac{k^2k_2 s_3' + k^3k_3 s_2'}{\Delta}\right)Q_v = 0, \tag{25}$$

$$\frac{8\pi G\mu}{\dot{s}^2}Q_v - \frac{1}{2}\bar{\Delta}B_v = \frac{k^2k_2}{(a\dot{s})^2}\frac{k^3k_3}{\Delta}\left(s_2' - s_3'\right)\left(A - \frac{3}{2}C + 2G\right). \tag{26}$$

We have the freedom to impose one temporal gauge condition, which removes one variable of $\bar{Q}$, C, $\delta\bar{K}$, and δ. These gauge conditions are called the comoving gauge ($\bar{Q} \equiv 0$), the uniform-curvature gauge ($C \equiv 0$), the uniform-expansion gauge ($\delta\bar{K} \equiv 0$), and the uniform-density gauge ($\delta \equiv 0$). Any of these four gauge conditions fixes the temporal gauge transformation property completely. Together with the spatial C gauge which we already took, any of these gauge conditions fixes all gauge modes completely. Thus using these gauge conditions all variables become gauge free. In the same sense we can construct a unique corresponding gauge-

invariant combination for any variable; see Sec. III C of [4]. The synchronous gauge condition ($A = 0$) cannot fix the temporal gauge condition completely. Thus in this gauge condition we may still have the remnant gauge mode appearing in the solutions; thus the equation should be higher order. In a dust medium, ignoring the rotation mode, the synchronous gauge condition coincides with the comoving gauge condition; see Eq. (19). The nonvanishing solution for $\bar{Q}$ of Eq. (19) under the synchronous gauge is the remaining gauge mode in that gauge. The rotation mode Q_v works only as a source; see Eq. (25). After fixing the gauge, and ignoring the rota-

tion mode, we have a fourth order differential equation for the coupled scalar and the tensor mode.

The gravitational wave has two polarization states. One state (G) couples with the scalar mode. The other state ($\bar{G}$) is completely decoupled from the scalar mode and evolves independently. However, $\bar{G}$ couples with one of the vector modes $\bar{Q}_v$ (and $\Delta\bar{B}_v$). The equations describing the evolution are

$$\bar{Q}_v' + \left(2 - 3w + \frac{\ddot{s}}{\dot{s}^2} + 2s_1'\right)\bar{Q}_v = 0, \tag{27}$$

$$\Delta\bar{B}_v = \frac{16\pi G\mu}{\dot{s}^2}\bar{Q}_v - 4\,(s_2' - s_3')\,\frac{k^2 k_2}{(a\dot{s})^2}\bar{G}, \tag{28}$$

$$\bar{G}'' + \left[3 + \frac{\ddot{s}}{\dot{s}^2} + 2\,(s_1' - s_2') + 2\,(s_2' - s_3')\,\frac{k^2 k_2}{\Delta}\right]\bar{G}'$$

$$+ \left[-\Delta + 4\,(s_2' - s_3')\,(s_1' - s_2')\,\frac{k^2 k_2}{\Delta}\right]\bar{G}$$

$$= (s_2' - s_3')\,\frac{k^3 k_3}{\Delta}\bar{B}_v. \tag{29}$$

The rotation mode $\bar{Q}_v$ works only as a source for $\bar{G}$. This decoupled gravitational wave is determined by a second order differential equation with $\bar{Q}_v$ as a source. From the gauge transformation properties presented in Appendix C of [4] we showed that $\bar{Q}_v$ and $\bar{G}$ are gauge invariant; see Eq. (15). $\Delta\bar{B}_v$ depends on the rotational gauge transformation; under the C-gauge condition it is gauge free; see Eqs. (15) and (16).

E. Parameter space

The background anisotropy $s_\alpha(t)$ can be normalized using S_α; see the Appendix. From Eq. (A3) we have

$$\sum_\alpha S_\alpha = 0, \quad \sum_\alpha S_\alpha^2 = 6. \tag{30}$$

Thus S_α can be parametrized using a parameter u as

$$\sqrt{3}\frac{-1}{\sqrt{1 + u + u^2}}, \quad \sqrt{3}\frac{-u}{\sqrt{1 + u + u^2}}, \quad \sqrt{3}\frac{1 + u}{\sqrt{1 + u + u^2}}. \tag{31}$$

We present $S_\alpha(u)$ in Fig. 1.

We denote $S_\alpha = (S_1, S_2, S_3)$, where the $\hat{x}_\alpha$'s are aligned with the background principal axes. Assuming that $S_\alpha = (S_1, S_2, S_3)$ is parametrized by Eq. (31), as we change $u \to -u - 1$ we have the symmetry

$$S_1(-u - 1) = S_1(u), \quad S_2(-u - 1) = S_3(u),$$

$$S_3(-u - 1) = S_2(u). \tag{32}$$

S_α can be related to the Kasner exponent p_α introduced in [7] as

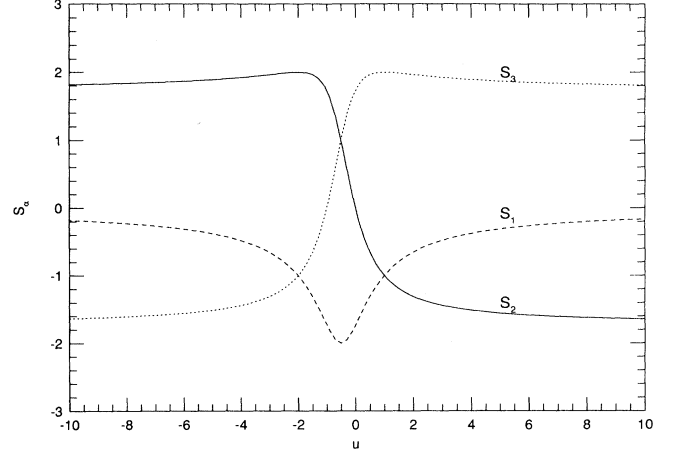$$p_\alpha = \frac{1 + S_\alpha}{3}, \tag{33}$$

FIG. 1. $S_\alpha(u)$: The background expansion rates in three directions $S_\alpha = (S_1, S_2, S_3)$ using a parameter u. In this plot we consider that S_α is assigned as in Eq. (31). We have $S_\alpha = (0, -\sqrt{3}, \sqrt{3})$, $(-1, -1, 2)$, $(-2, 1, 1)$ at $u \to \pm\infty$, 1, $-1/2$, respectively. For positive (or negative) S_α, the background in the α direction expands (or contracts) relative to the average expansion $a = e^s$.

where $\sum_\alpha p_\alpha = \sum_\alpha p_\alpha^2 = 1$. A different parametrization of p_α (thus S_α) can be found in [7,8].

We introduce

$$r \equiv \frac{k_2}{k_3}, \tag{34}$$

which indicates the direction of the wave vector; k_α is independent of time whereas $k^\alpha = \gamma^{\alpha\beta}k_\beta$ depends on time.

The configuration of the background symmetry can be characterized by u. Since we consider the wave vector lying in the plane of two principal axes, the direction and the size of the perturbed wave propagation vector can be characterized by using r and k_3.

In the following we make brief comments on some special situations of the background anisotropy and the perturbations. Since the decoupled set in Eqs. (27)–(29) is rather trivial, we will be mainly concerned with the set involving the scalar mode in Eqs. (17)–(26).

(a) If we cover all the wave vectors in a plane of $\hat{x}_2$ and $\hat{x}_3$, a given background configuration $S_\alpha = (S_1, S_2, S_3)$ is equivalently covered by investigating the one with S_2 and S_3 exchanged. In Sec. III we will derive the asymptotic solutions for $S_3 > S_2$ and $S_2 = S_3$. Thus the solutions for $S_3 < S_2$ can be obtained by investigating the situation for $S_\alpha = (S_1, S_3, S_2)$.

(b) For $S_2 = S_3$, we have two cases: $S_\alpha = (-2, 1, 1)$ and $(2, -1, -1)$. From Eqs. (17)–(26) we notice that the rotation mode completely decouples; the same is true for Eqs. (27)–(29). The system is independent of the direction of k_α, and thus independent of r.

(c) For $S_1 = S_2$, we have $S_\alpha = (-1, -1, 2)$ or $(1, 1, -2)$. In the former case, in the shear-dominated era the scalar mode decouples from the tensor mode. Such decoupling

also occurs for $k_2 = 0$ in the $S_1 = S_2$ case; see case (e) below. In this case, the asymptotic solutions are separately derived in Secs. III A 2 and Sec. III B. However, in the latter case, the scalar and the tensor modes remain coupled even in the shear-dominated stage. As mentioned in (a) the solutions in the latter case can be covered by the one with $S_\alpha = (1, -2, 1)$, where $S_3 > S_2$.

(d) For $r = 0$ ($k_2 = 0$) many terms vanish, but all three types of perturbations are coupled generally. From Eq. (26) B_v is completely determined by Q_v. A k_2 term always appears together with $(s'_2 - s'_3)$, but not vice versa. Thus the $k_2 = 0$ situation can be included as a special case of the $S_2 = S_3$ situation.

(e) For $k_2 = 0$ and $S_1 = S_2$, the scalar and the tensor modes decouple. However, B_v still works as a source for the scalar and the tensor modes.

(f) The system with a maximally different background anisotropy appears when $u = \pm\infty$; see Fig. 1. If we consider $S_3 > S_2$, we have three different cases: $S_\alpha = (0, -\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, 0, \sqrt{3})$, and $(\sqrt{3}, -\sqrt{3}, 0)$.

III. ASYMPTOTIC SOLUTIONS IN THE SHEAR-DOMINATED STAGE

We consider the shear-dominated stage in the Bianchi type-I model filled with an ideal fluid. We *assume* $w = $ const. The background solutions are presented in the Appendix. In the shear-dominated stage we have

$$\frac{\ddot{s}}{\dot{s}^2} = -3, \quad s'_\alpha = S_\alpha, \quad \sum_\alpha s'^2_\alpha = 6. \quad (35)$$

We *ignore* the rotation modes which have the role of sources. We *consider* the case with $S_3 > S_2$. In this case, in the early enough shear-dominated stage, we have

$$\sqrt{\frac{k^2 k_2}{k^3 k_3}} = a^{S_3 - S_2} \frac{k_2}{k_3} = (e^s)^{S_3 - S_2} r \ll 1. \quad (36)$$

We normalized the scale factor so that $a(\equiv e^s) \simeq 1$ near the transition epoch from the shear-dominated era to the matter-dominated era. Thus $a \ll 1$ in the shear-dominated era. Thus $\Delta = -k^3 k_3$ and

$$\frac{\Delta'}{\Delta} = -2S_3, \quad \frac{\bar{\Delta}'}{\bar{\Delta}} = 4 - 2S_3. \quad (37)$$

We can show the exact relation

$$\frac{4}{\Delta^2} (s'_2 - s'_3)^2 k^2 k_2 k^3 k_3 = \left(\frac{\Delta'}{\Delta}\right)' + \left(3 + \frac{\ddot{s}}{\dot{s}^2}\right) \frac{\Delta'}{\Delta}. \quad (38)$$

Except for terms in Eq. (25), the $k^2 k_2$ and $k^3 k_3$ terms appear together with $(s'_2 - s'_3)$ term in Eqs. (17)-(26). For $S_2 = S_3$, Eq. (25) leads to the same equation as for the $S_3 > S_2$ situation, and Eqs. (36) and (37) remain valid. Thus the $S_2 = S_3$ case belongs to the case with $S_3 > S_2$. For $S_2 = S_3$, apparently, the general evolution should not depend on r. As mentioned in case (a) of Sec.

II E, the case with $S_3 < S_2$ is effectively covered by the case with S_2 and S_3 exchanged.

In the following, we present the large scale asymptotic solutions in the shear-dominated stage. We consider the comoving gauge and the uniform-curvature gauge conditions. In principle, the solutions in the other gauge conditions can be derived from the known solutions in a gauge by using the gauge transformation; see below Eq. (66). The asymptotic solutions can be used as the initial conditions for numerical investigation which will be presented in subsequent work.

For comments on small scale solutions in the shear-dominated regime, see Sec. VIII of [2].

A. The comoving gauge

In the comoving gauge we set $\bar{Q} \equiv 0$.

1. Large scale limit with $S_3 \geq S_2$

In order to derive the large scale asymptotic solutions in the shear-dominated stage, since we assume $S_3 > S_2$, we neglect $k^2 k_2 / \Delta$ terms; see Eqs. (36) and (38). In the shear-dominated stage the $4\pi G\mu/\dot{s}^2$ term is negligible compared with unity. As mentioned below Eq. (38), the solutions in the following include the situation with $S_2 = S_3$. However, we keep $\bar{\Delta}$ terms in order to derive the solutions for every variable to nonvanishing order in the large scale expansion. Keeping $\bar{\Delta}$ order terms, from Eqs. (17)-(19), (22)-(24), and (26) we can derive

$$\delta'' + 3(1 - w)\delta' + \frac{8}{9}\frac{2 - S_3}{S_3^2}\bar{\Delta}\delta' + \frac{w}{3}\left(-7 + \frac{8}{S_3}\right)\bar{\Delta}\delta$$

$$= -\frac{8}{9}(1 + w)\frac{S_1 - S_2}{S_3^2}\bar{\Delta}\left(1 + \frac{4}{9}\frac{\bar{\Delta}}{S_3^2}\right)\left(G' + \frac{3}{2}S_3 G\right), \quad (39)$$

$$G'' - \bar{\Delta}G = \frac{S_1 - S_2}{2}\frac{1 - w}{1 + w}\delta'. \quad (40)$$

Combining these two equations, to the leading order in $\bar{\Delta}$, we can derive

$$[\delta' + 3(1 - w)\delta]''' - 4(2 - S_3)[\delta' + 3(1 - w)\delta]''$$

$$+ 4(2 - S_3)^2[\delta' + 3(1 - w)\delta]' = 0, \quad (41)$$

$$[G' + 3(1 - w)G]'' = 0. \quad (42)$$

For the derivation it is convenient to use $(S_1 - S_2)^2 = 3(4 - S_3^2)$.

The large scale asymptotic solutions in the shear-dominated stage are (we present to nonvanishing order in the expansion of $\bar{\Delta}$)

$$\delta(\mathbf{x},t) \equiv (1+w)a_1(\mathbf{x})e^{-3(1-w)s} + (1+w)\left[1 + \frac{w}{2(2-S_3)(3+w-2S_3w)}\bar{\Delta}\right]a_2(\mathbf{x})$$

$$+(1+w)\bar{\Delta}\left[1 + \frac{3+5w-4S_3w}{8(2-S_3)^2(11-4S_3-3w)}\bar{\Delta}\right]a_3(\mathbf{x}) + (1+w)s\bar{\Delta}a_4(\mathbf{x}),\tag{43}$$

$$C(\mathbf{x},t) = -\frac{1}{3}(2-S_3)a_1e^{-3(1-w)s} - w(1-w)\frac{2-S_3}{3+w-2S_3w}a_2 - \left[(2-S_3)(7-2S_3-3w) + \frac{3+w-2S_3w}{4(2-S_3)}\bar{\Delta}\right]a_3$$

$$-\frac{1}{2}\left[11-4S_3-3w+2(2-S_3)(7-2S_3-3w)s\right]a_4,\tag{44}$$

$$\delta\bar{K}(\mathbf{x},t) = -3a_1e^{-3(1-w)s} + w\left[-3 + \frac{2(2-S_3)-3w}{2(2-S_3)(3+w-2S_3w)}\bar{\Delta}\right]a_2$$

$$+\bar{\Delta}\left[4-2S_3-3w+\frac{3+5w-4S_3w}{8(2-S_3)^2(11-4S_3-3w)}(8-4S_3-3w)\bar{\Delta}\right]a_3 + \left[1+(4-2S_3-3w)s\right]\bar{\Delta}a_4,\tag{45}$$

$$\Delta\bar{B}(\mathbf{x},t) = -\frac{3}{2}(1-w)S_3a_1e^{-3(1-w)s} + \frac{w}{(3+w-2S_3w)}\bar{\Delta}a_2 + \frac{1}{4}(7-3w-8S_3+6S_3w)\bar{\Delta}a_3$$

$$-\frac{3}{2}(2-S_3)(7-2S_3-3w)a_4,\tag{46}$$

$$G(\mathbf{x},t) = -\frac{1}{6}(S_1-S_2)a_1e^{-3(1-w)s} + \frac{w}{2(S_1-S_2)(3-2S_3w+w)}\left[(2-S_3)(7S_3w-2w-6-3S_3) - w\bar{\Delta}\right]a_2$$

$$+\left\{-\frac{3}{2}\frac{S_3}{S_1-S_2}(2-S_3)(7-2S_3-3w) + \frac{3}{8(S_1-S_2)(2-S_3)}\left[8-7S_3+w\left(-8+3S_3+2S_3^2\right)\right]\bar{\Delta}\right\}a_3$$

$$+\frac{1}{2}\frac{1}{S_1-S_2}\left[4(7-2S_3-3w) + \frac{1}{2}S_3(-61+20S_3+21w) - 3S_3(2-S_3)(7-2S_3-3w)s\right]a_4,\tag{47}$$

where $a_1(\mathbf{x})$, $a_2(\mathbf{x})$, $a_3(\mathbf{x})$, and $a_4(\mathbf{x})$ are four integration constants. $a_i(\mathbf{x})$ determine the initial amplitudes of the four different modes. As long as we do not have the physical mechanism which determines the relation between the amplitudes, the amplitude of each mode can be given arbitrarily as the initial condition. A can be determined simply by using Eq. (19) as

$$A = -\frac{w}{1+w}\delta.\tag{48}$$

For a dust medium we have $A = 0$ which corresponds to the synchronous gauge condition.

For $S_1 = S_2$, some of the solutions in Eqs. (43)-(47) diverge. We have two cases $S_\alpha = (-1,-1,2)$ and $(1,1,-2)$. From Eqs. (39) and (40) we notice that for $S_1 = S_2$ the scalar and the tensor modes decouple; in Eqs. (39) and (40) we already assume $S_3 > S_2$ and thus $S_\alpha = (-1,-1,2)$. Solutions for the $S_\alpha = (-1,-1,2)$ case will be derived separately in Sec. III A 2. The case with $S_\alpha = (1,1,-2)$ is covered by $S_\alpha = (1,-2,1)$ where no divergence occurs in the asymptotic solutions.

From Eqs. (43) and (47) we have solutions for δ and G to the lowest order as

$$\delta \propto e^{-3(1-w)s}, \quad \text{const}, \quad \bar{\Delta}, \quad s\bar{\Delta},\tag{49}$$

$$G \propto e^{-3(1-w)s}, \quad \text{const}, \quad \text{const}, \quad s.\tag{50}$$

From Eq. (37) we have

$$\bar{\Delta} \propto e^{2(2-S_3)s}.\tag{51}$$

In terms of t, since $a \propto t^{1/3}$, we have

$$\delta \propto t^{-(1-w)}, \quad \text{const}, \quad t^{\frac{2}{3}(2-S_3)}, \quad t^{\frac{2}{3}(2-S_3)}\ln t.\tag{52}$$

$$G \propto t^{-(1-w)}, \quad \text{const}, \quad \text{const}, \quad \ln t.\tag{53}$$

In the FLRW case, for constant w, the solutions for δ and G in the comoving gauge are (see Appendix G of [4])

$$\delta \propto t^{-\frac{1-w}{1+w}}, \quad t^{\frac{2}{3}\frac{1+3w}{1+w}}, \qquad G \propto t^{-\frac{1-w}{1+w}}, \quad \text{const};$$

$$a \propto t^{\frac{2}{3(1+w)}}.\tag{54}$$

In the FLRW limit for $w = 1$ we have $a \propto t^{1/3}$. For δ, the first and the second solution in Eq. (52) reproduce the decaying mode of the FLRW solution, whereas the third one reproduces the growing mode. In the FLRW case we have $G \propto \text{const}$. The first three solutions in Eq. (53) remain constant.

2. $S_1 = S_2$ case

In the shear-dominated stage, for $S_1 = S_2$ we can show that the scalar mode decouples from the tensor mode. This decoupling occurs only for $S_3 > S_2$, where we have Eqs. (39) and (40). Thus for the decoupling we need $S_\alpha = (-1,-1,2)$; as mentioned in case (a) of Sec. II E, $S_\alpha = (1,1,-2)$ can be covered by $S_\alpha = (1,-2,1)$, where $S_3 > S_2$. From Eq. (37) we have $\bar{\Delta} \propto \text{const}$. For the tensor mode we have

$$G \propto \text{const}, \quad s.\tag{55}$$

These two solutions can be considered as a_2 and a_4 modes in Eq. (47). For the scalar mode we have

$$\delta = (1+w)a_1 e^{-3(1-w)s} + (1+w)\bar{\Delta}a_3, \qquad (56)$$

$$C = 0, \qquad (57)$$

$$\delta \bar{K} = -3a_1 e^{-3(1-w)s} - 3w\bar{\Delta}a_3, \qquad (58)$$

$$\bar{\Delta}\bar{B} = -3(1-w)a_1 e^{-3(1-w)s}. \qquad (59)$$

A follows from Eq. (48). These solutions can be included in the leading order general solutions in Eqs. (43)–(47); the a_3 mode of $\bar{\Delta}\bar{B}$ in Eq. (46) is the next leading order term. Thus the a_1 and a_3 modes correspond to the scalar mode, whereas the a_2 and a_4 modes correspond to the tensor mode.

B. The uniform-curvature gauge

The authors of [3] derived the asymptotic solutions using a set of gauge-invariant variables which correspond to the variables in the comoving gauge and the uniform-curvature gauge together with the C gauge. The main differential equations used in [3] are expressed by using the gauge-invariant variables corresponding to the uniform-curvature gauge ones. The gauge-invariant density perturbation variable ϵ_m in [3] corresponds to δ in the comoving gauge. The other gauge-invariant variables in [3] correspond to the ones in the uniform-curvature gauge where $C \equiv 0$; $\Phi_1 = A$, $\Phi = \bar{B}$, $G_1^1 = G$, and $\Psi = -(a\dot{s}/k)B_v$. In the FLRW context, the uniform-curvature gauge is known to be important for treating scalar field perturbations; see [6]. We expect that the special role of the uniform-curvature gauge for treating scalar field perturbation will be maintained even in the anisotropic background. The case for the scalar field will be presented in future work.

In the uniform-curvature gauge we set $C \equiv 0$. From Eqs. (17), (20), (21), (23), and (24) we can derive

$$A' + w\bar{\Delta}\bar{B} = (1-w)\frac{S_1 - S_2}{2 - S_3}G', \qquad (60)$$

$$\left(\bar{\Delta}\bar{B}\right)' + \frac{3}{2}S_3(1-w)\bar{\Delta}\bar{B} + \bar{\Delta}A$$
$$= -\frac{3}{2}(1-w)S_3\frac{S_1 - S_2}{2 - S_3}G', \qquad (61)$$

$$G'' - \frac{1-w}{2}\frac{(S_1 - S_2)^2}{2 - S_3}G' - \bar{\Delta}G = \frac{1-w}{2}(S_1 - S_2)\bar{\Delta}\bar{B}. \qquad (62)$$

Combining Eqs. (60)–(62), and keeping terms with higher order in $\bar{\Delta}$, we can derive

$$[G' - 2(2 - S_3)G]''' - 3(1-w)[G' - 2(2 - S_3)G]''$$
$$= \bar{\Delta}[(1+w)G'' + (1 - 2S_3 + 3w)G'] - w\bar{\Delta}^2 G. \qquad (63)$$

The large scale asymptotic solutions in the shear-dominated stage are

$$G(\mathbf{x}, t) \equiv c_1(\mathbf{x})e^{3(1-w)s} + \left[1 + \frac{\bar{\Delta}}{4(2 - S_3)^2}\right]c_2(\mathbf{x})$$
$$+ c_3(\mathbf{x})s + \bar{\Delta}c_4(\mathbf{x}), \qquad (64)$$

$$\bar{\Delta}\bar{B}(\mathbf{x}, t) = -9(1-w)\frac{S_3}{S_1 - S_2}c_1 e^{3(1-w)s}$$
$$-\frac{1}{2}\frac{S_1 - S_2}{(2 - S_3)^2}\bar{\Delta}c_2 - \frac{S_1 - S_2}{2 - S_3}c_3$$
$$-2(S_1 - S_2)\left[1 - \frac{4}{3}\frac{1}{1-w}\frac{2 - S_3}{2 + S_3}\right]\bar{\Delta}c_4, \qquad (65)$$

$$A(\mathbf{x}, t) = \frac{S_1 - S_2}{4 - S_3^2}(2 + S_3 - 2w)c_1 e^{3(1-w)s}$$
$$+\frac{S_1 - S_2}{2 - S_3}\left[1 + \frac{\bar{\Delta}}{4(2 - S_3)^2}\right]c_2 + \frac{S_1 - S_2}{2 - S_3}c_3 s$$
$$+(S_1 - S_2)\left[8\frac{2 - S_3}{2 + S_3}\left(1 - \frac{2}{3}\frac{2 - S_3}{1-w}\right)\right.$$
$$\left. +\bar{\Delta}\frac{1}{2 - S_3}\left(1 - \frac{4}{3}\frac{w}{1-w}\frac{2 - S_3}{2 + S_3}\right)\right]c_4, \qquad (66)$$

where $c_1(\mathbf{x})$, $c_2(\mathbf{x})$, $c_3(\mathbf{x})$, and $c_4(\mathbf{x})$ are the constant coefficients for solutions of the fourth order differential equation. The solutions for other variables δ, $\bar{Q}$, etc. can be obtained from the constraint equations; see Eqs. (21) and (22).

In principle, the solutions in Eqs. (64)–(66) can be derived from the known solutions in the comoving gauge in Eqs. (43)–(47) by using the gauge transformation. However, due to some cancellations in the series of the asymptotic expansion in the large scale limit, in practice we find it is more convenient to derive the solutions in each different gauge condition separately. The asymptotic solutions in these two gauge conditions in Sec. III A and Sec. III B are consistent with each other.

For $S_1 = S_2$, some solutions in Eqs. (64)–(66) diverge. As mentioned in Sec. III A 2, in the shear-dominated stage, for $S_1 = S_2$ the scalar mode decouples from the tensor mode. Since the decoupling occurs only for $S_3 > S_2$, we need $S_\alpha = (-1, -1, 2)$. The $S_\alpha = (1, 1, -2)$ case can be covered by $S_\alpha = (1, -2, 1)$. From Eq. (37) we have $\bar{\Delta} \propto$ const. For the tensor mode we have

$$G \quad \propto \quad \text{const}, \quad s, \qquad (67)$$

which is the same as the comoving gauge solution in Eq. (55). For the scalar mode, to the leading order the solutions cancel out; the solutions to the leading order in the comoving gauge are presented in Eqs. (56)–(59).

C. Decoupled gravitational wave mode

We neglect the rotation mode (thus let $\bar{Q}_v \equiv 0$) and higher order terms in $k^2 k_2/\Delta$. The asymptotic solution $\bar{G}$ can be determined from Eqs. (28) and (29) as

$$\bar{G}(\mathbf{x}, t) = g_1(\mathbf{x})e^{-2(S_1 - S_2)s} + g_2(\mathbf{x}), \qquad (68)$$

where $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ are constant coefficients of the second order differential equation. In the limit of vanishing

background shear, we have $\bar{G} \propto$ const. This is consistent with the FLRW solutions in Eq. (54) with $w = 1$.

D. Rotation modes

For Q_v in Eq. (25) we assume the shear-dominated era and $S_3 > S_2$, thus Eqs. (35) and (36) remain valid. The solution becomes

$$Q_v(\mathbf{x}, s) = e^{(-1-3w+2S_2)s} Q_v(\mathbf{x}, 0). \tag{69}$$

This solution can work as a source for the scalar mode coupled with the tensor mode G. For $\bar{Q}_v$ in Eq. (27) we only assume the shear-dominated era, and thus Eq. (35) remains valid. The solution becomes

$$\bar{Q}_v(\mathbf{x}, s) = e^{(-1-3w+2S_1)s} \bar{Q}_v(\mathbf{x}, 0). \tag{70}$$

This solution can work as a source for the decoupled gravitational wave $\bar{G}$. Q_v and $\bar{Q}_v$ are gauge invariant; see below Eq. (16). The solutions in Eqs. (69) and (70) are valid for the general scale.

IV. DISCUSSION

In this paper we investigated the evolution of perturbations in the Bianchi type-I universe supported by an ideal fluid. The equations are presented using dimensionless variables, and thus are suitable for numerical investigation. Large scale asymptotic solutions for perturbation variables are found in the shear-dominated anisotropic stage assuming $w = $ const. We obtained the solutions in the comoving gauge and the uniform-curvature gauge conditions.

We restricted our attention to the configuration of the perturbations where the wave propagation vector lies in the plane of two principal axes of the background anisotropy. The perturbation equations are classified into two sets. In one set, the scalar and the tensor modes are shown to be coupled to each other with a rotation mode as a source. In the other set, we have a decoupled tensor mode which evolves freely from other modes with one other rotation mode as a source. In both cases, the vector modes work only as the source terms, which is generally true as long as the anisotropic pressure vanishes; see Eq. (D4) of [4].

When the ideal fluid is a radiation ($w = \frac{1}{3}$) or a dust ($w = 0$) we have exact solutions describing the background evolution from the shear-dominated to matter-dominated FLRW stage; see Eqs. (A5) and (A6). Using the equations in Sec. II D and the initial conditions in Sec. III, we can numerically investigate the evolution of perturbations through the transition. These will be presented in subsequent work.

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APPENDIX: BACKGROUND EVOLUTION IN AN IDEAL FLUID CASE

In an ideal fluid case [thus $p = p(\mu)$] with $\Lambda = 0$ the background evolution is determined by Eqs.

(2)–(4). Thus $\dot{s}_\alpha \propto e^{-3s}$. For $w(\equiv p/\mu) = $ const we have $\mu \propto e^{-3(1+w)s}$. We introduce two constants:

$$c_1 \equiv \frac{8\pi G}{3} \mu e^{3(1+w)s}, \quad c_2 \equiv \frac{1}{6} \sum_\alpha \dot{s}_\alpha^2 e^{6s}. \tag{A1}$$

With $a \equiv e^s$ we can derive

$$t = \int^a \frac{a^2 da}{\sqrt{c_1 a^{3(1-w)} + c_2}}. \tag{A2}$$

Introducing

$$S_\alpha \equiv \frac{\dot{s}_\alpha}{\sqrt{\frac{1}{6} \sum_\beta \dot{s}_\beta^2}}, \tag{A3}$$

we can derive

$$e^{s_\alpha + s_{\alpha 0}} = \left(\frac{\sqrt{c_1 a^{3(1-w)} + c_2} - \sqrt{c_2}}{\sqrt{c_1 a^{3(1-w)} + c_2} + \sqrt{c_2}} \right)^{S_\alpha / [3(1-w)]}, \tag{A4}$$

where $s_{\alpha 0}$ is the integration constant which can be adjusted by coordinate redefinition. We may set $s_{\alpha 0} \equiv 0$.

1. Radiation and dust medium

For a radiation medium with $w = \frac{1}{3}$, it is convenient to use $d\eta \equiv a^{-1} dt$ as a time variable. Equations (A2) and (A4) lead to

$$e^s = \left(\frac{8\pi G}{3} \mu e^{4s} \right)^{1/2} \left[(\eta + \eta_0)^2 - \eta_s^2 \right]^{1/2},$$

$$e^{s_\alpha} = \left(\frac{\eta + \eta_0 - \eta_s}{\eta + \eta_0 + \eta_s} \right)^{S_\alpha / 2}. \tag{A5}$$

For a dust medium with $w = 0$, Eqs. (A2) and (A4) lead to

$$e^s = \left(6\pi G \mu e^{3s} \right)^{1/3} \left[(t + t_0)^2 - t_s^2 \right]^{1/3},$$

$$e^{s_\alpha} = \left(\frac{t + t_0 - t_s}{t + t_0 + t_s} \right)^{S_\alpha / 3}. \tag{A6}$$

We introduce

$$\eta_s \equiv \frac{3}{2} \frac{\sqrt{\frac{1}{6} \sum_\alpha \dot{s}_\alpha^2}}{4\pi G \mu e^s}, \quad t_s \equiv \frac{\sqrt{\frac{1}{6} \sum_\alpha \dot{s}_\alpha^2}}{4\pi G \mu}, \tag{A7}$$

which indicate the transition epochs from shear domination to matter (radiation or dust) domination. η_0 and t_0 are the integration constants which can be adjusted by temporal coordinate redefinitions.

2. Shear-dominated stage

In a shear-dominated stage we consider the case with $c_2 e^{-6s} \gg c_1 e^{-3(1+w)s}$. We can derive solutions consider-

ing the matter contribution perturbatively. To the zeroth order in the matter contribution, we have

$$a = (3\sqrt{c_2}\bar{t})^{1/3} , \quad \dot{s} = \frac{1}{3\bar{t}}, \quad \dot{s}_\alpha = \frac{S_\alpha}{3\bar{t}},$$

$$e^{s_\alpha} \propto (3\sqrt{c_2}\bar{t})^{S_\alpha/3} . \qquad (A8)$$

In terms of η as the time, we have

$$a = (2\sqrt{c_2}\bar{\eta})^{1/2} , \quad \frac{\partial s}{\partial \eta} = \frac{1}{2\bar{\eta}}, \quad \frac{\partial s_\alpha}{\partial \eta} = \frac{S_\alpha}{2\bar{\eta}},$$

$$e^{s_\alpha} \propto (2\sqrt{c_2}\bar{\eta})^{S_\alpha/2} , \qquad (A9)$$

where $\bar{t} = t + \text{const}$ and $\bar{\eta} = \eta + \text{const}$; the constant terms can be normalized to zero. The proportionality constant in e^{s_α} can be redefined through $s_{\alpha 0}$.

3. Energy-dominated stage

In an energy-dominated stage, we consider the case with $c_2 e^{-6s} \ll c_1 e^{-3(1+w)s}$. We can derive solutions con-

sidering the shear contribution perturbatively. To the zeroth order in the shear contribution, we have

$$e^s = \left[\frac{3(1+w)}{2}\sqrt{c_1}\bar{t} \right]^{\frac{2}{3(1+w)}} , \quad \dot{s} = \frac{2}{3(1+w)}\bar{t}^{-1},$$

$$\dot{s}_\alpha = \sqrt{c_2}S_\alpha \left[\frac{3(1+w)}{2}\sqrt{c_1}\bar{t} \right]^{-\frac{2}{1+w}} . \qquad (A10)$$

In terms of η we have

$$e^s = \left(\frac{1+3w}{2}\sqrt{c_1}\bar{\eta} \right)^{\frac{2}{1+3w}} , \quad \frac{\partial s}{\partial \eta} = \frac{2}{1+3w}\bar{\eta}^{-1},$$

$$\frac{\partial s_\alpha}{\partial \eta} = \sqrt{c_2}S_\alpha \left(\frac{1+3w}{2}\sqrt{c_1}\bar{\eta} \right)^{-\frac{4}{1+3w}} , \qquad (A11)$$

where $\bar{t} = t + \text{const}$ and $\bar{\eta} = \eta + \text{const}$; the constant terms can be normalized to zero. The proportionality constant in e^{s_α} can be redefined through $s_{\alpha 0}$.

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