Tetrad-based perturbative approach to inhomogeneous universes: A general relativistic version of the Zel'dovich approximation

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A new approximation scheme in general relativity is developed to describe nonlinear inhomogeneous universes containing irrotational dust. The parallel-transported basis of the orthonormal tetrad frame is employed and a second-order differential equation is obtained for the perturbations of the spatial basis vectors, with nonlinear corrections as a source term. The equation can be solved iteratively in a way very similar to that in the Lagrangian perturbation theory in Newtonian cosmology. The first-order solution is presented, which contains Szekeres' exact solution as a special case. A general relativistic version of the "Zel'dovich approximation" is proposed with emphasis on the formal similarity to the Newtonian treatment.

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I. INTRODUCTION

There is growing interest on the general relativistic treatment of nonlinear, inhomogeneous cosmology. Futamase [1–3] formulated a cosmological post-Newtonian approximation to describe the metric for a clumpy universe. (See also Tomita [4,5] for post-Newtonian equations of motion in an expanding universe.) Spatial averaging is introduced in his papers to get a background metric, and the Hubble expansion is generated collectively by the inhomogeneously distributed matter. A more rigorous argument on the spatial averaging is done by Kasai [6,7]. An example of exact solutions is presented which are nonlinearly inhomogeneous but homogeneous and isotropic on average. The relation to gauge-invariant perturbation theory is also discussed. Futamase's scheme may be classified as a "short wavelength approximation." It is applicable only when the characteristic scale of spatial fluctuations, say ℓ, is smaller than the Hubble radius L: $\ell/L \ll 1$. There are also works on the so-called "long wavelength approximation," such as the "anti-Newtonian" approximation [8–11] and the gradient expansion technique [12]. They rely on the assumption that the characteristic scale is much larger than the Hubble radius: $\ell/L \gg 1$. From the viewpoint of structure formation, however, those treatments are not very relevant because the characteristic size of interest is smaller than the horizon size at the later stage of structure formation where nonlinear effects become important. In the matter-dominated era, the horizon size L grows faster than ℓ. Therefore, even if we assume $\ell/L \gg 1$ in the early stage, the long wavelength approximations have a limited regime of validity until the horizon-crossing epoch $\ell/L = 1$. Tomita also developed the second-order perturbation theory as an extension of linearized theory in general relativity [13]. It, however, still relies on the assumption of the density fluctuation being small.

Matarrese *et al.* [14] developed another algorithm based on the fluid flow approach [15]. They claim some similarity to the Zel'dovich approximation in Newtonian theory, but the dynamics in their formulation is complicated and followed by a system of six coupled first-order differential equations even under the restriction of a vanishing magnetic part of the Weyl tensor $H_{ab} = 0$. The approach was extended up to the second order and the results were presented both in the limit $\ell/L \gg 1$ and $\ell/L \ll 1$ [16].

Although Newtonian theory may successfully apply in regions small compared to the Hubble radius, it is not only of theoretical importance but also of observational significance to have a general relativistic treatment for nonlinear evolution of inhomogeneous universes. (An interesting possibility was suggested by Bildhauer and Futamase [17] that the nonlinear back reaction might resolve the cosmic age problem without introducing a cosmological constant.) In this respect, it is desirable to have a more generic formalism without such restrictions as mentioned above. In this paper, we develop a tetrad-based perturbative approach to the nonlinear evolution of inhomogeneous universes. The spatial basis vectors (or the triad) are taken as parallel transported along the fluid lines, and it is derived that the deviations of the spatial basis from the "background" are essentially determined by the second-order ordinary differential equation. The plan of this paper is as follows. Section II gives a brief summary of the Newtonian treatment by Buchert [18–20] and Buchert and Ehlers [21]. It is instructive and important to emphasize the formal similarity of our fully relativistic treatment to the Newtonian one. Section III presents the tetrad-based perturbative approach in the framework of general relativity. The first-order solution is obtained, and it is pointed out that it contains Szekeres' exact solution as a special case. A relativistic version of the Zel'dovich approximation is proposed with emphasis on the formal similarity to the Newtonian treatment. Section IV contains concluding remarks. We use the following convention: Greek indices $\lambda, \mu, \ldots$ (and $a, b, \ldots$ for tetrad indices) run from 0 to 3, Latin indices $i, j, \ldots$ from 1 to 3, and the speed of light is 1.

II. SUMMARY OF THE NEWTONIAN TREATMENT

In this section, we briefly review the Newtonian treatment. It is not only helpful in order to make the paper self-contained but also important to emphasize the formal similarity between the Newtonian and relativistic treatments. Using expanding coordinates (see, e.g., Ref. [22])

$$\mathbf{r} = a(t)\mathbf{x}, \tag{2.1}$$

the basic equations in Newtonian cosmology are

$$\frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\nabla\cdot(\rho\mathbf{v}) = 0, \tag{2.2}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{g}, \tag{2.3}$$

$$\nabla\times\mathbf{g} = 0, \tag{2.4}$$

$$\nabla\cdot\mathbf{g} = -4\pi G a(\rho-\rho_b), \tag{2.5}$$

where

$$\mathbf{v} \equiv \dot{\mathbf{r}} - \frac{\dot{a}}{a}\mathbf{r} = a\dot{\mathbf{x}} \tag{2.6}$$

is the peculiar velocity which represents the deviation from the uniform Hubble flow, ρ is the density of dust matter, and the scale factor $a(t)$ satisfies the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_b = \frac{8\pi G}{3}\rho_b(t_i)\left(\frac{a_i}{a}\right)^3 \tag{2.7}$$

with the background density ρ_b. Throughout the paper, we assume the "background" is $k = \Lambda = 0$ Friedmann-Lemaître-Robertson-Walker (FLRW). The generalization to $k \neq 0, \Lambda \neq 0$ cases is straightforward. We can obtain, from Eqs. (2.2) and (2.5),

$$\nabla\cdot\left(\frac{\partial\mathbf{g}}{\partial t} + 2\frac{\dot{a}}{a}\mathbf{g} - 4\pi G\rho\mathbf{v}\right) = 0. \tag{2.8}$$

Therefore,

$$\frac{\partial\mathbf{g}}{\partial t} + 2\frac{\dot{a}}{a}\mathbf{g} = 4\pi G\rho_b\mathbf{v} - \frac{1}{a}(\nabla\cdot\mathbf{g})\mathbf{v} + \nabla\times\mathbf{t}. \tag{2.9}$$

In the following, we shall omit the curl term $\nabla\times\mathbf{t}$, which is not relevant to the purpose of this paper.

A. The Lagrangian approach

The Lagrangian perturbation theory of FLRW cosmologies has been thoroughly investigated by Buchert [18–20], and Buchert and Ehlers [21]. This theory does not rely on the density fluctuations being small. Furthermore, it was found that the Zel'dovich approximation [23,24], which is widely applied to the problems of the large scale structure formation, is contained in a subclass of the first-order solutions in the Lagrangian perturbation theory [19].

Introducing the Lagrangian time derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla, \tag{2.10}$$

Eqs. (2.2), (2.3), and (2.9) become

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho + \frac{\rho}{a}\nabla\cdot\mathbf{v} = 0, \tag{2.11}$$

$$\frac{d\mathbf{v}}{dt} + \frac{\dot{a}}{a}\mathbf{v} = \mathbf{g}, \tag{2.12}$$

$$\frac{d\mathbf{g}}{dt} + 2\frac{\dot{a}}{a}\mathbf{g} = 4\pi G\rho_b\mathbf{v} + \frac{1}{a}\left[(\mathbf{v}\cdot\nabla)\mathbf{g} - (\nabla\cdot\mathbf{g})\mathbf{v}\right]. \tag{2.13}$$

Differentiating Eq. (2.12) again and using Eqs. (2.6) and (2.13), we obtain

$$\frac{d}{dt}\left[a^3\left(\frac{d^2\mathbf{x}}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\mathbf{x}}{dt} - 4\pi G\rho_b\mathbf{x}\right)\right] = a\left[(\mathbf{v}\cdot\nabla)\mathbf{g} - (\nabla\cdot\mathbf{g})\mathbf{v}\right], \tag{2.14}$$

which is the key equation in the Newtonian treatment.

B. The first-order solutions

So far the treatment is exact. The left-hand side of Eq. (2.14) is already linearized with respect to $\mathbf{x}$ and the source term in the right-hand side contains higher-order terms. Therefore, it can be solved iteratively. Neglecting the source term, the equation for the first-order solutions is

$$a^3\left(\frac{d^2\mathbf{x}}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\mathbf{x}}{dt} - 4\pi G\rho_b\mathbf{x}\right) = \mathbf{C}, \quad \frac{d\mathbf{C}}{dt} = 0, \tag{2.15}$$

or, choosing $\mathbf{C} \equiv -4\pi G\rho_b a^3\mathbf{q}$,

$$\frac{d^2}{dt^2}(\mathbf{x}-\mathbf{q}) + 2\frac{\dot{a}}{a}\frac{d}{dt}(\mathbf{x}-\mathbf{q}) - 4\pi G\rho_b(\mathbf{x}-\mathbf{q}) = 0. \tag{2.16}$$

Note that Eq. (2.16) has the same form as that which governs the density contrast δ in linear perturbation theory. Using the growing mode and the decaying mode solutions $D^+(t) \propto t^{2/3}$, $D^-(t) \propto t^{-1}$, respectively, the first-order solutions are

$$\begin{aligned}\mathbf{x} &= \mathbf{q} + D^+(t)\,\nabla\Psi + D^-(t)\,\nabla\Phi \\ &= \mathbf{X} + \left[D^+(t) - D^+(t_i)\right]\nabla\Psi + \left[D^-(t) - D^-(t_i)\right]\nabla\Phi,\end{aligned} \tag{2.17}$$

where $\mathbf{X} = \mathbf{x}(t_i)$ are the Lagrangian coordinates, and Ψ and Φ are functions of $\mathbf{X}$.

The first-order solutions contain exact solutions as a special case when the collapse is locally one dimensional. It is most easily seen by setting Ψ and Φ are functions of, say, X^1 only. Then

$$\mathbf{v} = (v^1, 0, 0), \quad \mathbf{g} = (g^1, 0, 0) \tag{2.18}$$

and the source term in Eq. (2.14) vanishes exactly.

C. The Zel'dovich approximation

From the viewpoint of the Lagrangian perturbation theory, the Zel'dovich approximation is regarded as a subclass of the first-order solutions which take the growing mode in the first-order solutions

$$\mathbf{r} = a(t)\,\mathbf{x} = a(t)\,\left\{\mathbf{X} + [D(t) - D(t_i)]\,\nabla\Psi\right\},$$

$$D(t) \propto a(t) \propto t^{2/3}, \tag{2.19}$$

and use the exact expression for the density

$$\begin{aligned}\rho &= \rho_i \left(\frac{a_i}{a}\right)^3 \frac{1}{\det(\partial x^i/\partial X^j)} \\ &\simeq \rho_i \left(\frac{a_i}{a}\right)^3 \frac{1}{\det\left\{\delta^i_j + [D(t) - D(t_i)]\,\Psi^{,i}{}_{,j}\right\}}\end{aligned} \tag{2.20}$$

(until the quasinonlinear stage $\delta \sim 1$). The first part of Eq. (2.20) is directly obtained from Eq. (2.11).

D. Beyond the Zel'dovich approximation

Higher-order solutions in the Lagrangian perturbation theory have been obtained in order to overcome shortcomings of the Zel'dovich approximation. Here we will give a simple example of the second-order approach. For more extensive discussions including third-order calculations, see Refs. [20,21] and references therein. We assume the form

$$\mathbf{x} = \mathbf{q} + D(t)\nabla\Psi_{(1)} + E(t)\nabla\Psi_{(2)}. \tag{2.21}$$

Then we obtain from Eq. (2.14) the second-order equation

$$\left[\frac{d^2E}{dt^2} + 2\frac{\dot{a}}{a}\frac{dE}{dt} - 4\pi G\rho_b E\right]\nabla\Psi_{(2)}$$

$$= \frac{1}{a^3}\int^t dt\, a\left[\left(\mathbf{v}_{(1)}\cdot\nabla\right)\mathbf{g}_{(1)} - \mathbf{v}_{(1)}\left(\nabla\cdot\mathbf{g}_{(1)}\right)\right], \tag{2.22}$$

where $\mathbf{v}_{(1)}$ and $\mathbf{g}_{(1)}$ are the first-order quantities. This is solved to give

$$\begin{aligned}\mathbf{x} = \mathbf{X} &+ [D(t) - D(t_i)]\,\nabla\Psi_{(1)} \\ &+ \frac{3}{14}\left[D^2(t) - D^2(t_i)\right]\nabla\Psi_{(2)},\end{aligned} \tag{2.23}$$

and $\Psi_{(2)}$ is related to $\Psi_{(1)}$ by the equation

$$\nabla\Psi_{(2)} = \left(\nabla\Psi_{(1)}\cdot\nabla\right)\nabla\Psi_{(1)} - \left(\nabla^2\Psi_{(1)}\right)\nabla\Psi_{(1)}. \tag{2.24}$$

III. GENERAL RELATIVISTIC TREATMENT

In this section, we develop a general relativistic treatment. The models we consider are the same as in the previous section, i.e., that contain irrotational dust with density ρ and four-velocity u^μ. In comoving synchronous coordinates

$$ds^2 = -dt^2 + g_{ij}\,dx^i dx^j \tag{3.1}$$

with $u^\mu = (1, 0, 0, 0)$, the Einstein equations read

$$\frac{1}{2}\left\{{}^{(3)}R + (K^i{}_i)^2 - K^i{}_j K^j{}_i\right\} = 8\pi G\rho, \tag{3.2}$$

$$K^i{}_{j\|i} - K^i{}_{i\|j} = 0, \tag{3.3}$$

$$\dot{K}^i{}_j + K^\ell{}_\ell K^i{}_j + {}^{(3)}R^i{}_j = 4\pi G\rho\delta^i{}_j, \tag{3.4}$$

and the energy equation $u_\mu T^{\mu\nu}{}_{;\nu} = 0$ gives

$$\dot{\rho} + \rho K^i{}_i = 0, \tag{3.5}$$

where an overdot ($\dot{}$) denotes $\partial/\partial t$, $\|$ denotes the covariant derivative with respect to the three-metric g_{ij}, ${}^{(3)}R^i{}_j$ is the three-dimensional Ricci tensor, and

$$K^i{}_j = \frac{1}{2}g^{ik}\dot{g}_{kj} \tag{3.6}$$

is the extrinsic curvature. The propagation equation for the Ricci curvature tensor may be obtained from the Bianchi identity, which reads

$$\begin{aligned}{}^{(3)}\dot{R}^i{}_j + 2K^i{}_\ell\,{}^{(3)}R^\ell{}_j = &\, K^i{}_{\ell\|j}{}^{\|\ell} + K^\ell{}_j{}^{\|i}{}_{\|\ell} \\ &- K^i{}_j{}^{\|\ell}{}_{\|\ell} - K^\ell{}_\ell{}^{\|i}{}_{\|j}.\end{aligned} \tag{3.7}$$

[See the Appendix for a simple derivation of Eq. (3.7).]

A. Tetrad-based perturbative approach

Let us introduce the orthonormal tetrad by

$$g_{\mu\nu} = \eta_{(a)(b)}\,\bar{e}^{(a)}_\mu\,\bar{e}^{(b)}_\nu \tag{3.8}$$

with

$$\bar{e}^{(0)}_\mu = u_\mu = (-1, 0, 0, 0), \quad \bar{e}^{(\ell)}_\mu = \left(0, \bar{e}^{(\ell)}_i\right). \tag{3.9}$$

We take the parallel-transported spatial basis along the fluid lines: i.e.,

$$\bar{e}^{(\ell)}_{\mu;\nu}\,u^\nu = 0. \tag{3.10}$$

In our choice of the tetrad components, it gives

$$\dot{\bar{e}}^{(\ell)}_{\ i} = K^j_{\ i}\,\bar{e}^{(\ell)}_{\ j}. \tag{3.11}$$

Similar to the Newtonian case, it is convenient to use the conformally transformed basis

$$\bar{e}^{(\ell)}_{\ i} \equiv a(t)\, e^{(\ell)}_{\ i}. \tag{3.12}$$

Then

$$\dot{e}^{(\ell)}_{\ i} = \left(K^j_{\ i} - \frac{\dot a}{a}\delta^j_{\ i}\right) e^{(\ell)}_{\ j} \equiv V^j_{\ i}\, e^{(\ell)}_{\ j}. \tag{3.13}$$

$V^i_j = e_{(\ell)}^{\ i}\dot{e}^{(\ell)}_{\ j}$ is the peculiar deformation tensor which represents the deviation from the uniform Hubble expansion. It is also useful to define the covariant derivative with respect to $\gamma_{ij} \equiv a^{-2}g_{ij}$ denoted by $|$, and the corresponding curvature tensor

$$\mathcal{R}^i_{\ j} \equiv e_{(\ell)}^{\ i}\left(e^{(\ell)}_{\ k|j}{}^{|k} - e^{(\ell)}_{\ k}{}^{|k}{}_{|j}\right) = a^{2\,(3)}R^i_{\ j}. \tag{3.14}$$

Using these definitions, Eqs. (3.3), (3.4), and (3.7) are rewritten as

$$V^i_{\ j|i} - V^i_{\ i|j} = 0, \tag{3.15}$$

$$\dot{V}^i_j + \left(3\frac{\dot a}{a} + V^k_k\right)V^i_j + \frac{1}{a^2}\left(\mathcal{R}^i_{\ j} - \frac{1}{4}\mathcal{R}\,\delta^i_{\ j}\right) = \frac{1}{4}\left\{(V^k_k)^2 - V^k_\ell V^\ell_k\right\}\delta^i_{\ j}, \tag{3.16}$$

$$\dot{\mathcal{R}}^i_{\ j} + 2V^i_\ell\,\mathcal{R}^\ell_{\ j} = V^i_{\ \ell|j}{}^{|\ell} + V^\ell_{\ j}{}^{|i}{}_{|\ell} - V^i_{\ j}{}^{|\ell}{}_{|\ell} - V^\ell_{\ \ell}{}^{|i}{}_{|j}. \tag{3.17}$$

Differentiating Eq. (3.13) twice with respect to t and using Eq. (3.16), we obtain

$$\left[a^3\left(\ddot{e}^{(\ell)}_{\ i} + 2\frac{\dot a}{a}\,\dot{e}^{(\ell)}_{\ i} - 4\pi G\rho_b e^{(\ell)}_{\ i}\right)\right]^{\cdot} = a\left(a^2 S_1 - S_2\right)^{\cdot}, \tag{3.18}$$

where

$$S_1 = \frac{1}{4}\left[(V^k_k)^2 - V^k_m V^m_k\right]e^{(\ell)}_{\ i} + \left(V^k_i V^j_k - V^k_k V^j_i\right)e^{(\ell)}_{\ j} \tag{3.19}$$

and

$$S_2 = \left(\mathcal{R}^j_{\ i} - \frac{1}{4}\mathcal{R}\,\delta^j_{\ i}\right)e^{(\ell)}_{\ j}. \tag{3.20}$$

This is the relativistic correspondence to the key equation (2.14) in Newtonian theory.

B. The first-order solutions

Again, the treatment is exact so far. In this subsection, we shall assume the following form for the triad,

$$e^{(\ell)}_{\ i} = \delta^{(\ell)}_{\ i} + b^{(\ell)}_{\ i} \tag{3.21}$$

and obtain the first-order solutions for $b^{(\ell)}_{\ i}$.

Let us write the first-order quantities with subscript (1). The peculiar deformation tensor is expressed in the first order as

$$V^i_{(1)\,j} = \delta^i_{\ (\ell)}\,\dot{b}^{(\ell)}_{\ j}. \tag{3.22}$$

Hereafter, we shall concentrate our discussion on initial scalar perturbations. In the first-order level, scalar (vector) and tensor modes do not couple with each other, and can be discussed separately. (It is, however, of interest to study nonlinear coupling effects of these modes. This problem will be investigated elsewhere.) Then, we can write

$$V^i_{(1)\,j} = \dot{b}^{,i}_{\ \ ,j}. \tag{3.23}$$

It is apparent that the constraint equation (3.15), which reads

$$V^i_{(1)\,j,i} - V^i_{(1)\,i,j} = 0 \tag{3.24}$$

in the first order, is satisfied by use of this expression. This also means the right-hand side of Eq. (3.17) vanished in the first order, which in turn gives

$$\dot{\mathcal{R}}^i_{(1)\,j} = 0, \quad {}^{(3)}R_{(1)} \propto \frac{1}{a^2}. \tag{3.25}$$

It means that each perturbed region on a $k = 0$ background evolves as if it were a separate FLRW universe with small scalar curvature (cf. Refs. [7,25]). Strictly speaking, of course, it may not be true because a FLRW behavior requires

$${}^{(3)}R^i_{\ j} \propto \frac{1}{a^2}\,\delta^i_{\ j}. \tag{3.26}$$

Now let us take a look at our key equation (3.18). It is apparent that S_1 in the source term is a second-order quantity (and higher). From Eq. (3.25), S_2 is also found to be a higher-order quantity. Again, the source term vanishes in linear order, and Eq. (3.18) can be solved iteratively. Neglecting the source term, the first-order solutions satisfy

$$a^3\left(\ddot{e}^{(\ell)}_{\ i} + 2\frac{\dot a}{a}\,\dot{e}^{(\ell)}_{\ i} - 4\pi G\rho_b\, e^{(\ell)}_{\ i}\right) = C^{(\ell)}_{\ i}, \quad \dot{C}^{(\ell)}_{\ i} = 0. \tag{3.27}$$

Choosing $C^{(\ell)}_{\ i} = -4\pi G\rho_b a^3\, q^{(\ell)}_{\ i}(x^k)$, the solutions are

$$\begin{aligned} e^{(\ell)}_{\ i} &= \delta^{(\ell)}_{\ i} + q^{(\ell)}_{\ i}(x^k) + \delta^{(\ell)}_{\ j}\left[D^+(t)\Psi^{,j}_{\ \ ,i}(x^k) + D^-(t)\Phi^{,j}_{\ \ ,i}(x^k)\right] \\ &\equiv \delta^{(\ell)}_{\ i} + X^{(\ell)}_{\ i}(x^k) + \left[D^+(t) - D^+(t_i)\right]\Psi^{,(\ell)}_{\ \ ,i} + \left[D^-(t) - D^-(t_i)\right]\Phi^{,(\ell)}_{\ \ ,i}, \end{aligned} \tag{3.28}$$

and the metric is written as

$$\gamma_{ij} = \delta_{ij} + 2X_{ij} + 2\left[D^+(t) - D^+(t_i)\right]\Psi_{,ij} + 2\left[D^-(t) - D^-(t_i)\right]\Phi_{,ij} \quad (3.29)$$

assuming the symmetric property $X_{ij} = \delta_{i(\ell)}X^{(\ell)}_{j} = X_{ji}$. X_{ij} represents the initial metric perturbations. The relation between $X^{(\ell)}_{i}$ and Ψ is given by Eq. (3.16). In the first order, it reads

$$\dot{V}^i_{(1)\,j} + 3\frac{\dot{a}}{a}V^i_{(1)\,j} + \frac{1}{a^2}\left(\mathcal{R}^i_{(1)\,j} - \frac{1}{4}\mathcal{R}_{(1)}\,\delta^i{}_j\right) = 0, \quad (3.30)$$

where $\mathcal{R}^i_{(1)\,j}$ is the first-order quantity of the three-dimensional Ricci tensor $\mathcal{R}^i{}_j$ calculated from the initial metric $\gamma_{ij}(t_i, x^k) = \delta_{ij} + 2X_{ij}$:

$$\mathcal{R}^i_{(1)\,j} = X^{i\ ,k}_{\ k\ ,j} + X^{k\ ,i}_{\ j\ ,k} - X^{i\ ,k}_{\ j\ ,k} - X^{k\ ,i}_{\ k\ ,j}. \quad (3.31)$$

Then we obtain

$$\Psi^{,i}{}_{,j} = -\frac{9}{10}\frac{t_i^2}{D^+(t_i)a^2(t_i)}\left(\mathcal{R}^i_{(1)\,j} - \frac{1}{4}\mathcal{R}_{(1)}\,\delta^i{}_j\right). \quad (3.32)$$

In particular, its trace gives

$$\Psi^{,j}{}_{,j} = \frac{9}{20}\frac{t_i^2}{a^2(t_i)D^+(t_i)}\left(X^{j\ ,k}_{\ j\ ,k} - X^{j\ ,k}_{\ k\ ,j}\right). \quad (3.33)$$

Note that Φ in the decaying mode is independent of $X^{(\ell)}_{i}$, and does not couple to the spatial curvature fluctuations ${}^{(3)}R_{(1)}$. In this sense, the decaying mode may be considered as the purely "isocurvature" perturbation.

The first-order approximate solutions contain Szekeres' exact solutions [26,27] as a special case when the collapse is locally one dimensional. It can be most easily seen by taking Ψ and Φ are functions of, say, x only $\left[x^i = (x, y, z)\right]$, and $q^{(\ell)}_{i} = \mathrm{diag}[A(x^k), 0, 0]$. Then,

$$V^i_j = \mathrm{diag}\left(V^x_x,\ 0,\ 0\right) \quad (3.34)$$

and Eq. (3.33) reads

$$\Psi^{,x}{}_{,x} \equiv \beta(x) = \frac{9}{20}\left(A^{,y}{}_{,y} + A^{,z}{}_{,z}\right) \quad (3.35)$$

with a normalization $t_i^2/a^2(t_i)D^+(t_i) = 1$. It is of interest to note that Szekeres' exact solutions

$$ds^2 = -dt^2 + t^{4/3}\left[\left(A(x^k) + t^{2/3}\,\beta(x) + t^{-1}\,\mu(x)\right)^2 dx^2 + dy^2 + dz^2\right] \quad (3.36)$$

with

$$A(x^k) = \frac{5}{9}\beta(x)\left(y^2 + z^2\right) + \sigma(x)y + \nu(x)z + \omega(x), \quad (3.37)$$

satisfy the "linearized constraint equation" (3.35).

C. General relativistic version of the Zel'dovich approximation

From the formal similarity as shown in the above discussions, it is very natural to propose a general relativistic version of the Zel'dovich approximation defined in the following way: (a) take the growing mode in the first-order solutions

$$\bar{e}^{(\ell)}_{i} = a(t)\left\{\delta^{(\ell)}_{i} + X^{(\ell)}_{i} + \left[D(t) - D(t_i)\right]\Psi^{,(\ell)}{}_{,i}\right\}, \quad D(t) \propto a(t) \propto t^{2/3}, \quad (3.38)$$

and (b) use the exact expression for the density,

$$\rho = \rho_i\left(\frac{a_i}{a}\right)^3\frac{\det\left[e^{(\ell)}_{j}(t_i, x^k)\right]}{\det\left[e^{(\ell)}_{j}(t, x^k)\right]} \simeq \rho_i\left(\frac{a_i}{a}\right)^3\frac{\det\left(\delta^{(\ell)}_{j} + X^{(\ell)}_{j}\right)}{\det\left\{\delta^{(\ell)}_{j} + X^{(\ell)}_{j} + \left[D(t) - D(t_i)\right]\Psi^{,(\ell)}{}_{,j}\right\}} \quad (3.39)$$

(until the mildly nonlinear stage $\delta \sim 1$). The first line of this expression is directly obtained from Eq. (3.5). In this approximation, the metric is written as

$$g_{ij} = a^2(t)\left\{\delta_{ij} + 2X_{ij} + 2\left[D(t) - D(t_i)\right]\Psi_{,ij}\right\}. \quad (3.40)$$

The proposal is supported not only by the formal analogy but also by the following physical correspondence. In Newtonian theory, the Lagrangian coordinates $\mathbf{X}$ are used to label the fluid lines, and the Eulerian coordinates $\mathbf{r} = a(t)\,\mathbf{x}(\mathbf{X}, t)$ are to represent the actual positions. $a(t)\dot{\mathbf{x}}$ gives the peculiar velocity which represents the deviation from the uniform Hubble expansion, and the Zel'dovich approximation utilizes the first-order solution for the peculiar velocity. In our relativistic formulation, comoving coordinates x^i are by definition the Lagrangian because $dx^i/dt = 0$ along the fluid lines. Let us consider two nearby fluid lines labeled by x^i and $x^i + \delta x^i$. Then δx^i is also the Lagrangian displacement. The Eulerian displacement, which represents the physical length of the separation, is given by the triad components

$$\delta x^{(\ell)} \equiv a(t)e^{(\ell)}_{i}\,\delta x^i, \quad (3.41)$$

and $a(t)\dot{e}^{(\ell)}_{i}\,\delta x^i$ gives the relative peculiar velocity which represents the deviation from the uniform Hubble expansion. Therefore, the first-order solution for $e^{(\ell)}_{i}$ in our formalism precisely corresponds to the linearized solution for the peculiar velocity in Newtonian theory.

IV. CONCLUDING REMARKS

In this paper, we have developed a tetrad-based perturbative approach to the nonlinear evolution of an inhomogeneous universe containing irrotational dust. We have used the parallel-transported basis along the fluid lines, and obtained the second-order differential equation for the spatial basis vectors. The equation can be solved iteratively, and the first-order solutions are given in a

$k = 0, \Lambda = 0$ background. A general relativistic version of the Zel'dovich approximation is proposed as an extrapolation of the first-order solutions to the mildly nonlinear regime. The relativistic version possesses the following favorable features: (a) it does not rely on the smallness of the density fluctuations, (b) it contains exact solutions when the collapse is locally one dimensional, and (c) it coincides with the result of linear perturbation theory when $|\delta| \ll 1$. These are precisely the features of the original Zel'dovich approximation in Newtonian theory.

It is of interest to compare our results with those obtained in other methods. First, in the fluid flow approach by Matarrese *et al.* [14], the dynamics is quite complicated and followed by a system of six coupled first-order ordinary differential equations, whereas in our approach it is essentially determined by the second-order ordinary differential equation. In the successive works [16], they gave results of the second-order perturbation expansion both in the limit of $\ell/L \gg 1$ and $\ell/L \ll 1$. However, it should still be clarified whether such wavelength-specific assumptions are crucial or not to their approach. Next, in the gradient expansion method [28] they need the fifth-order calculations (and a computer algebra program) in order to reproduce the exact Szekeres solution, while in our approach it is already contained in the first-order solutions. Croudace *et al.* [29] applied the Zel'dovich approximation to general relativity. They rely on the long wavelength assumption, so it is unclear if their approximation is adequate for the description of subhorizon scale inhomogeneities. Our version of the relativistic Zel'dovich approximation is formally similar to theirs, but no wavelength-specific assumption is made in it. Therefore, it is hoped that our approach gives a simple but powerful tool to study the nonlinear evolution of inhomogeneous structures, not only on scales larger than the horizon but also on subhorizontal scales.

Another advantage of our method is the possibility of using in high resolution numerical studies. The visualization of numerical simulations using our method may be illustrated in the following way. (1) First, prepare a box and draw Cartesian grids inside with separation Δx^i. (2) At each grid point, place the orthonormal triad basis $e^{(\ell)}_i$ which deviations from the grid lines are given by $e^{(\ell)}_i \Delta x^i$. (3) The velocity components along the orthogonal frame of each particle on the grid are given by $v^{(\ell)} = \dot{e}^{(\ell)}_i \Delta x^i$. (4) After a time step of Δt, move each particle according to the rule $x^{(\ell)}(t + \Delta t) = x^{(\ell)}(t) + v^{(\ell)} \Delta t$. (5) Place the triad frame on each particle and repeat the procedures. So far relativistic simulations for the problems of structure formation in the Universe are quite limited. Therefore, it is hoped that our method might give some physical intuition in the direction of this study. Such applications, as well as the extension to $k \neq 0, \Lambda \neq 0$ cases and the inclusion of the transverse traceless modes, will be the subject of future investigation.

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APPENDIX

A simple derivation of Eq. (3.7) is given as follows. In a locally flat three-dimensional coordinate system with ${}^{(3)}\Gamma^i{}_{jk} = 0$,

$$\begin{aligned} {}^{(3)}\dot{\Gamma}^i{}_{jk} &= \frac{1}{2} g^{i\ell} \left(\dot{g}_{\ell j,k} + \dot{g}_{\ell k,j} - \dot{g}_{jk,\ell} \right) \\ &= K^i{}_{j\|k} + K^i{}_{k\|j} - K_{jk}{}^{\|i}, \end{aligned} \tag{A1}$$

and

$$\begin{aligned} {}^{(3)}\dot{R}_{ij} &= {}^{(3)}\dot{\Gamma}^\ell{}_{ij,\ell} - {}^{(3)}\dot{\Gamma}^\ell{}_{i\ell,j} \\ &= K_{i\ell\|j}{}^{\|\ell} + K^\ell{}_{j\|i\ell} - K_{ij}{}^{\|\ell}{}_{\|\ell} - K^\ell{}_{\ell\|ij}. \end{aligned} \tag{A2}$$

Then

$$\begin{aligned} {}^{(3)}\dot{R}^i{}_j + 2K^i{}_\ell {}^{(3)}R^\ell{}_j &= g^{i\ell\,(3)}\dot{R}_{\ell j} \\ &= K^i{}_{\ell\|j}{}^{\|\ell} + K^\ell{}_j{}^{\|i}{}_{\|\ell} - K^i{}_j{}^{\|\ell}{}_{\|\ell} \\ &\quad - K^\ell{}_\ell{}^{\|i}{}_{\|j}. \end{aligned} \tag{A3}$$

Because of the tensor nature of Eq. (A3), it holds in any other coordinate system in the three-dimensional sense.

[1] T. Futamase, Phys. Rev. Lett. **61**, 2175 (1988).
[2] T. Futamase, Mon. Not. R. Astron. Soc. **237**, 187 (1989).
[3] T. Futamase, Prog. Theor. Phys. **86**, 389 (1991).
[4] K. Tomita, Prog. Theor. Phys. **79**, 258 (1988).
[5] K. Tomita, Prog. Theor. Phys. **85**, 1041 (1991).
[6] M. Kasai, Phys. Rev. Lett. **69**, 2330 (1992).
[7] M. Kasai, Phys. Rev. D **47**, 3214 (1993).
[8] K. Tomita, Prog. Theor. Phys. **48**, 1503 (1972).
[9] K. Tomita, Prog. Theor. Phys. **54**, 730 (1975).
[10] K. Tomita, Phys. Rev. D **48**, 5634 (1993).
[11] K. Tomita and N. Deruelle, Phys. Rev. D **50**, 7216 (1994).
[12] D. S. Salopek and J. M. Stewart, Class. Quantum Grav. **9**, 1943 (1992).
[13] K. Tomita, Prog. Theor. Phys. **37**, 831 (1967); **45**, 1747 (1971); **47**, 416 (1972).
[14] S. Matarrese, O. Pantano, and D. Saez, Phys. Rev. D **47**, 1311 (1993).
[15] G. F. R. Ellis, in *General Relativity and Cosmology*, edited by R. K. Sachs (Academic, New York, 1971).
[16] S. Matarrese, O. Pantano, and D. Saez, Phys. Rev. Lett. **72**, 320 (1994); Mon. Not. R. Astron. Soc. **271**, 513

(1994).
[17] S. Bildhauer and T. Futamase, Gen. Relativ. Gravit. **23**, 1251 (1991).
[18] T. Buchert, Astron. Astrophys. **223**, 9 (1989).
[19] T. Buchert, Mon. Not. R. Astron. Soc. **254**, 729 (1992).
[20] T. Buchert, Mon. Not. R. Astron. Soc. **267**, 811 (1994).
[21] T. Buchert and J. Ehlers, Mon. Not. R. Astron. Soc. **264**, 375 (1993).
[22] P. J. E. Peebles, *The Large Scale Structure of the Universe* (Princeton Univ. Press, Princeton, 1980).
[23] Ya. B. Zel'dovich, Astron. Astrophys. **5**, 84 (1970).
[24] Ya. B. Zel'dovich, Astrophys. **6**, 164 (1973).
[25] D. H. Lyth, Phys. Rev. D **31**, 1792 (1985).
[26] P. Szekeres, Commun. Math. Phys. **41**, 55 (1975).
[27] W. B. Bonnor and N. Tomimura, Mon. Not. R. Astron. Soc. **175**, 85 (1976).
[28] J. Parry, D. S. Salopek, and J. M. Stewart, Phys. Rev. D **49**, 2872 (1994).
[29] K. M. Croudace, J. Parry, D. S. Salopek, and J. M. Stewart, Astrophys. J. **423**, 22 (1994).