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Anisotropic scalar-tensor cosmologies

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We examine homogeneous but anisotropic cosmologies in scalar-tensor gravity theories, including Brans-Dicke gravity. We present a method for deriving solutions for any isotropic perfect fluid with a barotropic equation of state ($p \propto \rho$) in a spatially flat (Bianchi type I) cosmology. These models approach an isotropic, general relativistic solution as the expansion becomes dominated by the barotropic fluid. All models that approach general relativity isotropize except for the case of a maximally stiff fluid. For stiff fluid or radiation or in vacuum we are able to give solutions for arbitrary scalar-tensor theories in a number of anisotropic Bianchi and Kantowski-Sachs metrics. We show how this approach can also be used to derive solutions from the low-energy string effective action. We discuss the nature, and possibly avoidance of, the initial singularity where both shear and non-Einstein behavior are important.

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I. INTRODUCTION

Scalar-tensor theories of gravity [1–3] allow the gravitational coupling to vary, becoming a dynamical field rather than a fixed constant. This occurs in a range of fundamental theories that seek to incorporate gravity with the other interactions. In Kaluza-Klein models this arises from the variation of the size of the internal dimensions. In string theories the dilaton is a scalar field that is necessary for the consistent description of the motion of a string in a curved spacetime [4]. In a cosmological context such theories allow one to seek dynamical answers for questions such as why the Planck mass is so much larger than other physical scales or why it should appear to be fixed today.

The general form of the extended gravitational action in scalar-tensor theories is

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[F(\varphi)R - \frac{1}{2}(\nabla\varphi)^2 - U(\varphi) + 16\pi\mathcal{L}_{\rm matter}\right], \qquad (1)$$

where in general relativity $m_{\rm Pl}^2 \equiv F(\varphi)$ remains a constant. The Brans-Dicke model of gravity [2] corresponds to the particular choice of $U = 0$ and $F(\varphi) = \varphi^2/8\omega$ where ω is a constant parameter. The more general Lagrangian can still be rewritten in terms of their Brans-Dicke field $\phi \equiv F(\varphi)$ with the Brans-Dicke parameter $\omega(\phi) = F/[2(dF/d\varphi)^2]$,

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}(\nabla\phi)^2 - U(\phi) + 16\pi\mathcal{L}_{\rm matter}\right], \qquad (2)$$

and it is this form that we shall use here with $U(\phi)$ set to zero. A nontrivial potential for the Brans-Dicke field in the early universe clearly could affect the dynamics but we shall neglect this here in order to study the effect solely of the coupling of the Brans-Dicke field to the metric and to other matter fields. Higher-order gravity theories are equivalent to a scalar-tensor theory where $U(\phi)$ is nonzero but $\omega = 0$ [5] which introduces Yukawa-type corrections to the Newtonian potential. Setting $U = 0$ ensures a strictly Newtonian weak field limit to lowest order. The post-Newtonian parameters of general relativity are then recovered in the limit that $\omega \to \infty$ and $(\phi/\omega^3)(d\omega/d\phi) \to 0$ [6].

Most analytic cosmological solutions until recently were restricted to the case of Brans-Dicke gravity [7–12], specific choices of $\omega(\phi)$ [13, 14], or relations between $\omega(\phi)$ and $U(\phi)$ [15, 16]. Here we present techniques for deriving anisotropic cosmological solutions for general scalar-tensor theories where ω may be an arbitrary function of ϕ. This can lead to radically different evolution of the Brans-Dicke field as well as the Brans-Dicke parameter. Throughout we draw heavily on results derived in the conformally related Einstein frame [17], introduced in Sec. II. In Sec. III we define the quantities we will use to describe the evolution of homogeneous spacetimes.

We give solutions for barotropic fluids, including dust and a false vacuum energy density in Sec. IV, extending to Bianchi type I models the method used recently by Barrow and Mimoso [18] in spatially flat Friedmann-Robertson-Walker (FRW) models. This also allows us to consider the role of general relativity as a cosmological attractor within a large class of $\omega(\phi)$ theories.

In Sec. V we extend the method introduced by Barrow to derive solutions for FRW models in vacuum or containing radiation [19] or stiff matter [20], to solve for homogeneous spacetimes including spatial curvature as well as shear. In particular we give general solutions for

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arbitrary $\omega(\phi)$ theories in vacuum, or with radiation or stiff fluid, in Bianchi types I and V, as well as the general vacuum or stiff fluid solutions in Bianchi types III and locally rotationally symmetric (LRS) type IX and Kantowski-Sachs models, by exploiting known solutions in general relativity.

II. CONFORMAL FRAMES

Our field equations, obtained by varying the action in Eq. (2) with respect to the metric and field ϕ, are

$$\left(R^{ab}-\frac{1}{2}g^{ab}R\right)\phi=8\pi T^{ab}+\left(g^{ac}g^{bd}-\frac{1}{2}g^{ab}g^{cd}\right)\frac{\omega\phi_{,c}\phi_{,d}}{\phi}+\left(g^{ac}g^{bd}-g^{ab}g^{cd}\right)\nabla_c\nabla_d\phi, \quad (3)$$

$$(3+2\omega)\Box\phi=8\pi T-g^{ab}\omega_{,a}\phi_{,b}, \quad (4)$$

where the energy-momentum tensor $T^{ab}=(2/\sqrt{-g})\partial(\sqrt{-g}\mathcal{L}_{\rm matter})/\partial g_{ab}$.

In the scalar-tensor gravity theories the weak equivalence principle is guaranteed by requiring that all matter fields are minimally coupled to the metric g_{ab}.[1] Henceforth we will refer to this as the Jordan metric. Thus energy momentum is conserved:

$$\nabla^a T_{ab}=0\ . \quad (5)$$

However scalar-tensor theories can be rewritten in terms of a theory with a fixed gravitational constant with respect to the conformally related "Einstein" metric:

$$\widetilde{g}_{ab}\equiv G\phi g_{ab}\ , \quad (6)$$

where G is in fact an arbitrarily chosen constant which becomes the gravitational constant in the conformal metric. Note that if ϕ =const then the two frames are identical (allowing for an arbitrary constant rescaling of coordinates) and so the scalar-tensor results must be the same as in general relativity whenever this occurs. Notice also that for $\phi<0$ we must pick a negative G to maintain a positive conformal factor. Henceforth we shall assume $\phi\geq 0$.

Instead of appearing in the gravitational Lagrangian, the Brans-Dicke field now appears as a scalar field interacting with matter:

$$S=\frac{1}{16\pi}\int d^4x\sqrt{-\widetilde{g}}\left[\frac{1}{G}\widetilde{R}-\frac{1}{2}\frac{(\widetilde{\nabla}\phi)^2}{\phi^2}+16\pi\widetilde{\mathcal{L}}_{\rm matter}\right]\ , \quad (7)$$

where the covariant derivative $\widetilde{\nabla}_a$ is taken with respect to the Einstein metric and

$$\widetilde{\mathcal{L}}_{\rm matter}\equiv\frac{\mathcal{L}_{\rm matter}}{(G\phi)^2}\ . \quad (8)$$

Thus, although we recover the familiar Einstein field equations, energy momentum is no longer conserved independently of the Brans-Dicke field

$$\widetilde{\nabla}^a\widetilde{T}_{ab}=-\frac{1}{2}\frac{\phi_{,b}}{\phi}\widetilde{T}^a_a\ , \quad (9)$$

except when the energy-momentum tensor is traceless, corresponding to vacuum or radiation. In general it is the difficulty of including this interaction between matter and the Brans-Dicke field which limits our ability to produce analytic solutions.

Of course the overall energy-momentum tensor is conserved (as guaranteed by the Ricci identities) as long as we include the Brans-Dicke field as a matter field with energy-momentum tensor

$$\hat{T}^{ab}=\left(\widetilde{g}^{ac}\widetilde{g}^{bd}-\frac{1}{2}\widetilde{g}^{ab}\widetilde{g}^{cd}\right)\psi_{,c}\psi_{,d}\ , \quad (10)$$

where we define

$$d\psi\equiv\sqrt{\frac{3+2\omega(\phi)}{16\pi G}}\ \frac{d\phi}{\phi}\ . \quad (11)$$

Here we take[2] $3+2\omega\geq 0$.

This allows us to deal most easily with other massless fields in the matter Lagrangian, exploiting known solutions in general relativity to produce scalar-tensor counterparts [20]. Short-wavelength modes of a field, $\varphi\propto\exp(ik_ax^a)$ where $k_ak^a=0$, act like radiation with a traceless energy-momentum tensor. Long-wavelength modes of a massless scalar field, $\varphi(t)$, act like a stiff fluid with density equal to pressure equal to $\dot{\varphi}^2/2$ [21]. Although this minimally coupled scalar field in the Jordan frame interacts with the Brans-Dicke field in the Einstein frame, they combine to give the same dynamical effect as that of single stiff fluid. We will also consider massless fields which occur in the low energy effective action of string theory.

We will present in Sec. V results for scalar-tensor gravity in a number of anisotropic cosmologies in vacuum, with a stiff fluid and, in some cases, radiation. Before that, in Sec. IV we will give solutions for other barotropic fluids but restricted to a Bianchi type I metric.

III. HOMOGENEOUS SPACETIMES

Spatially homogeneous spacetimes admit a group of isometries acting transitively on their spacelike hypersurfaces [22]. It is possible to write the metrics of these

[1] Note that in this context the Brans-Dicke field is considered a gravitational field as its coupling to the metric is nonminimal.

[2] If this is not so then the scalar field in the Einstein frame has a negative energy density and the Minkowski vacuum, for instance, will be unstable. This seems to be a strong physical argument for rejecting such models and henceforth we shall assume $3+2\omega\geq 0$.

models as

$$ds^2 = -dt^2 + \gamma_{\alpha\beta}(t) W^\alpha{}_a W^\beta{}_b dx^a dx^b \ , \tag{12}$$

where the $W^a{}_\alpha$ (dual to $W^\alpha{}_a$) are the invariant vector fields of the reciprocal group, $[W_\alpha{}^a, W_\beta{}^a] = C^\delta{}_{\alpha\beta} W_\delta{}^a$. The structure constants of the isometry group $C^\delta{}_{\alpha\beta}$, satisfy the Jacobi identities $C^\alpha_{[\beta\delta} C^\lambda_{\gamma]\alpha} = 0$.

These metrics fall into the two classes of Bianchi and Kantowski-Sachs models. The former have a three-dimensional group acting in a simple-transitive way on the spatial hypersurfaces. They separate into equivalence classes according to Bianchi's classification of the three-dimensional Lie groups [23, 22, 24]. The latter has a four-dimensional isometry group, but no three simple-transitive subgroup acting on the three space. Instead it acts on a two-dimensional subspace. This subspace has constant (positive) curvature.

Although in all spatially homogeneous models the unit normal to the spatial hypersurfaces is a geodesic vector field t^a (with $t_a t^a = -1$) invariant under the group [22], the matter flow might be tilted relative to this direction. For the sake of simplicity, in what follows we shall only be concerned with those models in which the velocity of matter is parallel to the unit normal. It follows that T^{ab} has a timelike eigenvector and, thus, both rotation and acceleration are vanishing.

The homogeneous three-dimensional spatial hypersurfaces thus have a metric h_{ab}, orthogonal to the unit vector field, such that the full four-dimensional metric in the Jordan frame

$$g_{ab} = h_{ab} - t_a t_b \ . \tag{13}$$

A perfect comoving fluid, with density ρ, isotropic pressure p, and (possibly) anisotropic stress π_a^b then has an energy-momentum tensor

$$T_a^b \equiv \rho t_a t^b + p h_a^b + \pi_a^b \ . \tag{14}$$

We will only consider isotropic matter in what follows with $\pi_a^b = 0$ (although one should bear in mind that an anisotropic expansion can induce anisotropic pressures in some fluids [25]). The extrinsic curvature of the hypersurfaces can be defined as

$$t_{a;b} \equiv \frac{1}{3}\theta h_{ab} + \sigma_{ab} \ , \tag{15}$$

where the expansion $\theta \equiv t^a_{;a}$ and the shear $\sigma^2 \equiv \sigma_a^b \sigma_b^a / 2$. We will also find it convenient to define a volume scale factor V such that

$$\theta \equiv \frac{1}{V} V_{,a} t^a \equiv \frac{1}{V}\frac{dV}{dt} \ . \tag{16}$$

Note that if we redefine the contribution of the derivatives of the Brans-Dicke field to the right-hand side of the scalar-tensor equations of motion in Eq. (3) to be an effective energy-momentum tensor $\bar{T}_{ab}$, so that

$$\left(R_{ab} - \frac{1}{2} g_{ab} R \right) \phi = 8\pi \left(T_{ab} + \bar{T}_{ab} \right) \ , \tag{17}$$

then we find that the nonminimal coupling of the scalar field to the spacetime curvature creates an anisotropic stress [26] which is proportional to the shear:

$$\bar{\pi}_{ab}^{(\phi)} = \frac{1}{8\pi} \frac{\dot{\phi}}{\phi} \sigma_{ab} \ . \tag{18}$$

Thus though the Brans-Dicke field will not produce shear where none exists, it does exert an anisotropic pressure in the presence of shear even when the other matter is isotropic. On the other hand, in the Einstein frame the definition in Eq. (10) of the energy-momentum tensor $\hat{T}_{ab}$, associated with the Brans-Dicke field, can exert no anisotropic stress for a homogeneous field.

It is straightforward to calculate conformally transformed quantities in the Einstein metric where we find

$$\widetilde{V} = (G\phi)^{3/2} V, \quad d\widetilde{t}^2 = (G\phi) dt^2, \quad \widetilde{\sigma}^2 = \frac{\sigma^2}{(G\phi)},$$
$$\widetilde{\rho} = \frac{\rho}{(G\phi)^2}, \quad \widetilde{p} = \frac{p}{(G\phi)^2}. \tag{19}$$

The metric field equations (3) then yield the familiar constraint equation

$$\widetilde{\theta}^2 = 24\pi G \left(\widetilde{\rho} + \hat{\rho} \right) + 3\widetilde{\sigma}^2 - \frac{3}{2}\, {}^{(3)}\widetilde{R} \ , \tag{20}$$

where ${}^{(3)}\widetilde{R}$ is the curvature scalar of the hypersurfaces of homogeneity, and the Raychaudhuri equation

$$\frac{d\widetilde{\theta}}{d\widetilde{t}} + \frac{1}{3}\widetilde{\theta}^2 = -4\pi G \left(3\widetilde{p} + 3\hat{p} + \widetilde{\rho} + \hat{\rho} \right) - 2\widetilde{\sigma}^2 \ . \tag{21}$$

The total energy momentum in this frame is conserved:

$$\frac{d}{d\widetilde{t}} \left(\widetilde{\rho} + \hat{\rho} \right) + \widetilde{\theta} \left(\widetilde{\rho} + \hat{\rho} + \widetilde{p} + \hat{p} \right) = 0 \ , \tag{22}$$

where we include the density $\hat{\rho}$ and pressure $\hat{p}$ of the Brans-Dicke field.

But we are also interested in how the Brans-Dicke field and other matter evolve separately. The continuity equation for isotropic matter minimally coupled in the Jordan frame remains simplest in that frame,

$$\frac{d\rho}{dt} + \theta \left(\rho + p \right) = 0 \ , \tag{23}$$

which can be integrated for a barotropic fluid with $p = (\gamma - 1)\rho$ to give $\rho \propto V^{-\gamma}$ or $\widetilde{\rho} \propto (G\phi)^{(3\gamma-4)/2} \widetilde{V}^{-\gamma}$ in the Einstein frame.

The equation of motion for the Brans-Dicke field is then

$$\frac{d^2\psi}{d\widetilde{t}^2} + \widetilde{\theta} \frac{d\psi}{d\widetilde{t}} = -\frac{1}{2} \sqrt{\frac{16\pi G}{3 + 2\omega}} \left(3\widetilde{p} - \widetilde{\rho} \right) \ . \tag{24}$$

This is the one equation of motion in the Einstein frame that still includes an explicit dependence on the form of $\omega(\phi)$. In vacuum or with radiation the right-hand side is zero and so the evolution of both ψ and the volume scale factor $\widetilde{V}$ in the Einstein frame *is independent of the choice of function* $\omega(\phi)$. We will show in Sec. V that these solutions for $\widetilde{V}(\widetilde{t})$ also include all the scalar-tensor cosmologies containing stiff matter. Analytic solutions for other barotropic fluids but restricted to spatially flat FRW models have recently been given by Barrow and Mimoso [18]. We will now extend this method to the case of Bianchi type I metrics.

IV. BAROTROPIC MATTER IN BIANCHI TYPE I

We will first consider the case of barotropic fluids [where $p = (\gamma - 1)\rho$], such as dust ($\gamma = 1$) or false vacuum ($\gamma = 0$), but excluding for the time being the exceptional cases of stiff fluid ($\gamma = 2$) or radiation ($\gamma = 4/3$) as well as vacuum. We will also restrict our analysis to the simplest case of a Bianchi type I metric, where the structure constants $C^{\alpha}_{\beta\gamma}$ all vanish and the spatial curvature is zero. The metric can then be written as

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2, \quad (25)$$

which includes the spatially flat FRW metric when $a_1 = a_2 = a_3$. The expansion is given by

$$\theta \equiv \frac{3}{a}\frac{da}{dt}, \quad (26)$$

where the volume scale factor

$$V \equiv a^3 \equiv a_1 a_2 a_3 . \quad (27)$$

The ψ-field equation of motion, Eq. (24), is driven by the barotropic fluid density in the Einstein frame

$$\tilde{\rho} = \frac{3}{8\pi G}\frac{M(G\phi)^{(3\gamma-4)/2}}{\tilde{a}^{3\gamma}}, \quad (28)$$

where $M = 8\pi G\rho a^{3\gamma}/3$ is a constant and $\tilde{a}^3 \equiv \tilde{V}$ is the volume scale factor in the Einstein frame. This same energy density drives the evolution of the three scale factors whose individual expansion rates $\tilde{\theta}_i \equiv \dot{\tilde{a}}_i/\tilde{a}_i$ obey the field equations

$$\dot{\tilde{\theta}}_i + \tilde{\theta}\tilde{\theta}_i = \frac{3(2-\gamma)}{2}\frac{M(G\phi)^{(3\gamma-4)/2}}{\tilde{a}^{3\gamma}} . \quad (29)$$

However the difference between the expansion rates in any two directions is not driven by the isotropic fluid. The shear is

$$\sigma^2 \equiv \frac{1}{3}\left(\theta_1^2 + \theta_2^2 + \theta_3^2 - \theta_1\theta_2 - \theta_2\theta_3 - \theta_3\theta_1\right), \quad (30)$$

so in the conformally transformed Einstein frame it evolves like the energy density of a minimally coupled stiff fluid

$$\frac{d\tilde{\sigma}^2}{d\tilde{t}} + 2\tilde{\theta}\tilde{\sigma}^2 = 0 . \quad (31)$$

Thus it behaves exactly like a stiff fluid with density

$$\frac{\tilde{\sigma}^2}{8\pi G} = \frac{3\Sigma^2}{32\pi G\tilde{a}^6}, \quad (32)$$

where Σ^2 is a constant, evolving in a flat FRW metric with scale factor $\tilde{a}$.

Note that this general relativistic result is not in general true in scalar-tensor gravity. Just as a minimally coupled scalar field in the Jordan frame interacts with the field ψ in the Einstein frame, so shear, which evolves freely in the Einstein frame, is coupled to the Brans-Dicke field back in the Jordan frame, and we have

$$\sigma^2 = \frac{3\Sigma^2}{4(G\phi)^2 a^6} . \quad (33)$$

This is a result of the effective anisotropic pressure (in the presence of shear) induced by the Brans-Dicke field in the Jordan frame, but absent in the Einstein frame.

We will use the approach developed by Barrow and Mimoso [18] for the scalar-tensor solutions in spatially flat FRW metrics, itself an extension of the method used by Gurevich, Finkelstein, and Ruban [8] for Brans-Dicke gravity. We introduce the time coordinate ξ defined by

$$\begin{aligned} d\xi &\equiv \tilde{a}^{3(1-\gamma)}(G\phi)^{(3\gamma-4)/2}\sqrt{\frac{3}{3+2\omega}}d\tilde{t} \\ &\equiv a^{3(1-\gamma)}\sqrt{\frac{3}{3+2\omega}}\,dt , \end{aligned} \quad (34)$$

and the variables[3]

$$y = \sqrt{\frac{16\pi G}{3}}\,\tilde{a}^3\frac{d\psi}{d\tilde{t}}, \quad (35)$$

$$z_i = \tilde{a}^3\tilde{\theta}_i , \quad (36)$$

$$z = \frac{1}{3}\sum_i z_i . \quad (37)$$

The equations of motion, Eqs. (24) and (29), reduce to

$$y' = M\,(4-3\gamma) , \quad (38)$$

$$z_i' = \sqrt{\frac{3+2\omega}{3}}\frac{3(2-\gamma)M}{2}, \quad (39)$$

plus the constraint Eq. (20), which becomes

$$4z^2 = y^2 + \Sigma^2 + 4M(G\phi)^{(3\gamma-4)/2}\,\tilde{a}^{3(2-\gamma)} . \quad (40)$$

Each term on the right-hand side is non-negative and so we can describe the expansion in the Einstein frame, z^2, at any time as being dominated by either the term arising from the Brans-Dicke field energy density y^2, the shear Σ^2, or the matter energy density $4M(G\phi)^{(3\gamma-4)/2}\,\tilde{a}^{3(2-\gamma)}$.

These equations of motion can be integrated to give

$$y = (4-3\gamma)M(\xi - \xi_*) , \quad (41)$$

$$z_i = \frac{3(2-\gamma)M}{2}\left[\int_{\xi_0}^{\xi}\sqrt{\frac{3+2\omega(\bar{\xi})}{3}}d\bar{\xi} + \sigma_i\right], \quad (42)$$

where ξ_0, ξ_*, and σ_i are integration constants. Henceforth we will set the constant ξ_* to zero without loss of

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[3]Note that these variables correspond to those defined by Barrow and Mimoso in terms of Jordan frame variables where $z \equiv \sqrt{(3+2\omega)/3}[(x/3)+(y/2)]$ with $G = 1$ in that paper [18].

generality. This merely amounts to a translation of the time origin. We have chosen ξ_0 so that

$$z = \frac{3(2-\gamma)M}{2} \int_{\xi_0}^{\xi} \sqrt{\frac{3+2\omega(\bar{\xi})}{3}} d\bar{\xi} \,. \tag{43}$$

The integration constants σ_i thus characterize the initial shear and obey

$$\sum_i \sigma_i = 0 \,, \qquad \sum_i \sigma_i^2 = 2\Sigma^2 \,. \tag{44}$$

We can use these results to rewrite the equation of motion for the Brans-Dicke field, Eq. (24), solely in terms of ϕ and our time coordinate ξ:

$$\left(\frac{\phi'}{\phi}\right)' + \left[-\frac{4-3\gamma}{2} + \frac{(z^2)'}{(4-3\gamma)\xi}\right] \left(\frac{\phi'}{\phi}\right)^2 = \frac{1}{\xi}\left(\frac{\phi'}{\phi}\right). \tag{45}$$

The solutions to the Bernoulli equation, Eq. (45), can be cast into the particularly simple form

$$\ln\left(\frac{\phi}{\phi_0}\right) = (4-3\gamma) \int_{\xi_0}^{\xi} \frac{\bar{\xi}}{g(\bar{\xi})} d\bar{\xi} \,, \tag{46}$$

where ϕ_0 is an integration constant, by absorbing $z(\xi)$ into another function,

$$g(\xi) \equiv \frac{z^2(\xi)}{M^2} - \left(\frac{4-3\gamma}{2}\xi\right)^2 - g_0 \,, \tag{47}$$

with g_0 another integration constant.

Comparing the expression for ϕ'/ϕ obtained from Eq. (46) with that from Eq. (41) shows that

$$g = \frac{(G\phi)^{(3\gamma-4)/2}\, \tilde{a}^{3(2-\gamma)}}{M} = \frac{(G\phi) a^{3(2-\gamma)}}{M} \,. \tag{48}$$

In other words, $4M^2 g(\xi)$ is just the energy density term on the right-hand side of the constraint Eq. (40), and the definition of g in Eq. (47) is precisely this constraint equation, where

$$g_0 \equiv \frac{\Sigma^2}{4M^2} \,. \tag{49}$$

Thus $g(\xi)$ must be non-negative.

The behavior of the individual scale factors in the Jordan frame, $a_i(\xi)$, is given by

$$a_i = \left(\frac{Mg}{G\phi}\right)^{\frac{1}{3(2-\gamma)}} \exp\left[-\frac{\sigma_i}{2M}\int \frac{\sqrt{3+2\omega(\bar{\xi})}}{g(\bar{\xi})} d\bar{\xi}\right] , \tag{50}$$

which gives an analytic expression dependent on our ability to perform the integration in the exponential. The average scale factor obeys

$$a^{3(2-\gamma)} = \left(\frac{Mg}{G\phi}\right) \,. \tag{51}$$

The shear term, Σ^2, in the constraint Eq. (40) will only remain dynamically important if both y^2 and g remain bounded. The former requires that ξ^2 remains bounded; whenever $|\xi| \to \infty$ the model isotropizes. As we shall see in following sections, $\xi \to \infty$ as $t \to \infty$ for $\omega > 2(\gamma - 5/3)/(2-\gamma)^2$.

Note that singularities in the Jordan frame, by which we mean here points at which the volume scale factor, a^3, vanishes, can occur only if $g \to 0$ or $\phi \to \infty$. In fact the latter case can be shown also to require that $g/z^2 \to 0$ and thus the dynamical role played by matter (excluding stiff fluid with $\gamma = 2$) at the singularity is negligible in anisotropic models. Thus we reserve a discussion of the nature of the singularity until Sec. V B where we discuss vacuum and stiff fluid models.

The behavior of the coupling $\omega(\phi)$ which defines the scalar-tensor theory is given in terms of $z(\xi)$ by

$$3 + 2\omega\left[\phi(\xi)\right] = \frac{4}{3(2-\gamma)^2}\,(z')^2 \,. \tag{52}$$

The $\omega(\phi)$ dependence is only obtained *after* we have solved for the evolution of ϕ and ω as functions of ξ, if we can invert Eq. (46) to find $\xi(\phi)$. In practice a theory is chosen by specifying $g(\xi)$ as a generating function from which $\phi(\xi)$ follows by Eq. (46), and $\omega(\xi)$ from Eqs. (47) and (52). We have $a(\xi)$ from Eq. (51), as well as $a_i(\xi)$ from Eqs. (50), and we can relate our time coordinate ξ to the proper time in the Jordan frame, $t(\xi)$, from Eq. (34).

A. Brans-Dicke gravity

Brans-Dicke theory is recovered when $\omega = \omega_0$ is a constant and we see from Eq. (52) that this implies that

$$z^2(\xi) = \frac{9(2-\gamma)^2 M^2}{4}\frac{3+2\omega_0}{3}(\xi - \xi_0)^2 \,. \tag{53}$$

In the isotropic case where the shear $\Sigma^2 = 0$, setting $\xi_0 = 0$ corresponds to what is often referred to as the solutions being matter (rather than ϕ) dominated at early times. In fact because both g and y^2 then evolve as ξ^2, the relative terms on the right-hand side of the constraint Eq. (40) are strictly proportional and so this could more accurately be described as a scaling solution. Clearly all Bianchi type I solutions approach this behavior, with $a \propto \xi^\mu$ and $\phi \propto \xi^\nu$, where

$$\mu = \frac{2(3+2\omega)(2-\gamma) - 2(4-3\gamma)}{3(3+2\omega)(2-\gamma)^2 - (4-3\gamma)^2} \,, \tag{54}$$

$$\nu = \frac{4(4-3\gamma)}{3(3+2\omega)(2-\gamma)^2 - (4-3\gamma)^2} \,, \tag{55}$$

as $\xi \to \infty$, giving Nariai's [7] isotropic power-law solutions, $a \propto t^m$ and $\phi \propto t^n$, where

$$m = \frac{2(3+2\omega)(2-\gamma) - 2(4-3\gamma)}{3(3+2\omega)\gamma(2-\gamma) - (4-3\gamma)(3\gamma-2)} \,, \tag{56}$$

$$n = \frac{4(4-3\gamma)}{3(3+2\omega)\gamma(2-\gamma) - (4-3\gamma)(3\gamma-2)} \,, \tag{57}$$

upon integration of the time transformation, Eq. (34), irrespective of the initial shear.

However, the $\xi \to \infty$ limit is never reached (for positive

ϕ) for $\omega < 2(\gamma - 5/3)/(2-\gamma)^2$ as in this case $g \to 0$ at finite ξ (and would become negative as $\xi \to \infty$). Nonetheless this universe does correspond to an infinite proper lifetime in the Jordan frame, but with a late time expansion driven by both shear and the Brans-Dicke field density, i.e., *the model does not isotropize.*

B. Approach to general relativity

Among the wider class of scalar-tensor gravity theories Brans-Dicke behavior looks atypical. It only occurs at late times where $g(\xi) \propto \xi^2$ in the limit $\xi \to \infty$. Otherwise the generalized Friedmann constraint equation becomes scalar field dominated if $g(\xi) < \alpha\xi^2$ for any constant α as $\xi \to \infty$, or matter dominated if $g(\xi) > \alpha\xi^2$ as $\xi \to \infty$. In this latter case we see from Eq. (52) that the Brans-Dicke parameter ω must diverge, and [from Eq. (46)] the Brans-Dicke field ϕ tends to a constant value; in other words *we recover the general relativistic behavior at late times.*

Recently Damour and co-authors [27, 28] have argued that the general relativistic limit acts as a cosmological attractor within the parameter space of more general scalar-tensor gravity theories. This occurs when the Brans-Dicke parameter diverges. In the notation favored by Damour and co-authors this corresponds to the Brans-Dicke field $\phi = F(\varphi)$ as defined in Eq. (1) having a local maximum with respect to the field φ (with $\phi \neq 0$). To see how this emerges in our notation, and in the Bianchi type I cosmologies, we will consider the limit

$$F(\varphi) = \phi_0 - \frac{1}{2}k\varphi^2 + O(\varphi^3) . \quad (58)$$

This simply corresponds to a pole in the function $\omega(\phi)$,

$$2\omega(\phi) + 3 = \frac{1}{2k}\left(\frac{\phi_0}{\phi_0 - \phi}\right)\left[1 + O\left(\sqrt{\frac{\phi_0 - \phi}{\phi_0}}\right)\right] . \quad (59)$$

This type of behavior occurs as $\xi \to \infty$ and we consider a generating function

$$g(\xi) = g_n\xi^{2n} + O(\xi^{2n-1}) , \quad (60)$$

with $n > 1$. Equation (46) then gives the evolution of the Brans-Dicke field as

$$\phi = \phi_0\left[1 - \frac{4-3\gamma}{2(n-1)g_n\xi^{2(n-1)}} + O(\xi^{-2n+1})\right] , \quad (61)$$

which, via Eq. (52), gives the above $\omega(\phi)$ behavior for

$$k = \frac{3(2-\gamma)^2}{4(4-3\gamma)}\frac{n-1}{n^2} . \quad (62)$$

Note that there is actually an upper bound on $k \leq 3(2-\gamma)^2/16(4-3\gamma)$. This corresponds to the condition that the Brans-Dicke field is overdamped and approaches ϕ_0 monotonically. For larger k it will execute damped oscillations about ϕ_0 [27].

Using Eq. (51) to find the limiting behavior of the average scale factor and using this to determine the relationship between the time coordinate ξ and the proper time in the Jordan frame, Eq. (34), we find (for $\gamma \neq 0$)

$$\xi \propto t^{(2-\gamma)/n\gamma} , \quad (63)$$

$$a \propto t^{2/3\gamma} , \quad (64)$$

$$\frac{\phi_0 - \phi}{\phi_0} \propto t^{-2(n-1)(2-\gamma)/n\gamma} \propto a^{-3(n-1)(2-\gamma)/n} . \quad (65)$$

For $\gamma = 0$ both ξ and thus a grow exponentially with respect to the proper time t leading to de Sitter expansion as $\xi \to \infty$.

Note that the shear must vanish relative to all the other terms in the constraint Eq. (47) as we approach general relativity. As $\xi \to \infty$ we find $\rho \propto t^{-2}$ but $\sigma^2 \propto t^{-4/\gamma}$.

V. MASSLESS FIELDS IN ANISOTROPIC COSMOLOGIES

We will in this section restrict ourselves only to vacuum or to matter consisting of the short-wavelength modes (radiation, $\gamma = 4/3$) and long-wavelength modes (stiff fluid, $\gamma = 2$) of massless fields [20]. Here we can integrate the equations of motion in the Einstein frame without specifying $\omega(\phi)$.

For radiation the fluid is conformally invariant and in this case, as we have seen, a perfect fluid, i.e., noninteracting, in the Jordan frame remains a perfect fluid in the Einstein frame. On the other hand, a noninteracting stiff fluid in the Jordan frame does not remain a perfect fluid in the Einstein frame, but we can still deal with the dynamics in this case as the homogeneous Brans-Dicke field also becomes a stiff fluid, ψ, and although there is an interaction between the two components their combined dynamical effect is the same as that of a single perfect stiff fluid. Thus if the stiff fluid in the Jordan frame is a homogeneous minimally coupled scalar field φ, we can define the composite scalar, χ, by

$$d\chi^2 \equiv d\psi^2 + G\phi d\varphi^2 , \quad (66)$$

which obeys the equation of motion for a homogeneous minimally coupled field

$$\frac{d^2\chi}{d\tilde{t}^2} + \tilde{\theta}\frac{d\chi}{d\tilde{t}} = 0 . \quad (67)$$

The corresponding energy density in the Einstein frame is

$$\tilde{\rho}_\chi = \frac{1}{2}\left(\frac{d\chi}{d\tilde{t}}\right)^2 = \frac{3}{8\pi G}\frac{A^2}{4\tilde{V}^2} , \quad (68)$$

where A is a constant of integration. In the absence of a second field φ then we simply have $\chi \equiv \psi$.

Thus with the conformally transformed matter density and pressure, for radiation and or stiff fluid in the Einstein frame

$$\frac{8\pi G}{3}\tilde{\rho} = \frac{\Gamma}{\tilde{V}^{4/3}} + \frac{MG\phi}{\tilde{V}^2} , \quad (69)$$

$$\frac{8\pi G}{3}\tilde{p} = \frac{\Gamma}{3\tilde{V}^{4/3}} + \frac{MG\phi}{\tilde{V}^2} , \quad (70)$$

where Γ and M are non-negative constants, we can obtain the energy density of the Brans-Dicke field in the Einstein frame as

$$\hat{\rho} = \hat{p} = \frac{1}{2}\left(\frac{d\psi}{d\tilde{t}}\right)^2 = \frac{3}{8\pi G}\,\frac{A^2 - 4MG\phi}{4\tilde{V}^2}\,. \tag{71}$$

Clearly we require $4MG\phi \leq A^2$ for this to correspond to a non-negative energy density. Thus in the presence of a stiff fluid, $M \neq 0$, this places an upper bound on the value of the Brans-Dicke field, or equivalently a lower limit on the effective gravitational coupling constant in the Jordan frame, $G_{\rm eff} \sim \phi^{-1} \geq (4M/A^2)G$.

The evolution of the volume scale $\tilde{V}(\tilde{t})$ can be given if we can solve the Einstein equations (20) and (21) with total energy density

$$\frac{8\pi G}{3}\,(\hat{\rho} + \tilde{\rho}) = \frac{A^2}{4\tilde{V}^2} + \frac{\Gamma}{\tilde{V}^{4/3}} \tag{72}$$

and pressure

$$\frac{8\pi G}{3}\,(\hat{p} + \tilde{p}) = \frac{A^2}{4\tilde{V}^2} + \frac{\Gamma}{3\tilde{V}^{4/3}}\,, \tag{73}$$

as well as the shear and spatial curvature which will depend on the Bianchi class. This Einstein frame solution for radiation and a stiff fluid, in a particular Bianchi or Kantowski-Sachs metric, is independent of $\omega(\phi)$ and describes the Einstein frame behavior for any scalar-tensor theory.

It is only at the final stage that we must specify $\omega(\phi)$ in order to invert

$$\sqrt{\frac{16\pi G}{3}}\chi(\phi) = \pm\int\sqrt{\frac{A^2}{A^2 - 4MG\phi}}\,\sqrt{\frac{3+2\omega(\phi)}{3}}\,\frac{d\phi}{\phi}\,, \\ = A\int\frac{d\tilde{t}}{\tilde{V}}\,, \tag{74}$$

and thus obtain the evolution of $\phi(\chi)$, which is both the Brans-Dicke field and the conformal factor relating the Einstein frame to the Jordan metric. We see that the effect of a stiff fluid in the Jordan frame (i.e., $M \neq 0$ for a given value of A^2) is to alter the relation between χ and ϕ. But this is the role, in vacuum, solely of $\omega(\phi)$ and so any scalar-tensor theory, defined by $\omega(\phi)$, plus a stiff fluid in the Jordan frame is equivalent to an effective theory with $\omega_{\rm vac}(\phi)$ in vacuum given by [20]

$$3 + 2\omega_{\rm vac}(\phi) \equiv \frac{A^2}{A^2 - 4MG\phi}\,\left[3 + 2\omega(\phi)\right]\,. \tag{75}$$

However, this equivalence is broken by the presence of any matter other than radiation (i.e., any matter with nonzero trace of the energy-momentum tensor) which interacts in the Einstein frame with the field ψ, rather than χ.

Even without knowing the specific form of $\omega(\phi)$ we can note a few general features of $\chi(\phi)$. In particular if χ diverges this must correspond to either $\omega \to \infty$ or $\ln(G\phi) \to \pm\infty$ (with only $\phi \to 0$ possible for $M \neq 0$). The former case may occur as ϕ approaches a finite, nonzero constant value in which case the Jordan frame coincides (up to an arbitrary constant factor) with the Einstein frame and we must have general relativistic behavior. The latter case corresponds to singular behavior of the conformal factor so we may expect radically different behavior in the Jordan frame from that in the Einstein frame. On the other hand as $\chi \to$const we must have either $\omega \to -3/2$ or $\phi \to$constant. Thus for any $\omega > -3/2$ we must also recover general relativistic behavior in this limit.

A. Scalar-tensor theories

We will give here only three particular examples for which we can perform the inversion to find $\phi(\chi)$ analytically in Eq. (74), though we can solve for any $\omega(\phi)$ using numerical integration.

1. Brans-Dicke gravity

When $\omega = \omega_0$ =const we find

$$\sqrt{\frac{16\pi G}{3}}\chi(\phi) = \pm\sqrt{\frac{3+2\omega_0}{3}}\ln G\phi \qquad \text{for } M = 0, \tag{76}$$

$$= \pm\sqrt{\frac{3+2\omega_0}{3}}\ln\left[\frac{|A| + \sqrt{A^2 - 4MG\phi}}{|A| - \sqrt{A^2 - 4MG\phi}}\right] \qquad \text{for } M \neq 0\,. \tag{77}$$

Notice that while the presence of a stiff fluid leaves the evolution of the scale factor in the Einstein frame unaltered (i.e., independent of the value of M for a given A^2) it affects the form of $\chi(\phi)$ and thus the evolution of ϕ and the conformal transform back to the Jordan frame.

Inverting these relations gives

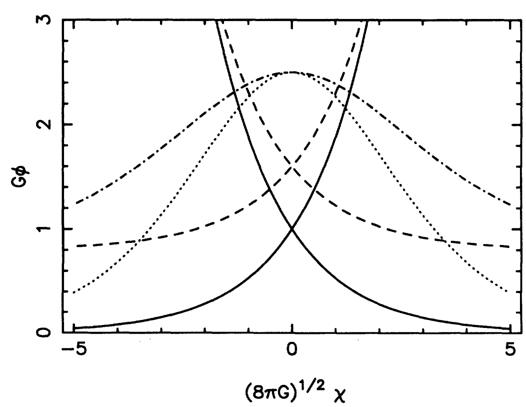

FIG. 1. The Brans-Dicke field ϕ, as a function of the minimally coupled field in the Einstein frame, χ. In Brans-Dicke gravity (with $\omega = 1$) $\phi(\chi)$ is given by solid lines (for two choices of sign of β_0) in vacuum or radiation ($A^2 = 2$, $M = 0$) or by dotted line in presence of a stiff fluid ($M = 0.2$). For $\omega(\phi)$ given in Eq. (81), with $\phi_0 = 0.8$, the dashed line gives the $\phi(\chi)$ for $M = 0$, or dotted-dashed line for $M = 0.2$.

$$G\phi(\chi) = \exp\left(\beta_0\sqrt{\frac{16\pi G}{3}}\chi\right) \quad \text{for } M = 0, \tag{78}$$

$$= \frac{A^2}{4M\cosh^2\left(\frac{\beta_0}{2}\sqrt{\frac{16\pi G}{3}}\chi\right)} \quad \text{for } M \neq 0, \tag{79}$$

as shown in Fig. 1, where

$$\beta_0 = \pm\sqrt{\frac{3}{3+2\omega_0}} \,. \tag{80}$$

Thus at an initial singularity in the Einstein frame where $\chi \to \infty$ we must have $\phi \to 0$ or $\phi \to \infty$ for $M = 0$. In the presence of a stiff fluid ($M \neq 0$) only $\phi \to 0$ is possible.

2. $\omega(\phi)$ with Brans-Dicke and general relativistic limits

An alternative choice of $\omega(\phi)$ that displays both a Brans-Dicke regime and general relativistic behavior is

$$2\omega(\phi) + 3 = (2\omega_0 + 3)\frac{\phi^2}{(\phi - \phi_0)^2} \,. \tag{81}$$

Clearly for $\phi \geq \phi_0$ we have $\omega \geq \omega_0$ and ω approaches this lower limit as $\phi \to \infty$, while as $\phi \to \phi_0$ we find $\omega \to \infty$.

Equation (74) then yields

$$\sqrt{\frac{16\pi G}{3}}\chi(\phi) = \pm\sqrt{\frac{2\omega_0+3}{3}}\ln\left(\frac{\phi-\phi_0}{\phi_0}\right) \quad \text{for } M = 0 \,, \tag{82}$$

$$= \pm\sqrt{\frac{2\omega_0+3}{3}}\sqrt{\frac{A^2}{A^2-4MG\phi}}\ln\left(\frac{\sqrt{A^2-4MG\phi_0}+\sqrt{A^2-4MG\phi}}{\sqrt{A^2-4MG\phi_0}-\sqrt{A^2-4MG\phi}}\right) \quad \text{for } M \neq 0 \,. \tag{83}$$

Note that $\chi \to \infty$ as $\phi \to \phi_0$, and also as $\phi \to \infty$ when $M = 0$. But because $\phi \leq A^2/4MG$ in the presence of the stiff fluid the latter Brans-Dicke limit cannot be reached.

This can be inverted to give

$$\phi(\chi) = \phi_0\left[1 + \exp\left(\sqrt{\frac{16\pi G}{3}}\beta_0\chi\right)\right] \quad \text{for } M = 0 \,, \tag{84}$$

$$= \frac{A^2}{4MG} - \left(\frac{A^2-4MG\phi_0}{4MG}\right)\tanh^2\left(\frac{1}{2}B_0\sqrt{\frac{16\pi G}{3}}\chi\right) \quad \text{for } M \neq 0 \,, \tag{85}$$

where β_0 was defined in Eq. (80) and

$$B_0 = \beta_0\sqrt{\frac{A^2-4MG\phi_0}{A^2}} \,. \tag{86}$$

The function $\phi(\chi)$ is plotted in Fig. 1.

3. Axion-dilaton string cosmologies

While the effective action derived in the low energy limit of string theory is sometimes referred to as Brans-Dicke gravity with $\omega = -1$, this is strictly only true when all other matter fields are minimally coupled to the Jordan, or "string," metric. This is generally not true in string theory, but we will show that at least for other massless fields such as the antisymmetric tensor field, $H_{\mu\nu\lambda}$, and moduli fields, β, appearing in the low energy effective action, the techniques developed for scalar-tensor gravity can also be applied to string theory. (For a more detailed discussion in the case of isotropic FRW cosmologies see [29].)

The background field equations of motion reduced to four dimensions can be derived from the action [4]

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}e^{-\varphi}\left[R + g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - ng^{\mu\nu}\beta_{,\mu}\beta_{,\nu} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \,. \tag{87}$$

Conventionally the variable gravitational coupling is represented by the dimensionless dilaton field φ, which is simply related to the Brans-Dicke field $\phi \equiv 1/(Ge^{\varphi})$. If the antisymmetric tensor field is a function only of the four-dimensional spacetime coordinates then it can be described by a single (pseudo-)scalar "axion" field h, where

$$H^{\mu\nu\lambda} = e^{\varphi}\epsilon^{\mu\nu\lambda\kappa}h_{,\kappa} \,, \tag{88}$$

and $\epsilon_{\mu\nu\lambda\kappa}$ is the antisymmetric volume form.

Conformally transforming to the Einstein frame the above action then becomes

$$S=\frac{1}{16\pi G}\int\sqrt{-\widetilde{g}}\left[\widetilde{R}-\frac{1}{2}\widetilde{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-n\widetilde{g}^{\mu\nu}\beta_{,\mu}\beta_{,\nu}\right.$$
$$\left.-\frac{1}{2}\widetilde{g}^{\mu\nu}e^{2\varphi}h_{,\mu}h_{,\nu}\right]. \tag{89}$$

Thus we see that the dilaton is simply related to our field ψ, defined in Eq. (11), as $d\psi\equiv-d\varphi/\sqrt{16\pi G}$. For homogeneous fields we can also define the composite scalar field $\chi(\varphi)$ such that

$$d\chi^2\equiv d\psi^2+\sqrt{\frac{n}{8\pi G}}d\beta^2+\frac{e^{2\varphi}}{16\pi G}dh^2\,, \tag{90}$$

whose total energy-momentum tensor is conserved. Although the axion field in particular is coupled directly to the dilaton, we can still integrate the equations of motion to deduce the relative energy densities in the Einstein frame:

$$\widetilde{\rho}_\chi=\frac{1}{2}\left(\frac{d\chi}{d\widetilde{t}}\right)=\frac{3}{8\pi G}\frac{A^2}{4\widetilde{V}^2}\,, \tag{91}$$

$$\widetilde{\rho}_\beta=\frac{n}{16\pi G}\left(\frac{d\beta}{d\widetilde{t}}\right)=\frac{3}{8\pi G}\frac{B^2}{4\widetilde{V}^2}\,, \tag{92}$$

$$\widetilde{\rho}_h=\frac{1}{32\pi G}\left(\frac{dh}{d\widetilde{t}}\right)=\frac{3}{8\pi G}\frac{Me^{-2\varphi}}{\widetilde{V}^2}\,, \tag{93}$$

and thus

$$\hat{\rho}=\frac{1}{2}\left(\frac{d\psi}{d\widetilde{t}}\right)=\frac{3}{8\pi G}\frac{A^2-B^2-4Me^{-2\phi}}{\widetilde{V}^2}\,. \tag{94}$$

Once again we see that the presence of other massless fields places an upper bound on the value of the Brans-Dicke field, or equivalently a lower bound on the dilaton:

$$e^{-\varphi}\le\sqrt{\frac{A^2-B^2}{4M}}\,. \tag{95}$$

The relation between χ and φ is then

$$\sqrt{\frac{16\pi G}{3}}\chi(\varphi)=\frac{1}{2\sqrt{3}}\int\frac{A}{\sqrt{A^2-B^2-4Me^{-2\varphi}}}\,d\varphi\,, \tag{96}$$

$$=\frac{1}{2\sqrt{3}}\sqrt{\frac{A^2}{A^2-B^2}}\,\varphi\quad\text{for } M=0\,, \tag{97}$$

$$=\frac{1}{2\sqrt{3}}\sqrt{\frac{A^2}{A^2-B^2}}\ln\left(\frac{\sqrt{A^2-B^2}+\sqrt{A^2-B^2-4Me^{-2\varphi}}}{\sqrt{A^2-B^2}-\sqrt{A^2-B^2-4Me^{-2\varphi}}}\right)\quad\text{for } M\neq 0\,. \tag{98}$$

Inverting this gives

$$e^{\varphi}(\chi)=\exp\left(2\sqrt{3}\sqrt{\frac{A^2-B^2}{A^2}}\sqrt{\frac{16\pi G}{3}}\chi\right)\quad\text{for } M=0\,, \tag{99}$$

$$=\left[\frac{2M}{A^2-B^2}\cosh\left(\sqrt{\frac{A^2-B^2}{A^2}}\sqrt{64\pi G}\chi\right)\right]^{1/2}\quad\text{for } M\neq 0\,. \tag{100}$$

For $M=0$ and $B=0$ this coincides with the Brans-Dicke result with $\omega=-1$. For $M=0$ but $B\neq 0$ the effective Brans-Dicke parameter lies in the range $-3/2<\omega<-1$. In the presence of the antisymmetric tensor field ($M\neq 0$) the form of $\phi\equiv(Ge^{\varphi})^{-1}$ differs from a purely Brans-Dicke result.

Thus we can use the same results as we will use for general scalar-tensor gravity models with stiff fluid (with or without radiation) in the Einstein frame to derive the general solutions to the low energy string effective action including homogeneous antisymmetric tensor and moduli fields (with or without radiation).

B. Anisotropic models

1. Bianchi type I

Considering again the spatially flat, anisotropic metric given in Eq. (25), we will introduce the variable $X\equiv\widetilde{a}^2\equiv(G\phi)a^2$ and the conformally invariant time coordinate $d\eta\equiv d\widetilde{t}/\widetilde{a}\equiv dt/a$. Remembering that the shear in the Einstein frame is given by $\widetilde{\sigma}^2=3\Sigma^2/4\widetilde{a}^6$, and using $\hat{\rho}$ as given in Eq. (71), so that

$$\frac{\phi'}{\phi}=\sqrt{\frac{3}{3+2\omega}}\,\frac{\sqrt{A^2-4MG\phi}}{X}\,, \tag{101}$$

and $\widetilde{\rho}$ given in Eq. (69), we find the constraint Eq. (20), in vacuum or with a stiff fluid in the Jordan frame (but for $\Gamma=0$), can be simply written as

$$X'^2=A^2+\Sigma^2\,. \tag{102}$$

This is precisely the same constraint equation as solved in the case of scalar-tensor gravity in flat FRW models in vacuum [19] or with stiff fluid [20] and we have

$$X=\widetilde{a}^2=\sqrt{A^2+\Sigma^2}\,|\eta-\eta_0|\,. \tag{103}$$

In the presence also of radiation in the Jordan frame the constraint Eq. (20) becomes

$$X'^2 = A^2 + \Sigma^2 + 4\Gamma X \,, \tag{104}$$

and so

$$X = \widetilde{a}^2 = |\eta - \eta_0| \left(\sqrt{A^2 + \Sigma^2} + \Gamma\, |\eta - \eta_0| \right) . \tag{105}$$

These are well-known general relativistic results for the evolution of Bianchi type I models in the presence of radiation and stiff fluid. As one would expect the stiff fluid and shear dominate the evolution near the singularity ($\widetilde{a} \to 0$) but the radiation term will dominate as $\widetilde{a} \to \infty$.

The behavior of this averaged scale factor, together with the shear

$$\widetilde{\sigma}^2 = \frac{3\Sigma^2}{4|\eta - \eta_0|^3(\sqrt{A^2 + \Sigma^2} + \Gamma|\eta - \eta_0|)^3} \,, \tag{106}$$

describe the general evolution. However, the Bianchi type I metric has three degrees of freedom so there is a degeneracy within the evolution we have described so far depending on how the expansion and shear are divided among the three scale factors. Solving the equations of motion for each scale factor in the Einstein frame we find

$$\widetilde{a}_i = |\eta - \eta_0|^{3c_i/2} \left(\sqrt{A^2 + \Sigma^2} + \Gamma|\eta - \eta_0| \right)^{1-(3c_i/2)} \,, \tag{107}$$

where the definitions of the overall expansion and shear give two constraints on the three new integration constants:

$$\sum_i c_i = 1 \,, \qquad \sum_i c_i^2 = 1 - \frac{2A^2}{3(A^2 + \Sigma^2)} \,. \tag{108}$$

Thus although there is a unique (isotropic) late-time behavior in the presence of radiation ($\Gamma \neq 0$), where $\widetilde{a}_i \propto |\eta - \eta_0|$, the initial behavior (for $|\eta - \eta_0| \ll \sqrt{A^2 + \Sigma^2}/\Gamma$) is dependent upon the choice of integration constants c_i (see Fig. 2).

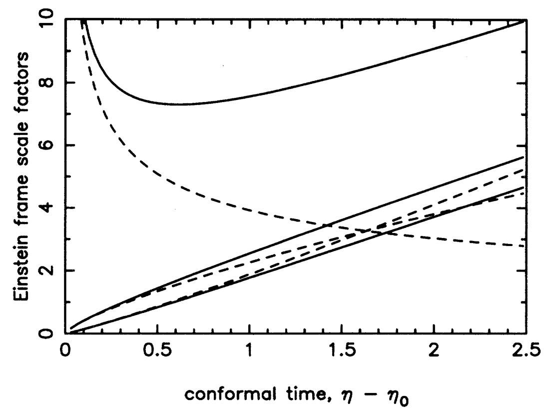

FIG. 2. Evolution of the three independent scale factors $\widetilde{a}_i$, in the Einstein frame, in Bianchi type I metric as given in Eq. (107) with $A^2 = 2$ and with radiation (solid line, $\Gamma = 2$) or without (dashed line, $\Gamma = 0$). We have chosen $c_1 = -1/4$, $c_2 = 1/2$, and $c_3 = 3/4$ in both cases.

This early power-law evolution of the scale factors includes the pure general relativistic "Kasner" vacuum solution with $A = 0$. Although the averaged scale factor, $\widetilde{a}$, always vanishes at η_0, this singularity can be pointlike (all $c_i > 0$ for all i) or linear (only one $c_i < 0$) unlike the Kasner behavior for which only the linear singularity is possible (except when $c_j = 1$ and $c_{i \neq j} = 0$ and the singularity is planar). Indeed for $A^2 > 3\Sigma^2$ only the pointlike singularity is possible [11].

To recover the full scalar-tensor results in the original Jordan frame we must calculate χ as a function of time from Eq. (68),

$$\sqrt{\frac{16\pi G}{3}}\, [\chi(\eta) - \chi_0] = \frac{A}{\sqrt{A^2 + \Sigma^2}} \ln|\eta - \eta_0| \qquad \text{for } \Gamma = 0, \tag{109}$$

$$= \frac{A}{\sqrt{A^2 + \Sigma^2}} \ln \frac{\Gamma|\eta - \eta_0|}{\sqrt{A^2 + \Sigma^2} + \Gamma|\eta - \eta_0|} \qquad \text{for } \Gamma \neq 0, \tag{110}$$

and then use $\phi(\chi)$, dependent on the form of $\omega(\phi)$, to specify the evolution of the Brans-Dicke field, and thus the conformal factor relating these solutions back to the Jordan frame variables. Note that the field χ must always diverge as $\eta \to \eta_0$. (In the absence of radiation it also diverges at $\eta \to \pm\infty$.)

As noted earlier, the definition of $\chi(\phi)$ in Eq. (74) requires either $\omega \to \infty$ or $\ln(G\phi) \to \pm\infty$. The former case where ϕ tends to a finite, nonzero value will lead to a purely general relativistic result as $X \to 0$. In the latter case we can expand about the point $X \to 0$ and we find that in the Jordan frame we have

$$\phi \propto |\eta - \eta_0|^n \,, \tag{111}$$

$$a \propto |\eta - \eta_0|^{(1-n)/2} \,, \tag{112}$$

$$a_i \propto |\eta - \eta_0|^{(3c_i - n)/2} \,, \tag{113}$$

$$t \propto |\eta - \eta_0|^{(3-n)/2} \,, \tag{114}$$

if $\omega \to \omega(0) \neq -3/2$ and $n \neq 3$, where, from Eq. (101),

$$n = \pm\sqrt{\frac{3}{3+2\omega}}\sqrt{\frac{A^2}{A^2+\Sigma^2}} \, . \qquad (115)$$

Thus without specifying the full $\omega(\phi)$ we can deduce a number of features of the possible behavior in the Jordan frame.

(1) In the presence of shear and/or a stiff fluid in the Jordan frame, a "bounce" (where the volume factor V is stationary, $dV/dt = 0$) will occur whenever

$$\frac{da}{dt} = \frac{1}{2}\left(\frac{X'}{X} - \frac{\phi'}{\phi}\right) = 0 \, , \qquad (116)$$

which, from Eqs. (101) and (104), requires

$$\omega = -\frac{3}{2} + \frac{3}{2}\left(\frac{A^2 - 4MG\phi}{A^2+\Sigma^2}\right) \, . \qquad (117)$$

We thus see that a bounce is only possible when

$$-\frac{3}{2} \leq \omega \leq 0 \, . \qquad (118)$$

In a flat FRW metric ($\Sigma^2 = 0$) and in the absence of a stiff fluid ($M = 0$) we require $\omega = 0$ [20]. The presence of shear or a stiff fluid requires a negative value of ω, producing a negative effective energy density for the Brans-Dicke field in the Jordan frame, to produce a bounce. For $\omega(\phi)$ less than that given in Eq. (117), the averaged scale factor in the Jordan frame actually grows as both X and ϕ approach zero.

(2) The singularity at $X = 0$ is always present in the Einstein frame, and is bound to be anisotropic for $\Sigma^2 \neq 0$, with $\widetilde{\sigma}^2 \to \infty$ as $X \to 0$. However, we noted in Sec. III that in the Jordan frame the Brans-Dicke field exerts an anisotropic pressure proportional to the shear and one might wonder whether it is possible for this to suppress the shear as one approaches the singularity. This is indeed possible. The shear in the Jordan frame in a Bianchi type I model is given by

$$\sigma^2 = \frac{3\Sigma^2 G\phi}{4X^3} \, , \qquad (119)$$

so if ϕ vanishes faster than X^3 the anisotropic initial singularity in the Einstein frame becomes isotropic in the Jordan frame. Again from Eqs. (74) and (104) we find that this requires $\phi \to 0$ and

$$\omega < -\frac{3}{2} + \frac{1}{6}\left(\frac{A^2}{A^2+\Sigma^2}\right) \, . \qquad (120)$$

(3) Is it then possible that the singularity as $X \to 0$ in the Einstein frame can be avoided completely in the Jordan frame? First note that the condition that $da/dt = 0$ [Eq. (117)] is incompatible with the condition that the shear should vanish [Eq. (120)], or even remain finite, as X and ϕ both approach zero. However, the expansion $\theta \equiv 3(da/dt)/a$ can remain finite even though da/dt diverges if the average scale factor a grows fast enough. Using the above results for a general scalar-tensor theory in a Bianchi type I metric with a stiff fluid, we can write the expansion as

$$\theta^2 = \frac{3(A^2+\Sigma^2)}{4}\,\frac{G\phi}{X^3}\left(1 \pm \sqrt{\frac{3}{3+2\omega}}\sqrt{\frac{A^2-4MG\phi}{A^2+\Sigma^2}}\right) \, . \qquad (121)$$

We see that whenever $\phi \to 0$ as $X \to 0$ such that the shear $\sigma^2 \to 0$, then the expansion will also vanish (for $\omega \neq -3/2$). This also coincides with the case where $\eta \to \eta_0$ takes an infinite proper time in the Jordan frame. Thus although these models reach an anisotropic singularity in a finite time in the Einstein frame, *this corresponds to a nonsingular, shear-free infinite proper lifetime in the Jordan frame.* The condition for this to occur is simply the condition for the shear to vanish as $X \to 0$ given above, which can occur for

$$-\frac{3}{2} < \omega < -\frac{4}{3} \qquad (122)$$

in a flat FRW metric ($\Sigma^2 = 0$), or for the more limited range given in Eq. (120) depending on the relative strength of the anisotropy.[4] Remember that $\phi = 0$ coincides with $\chi(\phi) \to \infty$ for $\omega \neq -3/2$ and so $\phi \to 0$ can be an attractor solution as $X \to 0$ where we have shown χ must diverge.

(4) If ϕ grows sufficiently rapidly as we approach $X \to 0$ then the dynamical effect of other matter, neglected here, may no longer remain negligible. The condition for the density of isotropic matter with barotropic index γ to decrease with respect to the shear or Brans-Dicke field density in the Einstein frame as $\eta \to \eta_0$ is that

$$\omega > -\frac{3}{2} + \frac{3}{2}\left(\frac{4-3\gamma}{2-\gamma}\right)^2\left(\frac{A^2}{A^2+\Sigma^2}\right) \, . \qquad (123)$$

For smaller values of ω it may no longer be possible to neglect the effect of this matter as $\widetilde{a} \to 0$. In particular in the case of the nonsingular solutions given above with vanishing shear and expansion (in the Jordan frame) as $\eta \to \eta_0$, the relative density of barotropic matter always grows as we approach η_0 for fluids with $\gamma \leq 1$. We have also neglected here any possible anisotropy in the matter content. This, of course, may well play an important role in anisotropic solutions, but is beyond the scope of this paper and we leave this for future work.

2. Bianchi type V

The next simplest case to consider is that of a Bianchi type V metric which can be written as [30, 24]

$$ds^2 = -dt^2 + a^2(t)\{dx^2 + e^{2x}\left[L^2(t)dy^2 + L^{-2}(t)dz^2\right]\}. \qquad (124)$$

---

[4]The requirement that $\omega < -4/3$ to avoid the singularity was pointed out by Nariai [9] in the case of Brans-Dicke gravity.

Just as Bianchi type I includes the flat FRW model, Bianchi type V includes the open FRW model in the isotropic limit, L =const. Here we have already introduced the averaged scale factor a, so that the expansion is simply

$$\theta = \frac{3}{a}\frac{da}{dt} , \tag{125}$$

and the shear is then

$$\sigma = \frac{1}{L}\frac{dL}{dt} . \tag{126}$$

Like open FRW models, the homogeneous hypersurfaces have a negative spatial curvature

$$^{(3)}R = -\frac{3}{a^2} , \tag{127}$$

but because this is a function of the averaged scale factor, i.e., does not select out a particular direction, it does not drive the shear.

Thus transforming to the Einstein frame where the Brans-Dicke field decouples from the spacetime curvature, we find that the shear again evolves as a free field, $\widetilde{\sigma}^2 = 3\Sigma^2/4\widetilde{a}^6$, just as in Bianchi type I. Introducing $X \equiv \widetilde{a}^2 \equiv G\phi a^2$ and $d\eta \equiv d\widetilde{t}/\widetilde{a} \equiv dt/a$ as before, the constraint equation, including stiff fluid and radiation, simply becomes

$$X'^2 - 4X^2 = A^2 + \Sigma^2 + 4\Gamma X . \tag{128}$$

This is mathematically identical to the constraint equation solved in the case of scalar-tensor gravity for an open FRW model [20]. We can integrate this to obtain

$$X = \widetilde{a}^2 = \frac{\tau\left(\sqrt{A^2+\Sigma^2} + \Gamma\tau\right)}{1-\tau^2} , \tag{129}$$

where we have written

$$\tau \equiv \tanh|\eta - \eta_0| . \tag{130}$$

This variable, τ, turns out to play much the same role in the presence of negative spatial curvature as the conformal time, η, does in the spatially flat case, as is the case in FRW models [20].

At late times ($\eta \to \pm\infty$) $\tau \to 1$ and the evolution becomes curvature dominated, as we would expect in a noninflationary universe. Because the curvature does not drive the shear,

$$\widetilde{\sigma}^2 = \frac{3(1-\tau^2)^3\Sigma^2}{4\tau^3\left(\sqrt{A^2+\Sigma^2}+\Gamma\tau\right)^3} , \tag{131}$$

it vanishes as $\tau \to 1$.

At early times $\tau \simeq |\eta - \eta_0| \ll 1$ and so the curvature is irrelevant and we recover a Bianchi type I solution. However, unlike the general Bianchi type I, the metric has only two degrees of freedom and so its evolution in the Einstein frame is completely described by the expansion and shear. Integrating the expression for $\widetilde{\sigma}$ gives

$$L^2 = L_0^2 \tau^{\sqrt{3}\Sigma/\sqrt{A^2+\Sigma^2}} \qquad \text{for } \Gamma = 0 , \tag{132}$$

$$= L_0^2\left(\frac{\tau}{\sqrt{A^2+\Sigma^2}+\Gamma\tau}\right)^{\sqrt{3}\Sigma/\sqrt{A^2+\Sigma^2}} \qquad \text{for } \Gamma \neq 0 . \tag{133}$$

The behavior near the singularity in Bianchi type V metrics is thus only a subset of the Bianchi type I solutions, given in Eqs. (107) and (108), restricted to $c_2 - c_3 = \sqrt{3}\Sigma/(A^2+\Sigma^2)$. Thus (for a given A and Γ) they are parametrized solely by the choice of Σ.

Similarly, because the Brans-Dicke field is decoupled from the spacetime curvature in the Einstein frame, the field

$$\sqrt{\frac{16\pi G}{3}}(\chi - \chi_0) = \frac{A}{\sqrt{A^2+\Sigma^2}}\ln\tau \text{ for } \Gamma = 0 , \tag{134}$$

$$= \frac{A}{\sqrt{A^2+\Sigma^2}}\ln\frac{\Gamma\tau}{\sqrt{A^2+\Sigma^2}+\Gamma\tau} \qquad \text{for } \Gamma \neq 0 , \tag{135}$$

and thus $\phi(\chi)$ approaches a fixed value ϕ_∞ as $\tau \to 1$ [unless $\chi(\phi)$ is singular at this point]. Thus the evolution of the metric in the original Jordan frame will also approach that in the Einstein frame at late times.

3. Bianchi type III and Kantowski-Sachs models

Here we will write the metric in the Jordan frame as

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)\left[dy^2 + s^2(y)dz^2\right] , \tag{136}$$

where

$$s(y) \equiv \begin{cases} \sin y & \text{for Kantowski-Sachs models,} \\ y & \text{for axisymmetric Bianchi I,} \\ \sinh y & \text{for Bianchi III.} \end{cases} \tag{137}$$

The volume scale factor $V = a_1 a_2^2$. If we introduce a conformally invariant time coordinate $d\xi \equiv \frac{dt}{a_2} = \frac{d\widetilde{t}}{\widetilde{a}_2}$ and let $X \equiv \widetilde{a}_1\widetilde{a}_2$, we can write the Einstein equations as

$$X'' + 4kX = 8\pi G(\widetilde{\rho} - \widetilde{p})X\widetilde{a}_2^2, \tag{138}$$

$$\left(\frac{\widetilde{a}_1'}{\widetilde{a}_1}\right)' + \frac{X'}{X}\frac{\widetilde{a}_1'}{\widetilde{a}_1} = 4\pi G(\widetilde{\rho} - \widetilde{p})\widetilde{a}_2^2, \tag{139}$$

$$\left(\frac{X'}{X}\right)^2 - \left(\frac{\widetilde{a}_1'}{\widetilde{a}_1}\right)^2 + 4k = 8\pi G\widetilde{\rho} + \hat{\rho}\widetilde{a}_2^2 , \tag{140}$$

where $' \equiv d/d\xi$ and $k = +1, 0, -1$ corresponds to Kantowski-Sachs, Bianchi I, or Bianchi III, respectively, in analogy with FRW models.

We have only been able to solve these equations analytically in the presence of a stiff fluid plus the Brans-Dicke field. Even in the case of radiation, where one can obtain both $\widetilde{\rho}$ and $\hat{\rho}$ as functions of $\widetilde{a}_1$ and $\widetilde{a}_2$, the resulting equations for $\widetilde{a}_1$ and $\widetilde{a}_2$ still cannot be integrated. However, we see that a stiff fluid in the Einstein frame, like the Brans-Dicke field, does not enter the first two equations and so (for $\Gamma = 0$) we can integrate both of these directly to give

$$X = \frac{\bar{A}}{2}s(2[\xi - \xi_0]) , \tag{141}$$

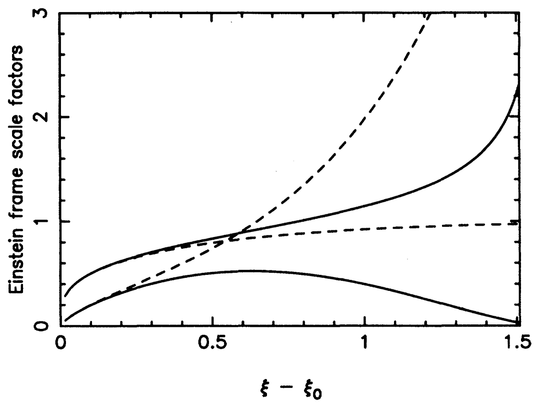

FIG. 3. Evolution with respect to time coordinate ξ of the two independent scale factors $\widetilde{a}_1$ and $\widetilde{a}_2$, in the Einstein frame, in Kantowski-Sachs (solid line) or Bianchi type III metric (dashed) as given in Eq. (143). We have taken $C = 0.3$.

with the appropriate function $s(x)$ defined above and thus

$$\widetilde{a}_1 = \widetilde{a}_{1*}\tau^C , \tag{142}$$

$$\widetilde{a}_2 = \widetilde{a}_{2*}\frac{\tau^{1-C}}{1+k\tau^2} , \tag{143}$$

where we have defined

$$\tau \equiv \begin{cases} |\tan(\xi-\xi_0)| & \text{for Kantowski-Sachs models,} \\ |\xi-\xi_0| & \text{for axisymmetric Bianchi I,} \\ |\tanh(\xi-\xi_0)| & \text{for Bianchi III.} \end{cases} \tag{144}$$

$\bar{A}$, C, and $\widetilde{a}_{1*}$ are integration constants and $\widetilde{a}_{2*} = \bar{A}/\widetilde{a}_{1*}$. The evolution of the two scale factors is shown in Fig. 3.

The energy density of the stiff fluid in the Einstein frame is given by Eq. (68) where the constraint Eq. (140) requires

$$A^2 = \frac{4}{3}\widetilde{a}_{1*}^2\widetilde{a}_{2*}^2(1-C^2) , \tag{145}$$

and so clearly we must have $C^2 < 1$ in the presence of a stiff fluid in the Einstein frame, or $C = \pm 1$ in general relativistic (i.e., ϕ =const) vacuum.

Note that we again obtain Kasner type solutions near the singularity at $\xi \to \xi_0$, independent of the sign of k. The curvature becomes irrelevant and we recover an axisymmetric subset of the Bianchi type I solutions, $a_i \propto |\eta - \eta_0|^{3c_i/2}$, where $d\eta \equiv d\widetilde{t}/(\widetilde{a}_1\widetilde{a}_2^2)^{1/3}$ and

$$c_1 = \frac{C}{2-C} \qquad \text{and} \qquad c_2 = c_3 = \frac{1-C}{2-C} > 0 . \tag{146}$$

These c_i obey the relations given in Eq. (108) with

$$\Sigma^2 = \left(\frac{2a_{1*}a_{2*}(1-2C)}{3}\right)^2 . \tag{147}$$

The area of the two-dimensional subspace always vanishes as we approach the singularity, while the remaining spatial dimension is free to diverge (for $C < 0$) or collapse ($C > 0$). Thus the singularity is linear or pointlike. In the Kantowski-Sachs case we find another singularity at $\xi \to \xi_0 + \pi/2$. This is like the initial singularity but with $C \to -C$, and thus a pointlike singularity is followed by a linear singularity, or vice versa (unless $C = 0$ in which case $\widetilde{a}_1$ remains constant throughout).

We can then integrate Eq. (68) to obtain

$$\sqrt{\frac{16\pi G}{3}}(\chi - \chi_0) = \int \frac{Ad\xi}{X}, \tag{148}$$

$$= \sqrt{\frac{4(1-C^2)}{3}}\ln\tau . \tag{149}$$

Thus χ must diverge both at early and late times in the Kantowski-Sachs (or Bianchi type I) metric, but the expanding universe becomes curvature dominated in the Bianchi type III metric with positive spatial curvature and χ coasts to a fixed value $\chi \to \sqrt{4(1-C^2)/3}$.

4. LRS Bianchi type IX

To give an example of an anisotropic cosmology with closed spatial hypersurfaces we consider a Bianchi type IX metric [9, 10]

$$\begin{aligned} ds^2 = -dt^2 &+ a_1^2(t)\ (\sin\psi d\theta - \cos\psi\sin\theta d\varphi)^2 \\ &+ a_2^2(t)\ (\cos\psi d\theta + \sin\psi\sin\theta d\varphi)^2 \\ &+ a_3^2(t)\ (d\psi + \cos\theta d\varphi)^2 , \end{aligned} \tag{150}$$

whose homogeneous spatial hypersurfaces have volume $V = 16\pi a_1a_2a_3$, so the expansion

$$\theta = \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} . \tag{151}$$

When $\dot{a}_1/a_1 = \dot{a}_2/a_2 = \dot{a}_3/a_3$ we recover the closed FRW model.

As the curvature terms in anisotropic models become more complicated, our ability to give analytic solutions becomes more restricted. In the case of Bianchi type IX we can only give analytic solutions for the locally rotationally symmetric (LRS) case (where $a_2 = a_3$) plus stiff fluid in the Einstein frame [31], corresponding to vacuum or stiff fluid solutions in scalar-tensor gravity. This will not show the chaotic behavior of the more general Bianchi IX metric [32] nor the isotropizing effect of matter such as isotropic radiation. Such issues would require a phase-space analysis and is beyond the scope of this paper but is perhaps a topic worthy of investigation in its own right.

The equations of motion for the scale factors in the Einstein frame are then

$$\frac{d}{d\widetilde{t}}\left(\frac{\dot{\widetilde{a}}_1}{\widetilde{a}_1}\right) + \widetilde{\theta}\frac{\dot{\widetilde{a}}_1}{\widetilde{a}_1} = -\frac{\widetilde{a}_1^2}{2\widetilde{a}_2^4} , \tag{152}$$

$$\frac{d}{d\widetilde{t}}\left(\frac{\dot{\widetilde{a}}_2}{\widetilde{a}_2}\right) + \widetilde{\theta}\frac{\dot{\widetilde{a}}_2}{\widetilde{a}_2} = \frac{\widetilde{a}_1^2 - 2\widetilde{a}_2^2}{2\widetilde{a}_2^4} , \tag{153}$$

plus the constraint equation

$$\left(\frac{\dot{\tilde{a}}_2}{\tilde{a}_2}\right)^{\cdot} + 2\frac{\dot{\tilde{a}}_1}{\tilde{a}_1}\frac{\dot{\tilde{a}}_2}{\tilde{a}_2} + \frac{4\tilde{a}_1^2\tilde{a}_2^2 - \tilde{a}_1^4}{4\tilde{a}_1^2\tilde{a}_2^4} = \frac{3A^2}{4\tilde{a}_1^2\tilde{a}_2^4} \ , \qquad (154)$$

where the only energy density is $\tilde{\rho}_\chi$ given in Eq. (68).

Introducing the volume weighted time coordinate $d\zeta = d\tilde{t}/(\tilde{a}_1\tilde{a}_2^2)$ and using the variables $x = 4\tilde{a}_1^4$ and $y = 4\tilde{a}_1^2\tilde{a}_2^2$, we can rewrite the equations of motion as

$$2\left(\frac{x'}{x}\right)' + x = 0 \ , \qquad (155)$$

$$2\left(\frac{y'}{y}\right)' + y = 0 \ , \qquad (156)$$

which can immediately be integrated to give

$$\tilde{a}_1^2 = \frac{w_1}{\cosh w_1(\zeta - \zeta_1)} \ ,$$
$$\tilde{a}_2^2 = \frac{w_2^2 \cosh w_1(\zeta - \zeta_1)}{w_1 \cosh^2 w_2(\zeta - \zeta_2)} \ , \qquad (157)$$

as shown in Fig. 4, where ζ_1, ζ_2, w_1, and w_2 are integration constants. Clearly, w_1 must be positive and we can also take w_2 to be positive without loss of generality.

The constraint equation requires

$$3A^2 = 4w_2^2 - w_1^2 \ . \qquad (158)$$

This is sufficient to ensure that $w_1 \leq 2w_2$ and both $\tilde{a}_2$ and $\tilde{a}_1$ approach zero as $\zeta \to \pm\infty$ (for nonzero w_1 and w_2), the interval between this big bang and big crunch taking only a finite proper time in the Einstein frame. Near these singularities the scale factors evolve as power laws, with respect to proper time (or conformal time), and we recover another one-parameter (for given A) subset of the Bianchi type I solutions, with

$$c_1 = \frac{w_1}{4w_2 - w_1} \ , \qquad c_2 = c_3 = \frac{2w_1 - w_1}{4w_2 - w_1} \ . \qquad (159)$$

The initial (and final) shear then approaches $\tilde{\sigma}^2 \to 3\Sigma^2/4\tilde{a}^6$ with $\Sigma^2 \equiv 4(w_2 - w_1)^2/9$.

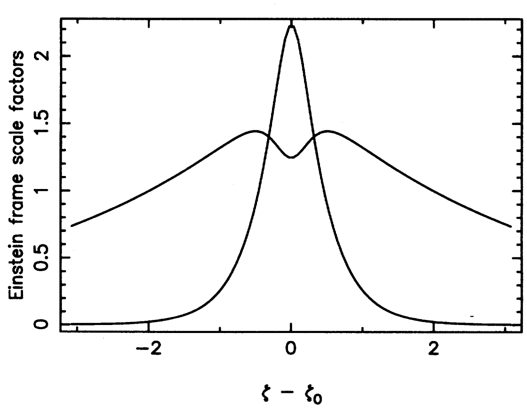

FIG. 4. Evolution with respect to time coordinate ζ of two independent scale factors $\tilde{a}_1$ and $\tilde{a}_2$, in Einstein frame, in LRS Bianchi type IX metric as given in Eq. (157). We have taken $A^2 = 2$ and $w_1 = 5$.

Note that the equation for the scalar field energy density, Eq. (68), shows that χ is just proportional to the volume weighted time, ζ, and so we have

$$\sqrt{\frac{16\pi G}{3}}(\chi - \chi_0) = \pm A\zeta \ , \qquad (160)$$

which must also diverge as we approach both the initial and final singularities.

VI. CONCLUSIONS

We have derived a number of new exact solutions for anisotropic cosmologies in scalar-tensor gravity theories. Previous studies of general scalar-tensor theories have been restricted to isotropic (FRW) models [19, 27, 18, 20]. We have extended the method used for deriving exact solutions for barotropic fluids in spatially flat FRW models [18] to Bianchi type I spacetimes and demonstrated that the ability of general relativity to act as a cosmological attractor [27] also extends to these models. Earlier studies of anisotropic scalar-tensor cosmologies concentrated on the particular case of Brans-Dicke gravity [9–11, 33, 30, 13]. In the Brans-Dicke gravity theory the deviation from general relativistic behavior is strictly limited due to present day observational limits on the Brans-Dicke parameter ω [6]. However, in more general scalar-tensor gravity the present day value need not constrain the value of ω near an initial singularity. In the Jordan frame the nonminimal coupling of the Brans-Dicke field introduces an effective anisotropic pressure proportional to the existing shear which can significantly modify the evolution even in vacuum, especially near a singularity. It is possible for the Brans-Dicke field to reverse the collapse as we approach an anisotropic singularity and lead to a nonsingular, isotropic expanding universe.

We have emphasized the importance of using the conformally related Einstein metric where the Brans-Dicke field is minimally coupled with respect to the metric and so the usual general relativistic results remain valid. Here, for instance, there is no anisotropic pressure due to the Brans-Dicke field. Non-Einstein behavior only appears in the transformation back to the Jordan frame.

The analytic complexity of the evolution in the Einstein frame lies in the interaction introduced directly between the Brans-Dicke field and ordinary matter. Except in the particular cases of vacuum, radiation, or a stiff fluid, this forces us to restrict our results for general barotropic fluids [$p = (\gamma - 1)\rho$] to the spatially flat Bianchi type I metric. Here the solutions for general $\omega(\phi)$ theories are characterized by a generating function $g(\xi)$, where ξ is a time coordinate. Brans-Dicke gravity corresponds to the case where $g \propto (\xi + \xi_0)^2$.

The shear vanishes relative to the density of matter and/or the Brans-Dicke field (i.e., the models isotropize) whenever $|\xi| \to \infty$. This requires simply that g remains positive, or equivalently that the Brans-Dicke parameter $\omega \geq 2(\gamma - 5/3)/(2-\gamma)^2$, as $|\xi| \to \infty$. If $g < \alpha\xi^2$ for any constant $\alpha > 0$ as $\xi \to \infty$ then the dynamics become dominated by the Brans-Dicke field density at late times. Conversely, if $g > \alpha\xi^2$ then the models become dominated by the barotropic matter and we recover the general relativistic behavior where the Brans-Dicke field $\phi \to$const and $\omega \to \infty$. Thus models that approach general relativity must isotropize. Brans-Dicke gravity is seen to correspond to the particular case where the relative densities of the Brans-Dicke field and ordinary matter remain proportional.

Singularities in the Jordan frame (where the volume scale factor vanishes) occur only when the effect of ordinary matter (with $p < \rho$) becomes negligible compared with the shear and Brans-Dicke field energy density. Thus it is sufficient to consider only the vacuum or stiff fluid ($p = \rho$) models to discuss the approach to the singularity.

We have shown that in the presence only of radiation and a stiff fluid (equivalent to the short and long wavelength modes of massless fields), or in vacuum, the evolution of the scale factor in the Einstein frame is independent of the form of $\omega(\phi)$ and corresponds exactly to the standard general relativistic evolution with radiation and a stiff fluid [20]. This enables us to give results for arbitrary $\omega(\phi)$ theories in Bianchi type I, III, V and LRS type IX and Kantowski-Sachs metrics. At singularities in the Einstein frame these all approach a Bianchi type I solution as the spatial curvature becomes negligible. As has previously been shown in Brans-Dicke gravity [9–11, 33, 30], the presence of the Brans-Dicke field, acting as a stiff fluid in the Einstein frame can change the nature of the singularity, admitting the possibility of a pointlike, anisotropic singularity. Transforming back to the Jordan metric may further modify the evolution dependent upon $\omega(\phi)$. If $\omega < -4/3$ then the initial singularity can be avoided in the Jordan frame, as found by Nariai in the case of Brans-Dicke gravity [9].

Approaching the singularity, the energy density of the scalar field in the Einstein frame, χ, must diverge. The relation between this field and the original Brans-Dicke field depends on the form of $\omega(\phi)$, but in any case a necessary condition for the divergence of χ is that $\ln(\phi)$ diverges or $\omega \to \infty$. However, the simplest general relativistic limit, considered above as a late-time attractor in Bianchi type I, in vacuum, or in the presence solely of radiation or stiff fluid, is not sufficient to lead to a divergence of χ and does not act as an attractor solution at the initial singularity. On the other hand $\phi \to 0$ is an attractor which can lead to nonsingular behavior in the Jordan frame.

Thus while many scalar-tensor gravity theories do approach general relativity at late times, where the role of other barotropic matter is dominant and anisotropy vanishes, they may nonetheless give markedly non-Einstein behavior near the initial big bang where anisotropy is important.

Note added. After completing this paper we became aware of similar work [34] considering cosmological solutions in scalar-tensor gravity with massless fields in spatially flat FRW and Bianchi type I metrics.

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