

Open inflation with an arbitrary false vacuum mass

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We calculate the power spectrum of adiabatic density perturbations in an open inflationary model in which inflation occurs in two stages. First an epoch of old inflation creates a large, smooth universe, solving the horizon and homogeneity problems. Then an open universe emerges through the nucleation of a single bubble, with constant density hypersurfaces inside the bubble having constant negative spatial curvature. An epoch of “slow roll” inflation, shortened to give $\Omega_0 < 1$ today, occurs within the bubble, which contains our entire observable universe. In this paper we compute the resulting density perturbations in the same “new thin wall” approximation used in a previous paper, but for an arbitrary positive mass of the inflaton field in the false vacuum satisfying $m^2/H^2 \geq 2$ where H is the Hubble constant during old inflation.

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I. INTRODUCTION

If the current density parameter Ω_0 is smaller than unity as some observations suggest, then we live in an open universe, described to a first approximation by a Friedmann-Robertson-Walker (FRW) metric of the form

$$ds^2 = -dt^2 + a^2(t)[d\xi^2 + \sinh^2[\xi]d\Omega_{(2)}^2]. \quad (1.1)$$

Slices of constant cosmic time are maximally symmetric three-dimensional manifolds of constant negative spatial curvature. The symmetry group of such a universe is $SO(3,1)$, with “boosts” corresponding to what we commonly regard as spatial translations.

Most inflationary models [1–3] predict a value of Ω_0 extremely close to unity. However, this is not a necessary consequence of inflation. As noted by Coleman and de Luccia [4] and by Gott [5], when a bubble nucleates in de Sitter space, inside the forward light cone of the materialization center, spatial hypersurfaces on which the scalar field is constant are spaces of constant negative spatial curvature. In other words, the bubble contains an expanding open FRW universe. Consequently, one can create an open universe from inflation through a two-stage process [5,6]. During an initial epoch of *old* inflation the inflaton field is stuck in a false vacuum. In this epoch the smoothness and horizon problems are solved; whatever inhomogeneities may have existed prior to inflation are erased. Then *old* inflation is exited through the nucleation of a single bubble. Instead of tunneling directly to the true vacuum, the inflaton field tunnels onto a “slow-roll” potential, and a shortened epoch of *new* inflation occurs inside the bubble. By *new* inflation, we mean here slow-roll inflation and do not refer to the way in which inflation began. For our purposes it is sufficient to assume that a sufficiently large volume became stuck in the false vacuum. This may have happened *chaotically* [25], or perhaps in some other way. Formally Ω is exactly zero on the forward light cone of the materialization center, and Ω flows toward one during the epoch of

new inflation inside the bubble until that era ends at reheating. In the subsequent evolution Ω flows away from unity. Its present value, Ω_0 , is determined by the total expansion factor during the *new* inflationary epoch and by the final reheat temperature. The usual fine-tuning argument against $\Omega_0 \neq 1$ is not valid here, because it is not $[\Omega^{-1} - 1]$ at reheating but rather $\ln[\Omega^{-1} - 1]$ at reheating that is proportional to the length of the new inflationary epoch. Thus to obtain interesting values for Ω_0 , one has to tune not a very small number, but rather the logarithm of a very small number. Numerically, this turns out to be a rather mild requirement [6].

Prior to bubble nucleation the geometry of spacetime is that of pure de Sitter space, with $H^2 = (8\pi G/3)V[\phi_{fv}]$ where ϕ_{fv} is the expectation value of the inflaton field in the false vacuum. To exploit the $SO(3,1)$ symmetry of the expanding bubble solution, it is advantageous to work in hyperbolic coordinates, which divide maximally extended de Sitter space into five coordinate patches, shown in Fig. 1, only two of which shall concern us here.

The line element for region I is given in Eq. (1.1), and for de Sitter space $a(t) = H^{-1}\sinh[Ht]$. The line element for region II is

$$ds^2 = d\sigma^2 + b^2(\sigma)[-d\tau^2 + \cosh^2[\tau]d\Omega_{(2)}^2], \quad (1.2)$$

where for de Sitter space $b(\sigma) = H^{-1}\sin[H\sigma]$, with $0 < H\sigma < \pi$. [Regions III–V have line elements of the form given in Eq. (1.1).]

Hyperbolic coordinates are useful for describing the expanding bubble solution because the inflaton field (in the background solution) is constant on slices of constant t (in region I) and on slices of constant σ (in region II). In Fig. 2 is sketched a bubble nucleation event, the dashed lines indicating the surfaces on which the scalar field is constant. The horizontal solid line separates the Euclidean (classically forbidden) region below from the Lorentzian (classically allowed) region above.

To compute the spectrum of density perturbations produced by quantum fluctuations of the inflation field, one

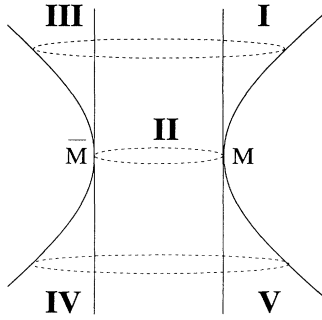


FIG. 1. A diagram of maximally extended de Sitter space, divided into the five hyperbolic coordinate patches. The vertical dashed lines indicate radial coordinate singularities. M is the materialization center and \bar{M} is its antipodal point. The forward and backward light cones of M contain regions I and V, and the forward and backward light cones of \bar{M} contain regions III and IV, respectively. Note that of the five regions, only region II contains a complete Cauchy surface for all of de Sitter space.

has to evolve the mode functions from the external de Sitter space across the bubble wall and into the bubble's interior. In a previous paper [6], we performed this calculation in three stages. First we expressed the Bunch-Davies vacuum modes (the natural vacuum modes for de Sitter space, see below) in hyperbolic coordinates. We then matched these modes across the bubble wall, taken to be very thin. Finally we included the coupling to gravity and computed the spectrum of density perturbations including gravity in the interior of the bubble. Because of the technical difficulty of the calculations, we restricted ourselves to a special case, where the mass m^2 of the inflaton field in the false vacuum equals $2H^2$, with H the Hubble constant during old inflation. As we emphasized there, this was an assumption without physical basis, solely made to simplify the computation. In this paper we generalize the result to arbitrary $m^2/H^2 \geq 2$. For those parts of the computation that are identical, we cite the results from Ref. [6]. Some of these issues have been investigated using a technique involving analytic continuation of Euclidean modes in a series of recent papers [7,8].

The organization of the paper is the following. In Sec. II we discuss initial conditions. The Bunch-Davies vacuum for a massive scalar field is expanded in terms of region II hyperbolic modes. In Sec. III, we continue these modes into the bubble coupling to the scalar component of linearized gravity and calculate the power spectrum. In Sec. IV we present some concluding remarks.

II. INITIAL CONDITIONS

Initially, prior to bubble materialization, the fluctuations of the inflaton field about the false vacuum may

$$\begin{aligned}
 -[\square + m^2(\sigma)]\phi(\sigma, \tau, \theta, \phi) &= \left[\frac{\partial^2}{\partial \sigma^2} + 3 \cot[\sigma] \frac{\partial}{\partial \sigma} - \frac{1}{\sin^2[\sigma]} \left(\frac{\partial^2}{\partial \tau^2} + 2 \tanh[\tau] \frac{\partial}{\partial \tau} + \frac{L^2}{\cosh^2[\tau]} \right) - m^2(\sigma) \right] \\
 &\times \phi(\sigma, \tau, \theta, \phi) = 0,
 \end{aligned} \tag{2.1}$$

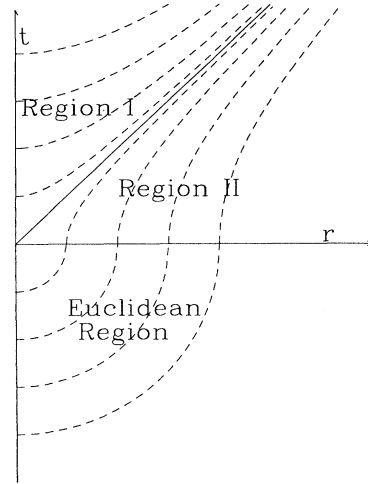


FIG. 2. Bubble nucleation process.

be regarded as a free scalar field of mass $m^2 = V''[\phi_{fv}]$. The so-called Bunch-Davies vacuum is the natural initial quantum state for these fluctuations for the following reason. Let us imagine there is some state prior to the onset of old inflation. This state may be quite inhomogeneous, but if it is to have a finite (renormalized) energy density, the very short-wavelength field modes must be taken to be in their ground state. At very short distances the spacetime approaches Minkowski spacetime, and the effects of spacetime curvature can be ignored (at least in the naive approach to quantizing fields in curved backgrounds that we shall follow here—see, for example, [9]). Ground state here means Minkowski space vacuum. As inflation begins, the co-moving wavelengths of all field modes are exponentially stretched. After a certain amount of *old* inflation (the same amount needed to make the universe homogeneous and isotropic) the only modes of interest are those which were exponentially far within the horizon when inflation began. It is natural to assume these modes are in the state forced upon them by the finite initial density constraint (i.e., the state corresponding to the *Minkowski space vacuum* at early times). This is the so-called Bunch-Davies vacuum. (For a discussion, see [9].) Needless to say, we should be reluctant to drop this assumption, because without it the mechanism of quantum-fluctuation generated perturbations would be likely to lose any predictive power.

We shall imagine that enough old inflation occurred to produce a homogeneous and isotropic universe, and to “drive” the scalar field modes of interest into the Bunch-Davies vacuum, via the constraint explained above. Our first task then is to express the Bunch-Davies vacuum in terms of region II hyperbolic modes, so that we can continue these modes into region I and after coupling to the scalar component of gravity calculate the power spectrum from open inflation. The wave equation for the inflaton field in terms of the region II hyperbolic coordinates is

where we work in units with $H = 1$ and where L^2 is the usual angular-momentum operator. Using the “new thin-wall approximation,” discussed in detail in [6], we set m^2 equal to $V''[\phi_{fv}]$ everywhere in region II and equal to zero everywhere in region I. To compute the power spectrum, it is sufficient to consider only the s -wave sector, so we set $L^2 = 0$. Equation (2.1) can be solved by separation of variables, where

$$\phi^{(\pm)}(\sigma, \tau; \zeta) = S(\sigma) \frac{e^{\mp i\zeta|\tau}}{\cosh[\tau]}, \tag{2.2}$$

where $S(\sigma)$ satisfies

$$S''(\sigma) + 3 \cot[\sigma] S'(\sigma) + \left[\frac{\zeta^2}{\sin^2[\sigma]} - m^2 \right] S(\sigma) = 0. \tag{2.3}$$

After the change of dependent variable $S(\sigma) = F(\sigma)/\sin[\sigma]$ and the change of independent variable $x = \cos[\sigma]$, Eq. (2.3) becomes the Legendre equation and

$$S(\sigma; \zeta) = \frac{P_{\nu'}^{i\zeta}(\cos[\sigma])}{\sin[\sigma]}, \tag{2.4}$$

where $\nu' = \sqrt{\frac{9}{4} - m^2} - \frac{1}{2}$. (There are two linearly independent solutions, one with $P_{\nu'}^{i\zeta}$ and the other with $P_{\nu'}^{-i\zeta}$.) We define, as usual (see, e.g., [10], p. 143),

$$P_{\nu'}^{i\zeta}(x) = \frac{1}{\Gamma(1 - i\zeta)} \left(\frac{1+x}{1-x} \right)^{i\zeta/2} {}_2F_1 \left(-\nu', \nu' + 1; 1 - i\zeta; \frac{1-x}{2} \right). \tag{2.5}$$

In terms of u , where $\tanh[u] = x = \cos[\sigma]$,

$$S_{\zeta} = \frac{1}{\operatorname{sech}[u]} \frac{1}{\Gamma(1 - i\zeta)} e^{+i\zeta u} {}_2F_1 \left(-\nu', \nu' + 1; 1 - i\zeta; \frac{1}{1 + e^{2u}} \right). \tag{2.6}$$

From the self-adjointness properties of the Legendre equation, it follows that

$$\Gamma(1 - \zeta)\Gamma(1 - \zeta') \int_{-\infty}^{+\infty} du P_{\nu'}^{i\zeta}(\tanh[u]) P_{\nu'}^{i\zeta'}(\tanh[u]) = C_1(\zeta)\delta(\zeta + \zeta') + C_2(\zeta)\delta(\zeta - \zeta') \tag{2.7}$$

where the functions $C_1(\zeta)$ and $C_2(\zeta)$ are to be determined from the asymptotic behavior of $P_{\nu'}^{i\zeta}$ as $u \rightarrow +\infty$ and $u \rightarrow -\infty$. Note that since ν' is either real or of the form $-\frac{1}{2} + i\gamma$ with γ real, it follows that $[P_{\nu'}^{i\zeta}(\tanh[u])]^* = P_{\nu'}^{-i\zeta}(\tanh[u])$. Clearly, as $u \rightarrow +\infty$,

$$\Gamma(1 - i\zeta) P_{\nu'}^{i\zeta} \approx e^{+i\zeta u}. \tag{2.8}$$

From the relation

$$\begin{aligned} {}_2F_1(-\nu', \nu' + 1; 1 - i\zeta; 1 - w) &= \frac{\Gamma(1 - i\zeta)\Gamma(-i\zeta)}{\Gamma(1 - i\zeta + \nu')\Gamma(-i\zeta - \nu')} {}_2F_1(-\nu', \nu' + 1; 1 + i\zeta; w) \\ &+ w^{-i\zeta} \frac{\Gamma(1 - i\zeta)\Gamma(+i\zeta)}{\Gamma(1 + \nu')\Gamma(-\nu')} {}_2F_1(-\nu', \nu' + 1; 1 - i\zeta; w), \end{aligned} \tag{2.9}$$

it follows that, as $u \rightarrow -\infty$,

$$\Gamma(1 - i\zeta) P_{\nu'}^{i\zeta} \approx \frac{\Gamma(1 - i\zeta)\Gamma(-i\zeta)}{\Gamma(1 - i\zeta + \nu')\Gamma(-i\zeta - \nu')} e^{+i\zeta u} + \frac{\Gamma(1 - i\zeta)\Gamma(+i\zeta)}{\Gamma(1 + \nu')\Gamma(-\nu')} e^{-i\zeta u}. \tag{2.10}$$

Consequently, using

$$\int_0^{\infty} du e^{i\zeta u} = \pi \delta(\zeta) + i\text{P} \left(\frac{1}{\zeta} \right), \tag{2.11}$$

where P indicates the principal part, one obtains

$$\begin{aligned} C_1(\zeta) &= \pi \left[1 + \frac{\Gamma(1 - i\zeta)\Gamma(1 + i\zeta)\Gamma(-i\zeta)\Gamma(+i\zeta)}{\Gamma(1 - i\zeta + \nu')\Gamma(1 + i\zeta + \nu')\Gamma(-i\zeta - \nu')\Gamma(+i\zeta - \nu')} + \frac{\Gamma(1 - i\zeta)\Gamma(1 + i\zeta)\Gamma(-i\zeta)\Gamma(+i\zeta)}{\Gamma^2(1 + \nu')\Gamma^2(-\nu')} \right] \\ &= (2\pi) \left[1 + \frac{\sin^2[\pi\nu']}{\sinh^2[\pi\zeta]} \right], \end{aligned} \tag{2.12}$$

and similarly

$$\begin{aligned} C_2(\zeta) &= \pi \frac{2\Gamma(1 - i\zeta)\Gamma(1 - i\zeta)\Gamma(-i\zeta)\Gamma(+i\zeta)}{\Gamma(1 + \nu')\Gamma(-\nu')\Gamma(1 - i\zeta + \nu')\Gamma(-i\zeta - \nu')} \\ &= (2\pi) \frac{\sin[\pi\nu']}{\sinh^2[\pi\zeta]} \frac{\Gamma(+i\zeta - \nu')}{\Gamma(-i\zeta - \nu')} \frac{\Gamma(1 - i\zeta)}{\Gamma(1 + i\zeta)} \times [\cosh[\pi\zeta]\sin[\pi\nu'] - i \sinh[\pi\zeta]\cos[\pi\nu']]. \end{aligned} \tag{2.13}$$

We wish to define spatial mode functions $F^{i\zeta}$ that are linear combinations of $P_{\nu'}^{i\zeta}$ and $P_{\nu'}^{-i\zeta}$ chosen so that

$$\int_{-\infty}^{+\infty} du F^{i\zeta} F^{i\zeta'} = \frac{1}{8\pi|\zeta|} \delta(\zeta + \zeta'). \quad (2.14)$$

The functions $C_1(\zeta)$ and $C_2(\zeta)$ have the form

$$\begin{aligned} C_1(\zeta) &= (2\pi) \cosh^2[\bar{\xi}(\zeta)], \\ C_2(\zeta) &= (2\pi) \cosh[\bar{\xi}(\zeta)] \sinh[\bar{\xi}(\zeta)] e^{i\bar{\varphi}(\zeta)}, \end{aligned} \quad (2.15)$$

where $\bar{\xi}(\zeta)$ and $\bar{\varphi}(\zeta)$ are real. Therefore, we may choose $F^{i\zeta}$ according to

$$F^{+i\zeta}(u) = \frac{1}{4\pi\sqrt{|\zeta|} \cosh[\bar{\xi}]} \left[\cosh[\bar{\xi}/2] \Gamma(1 - i\zeta) P_{\nu'}^{+i\zeta}(\tanh[u]) - e^{i\bar{\varphi}} \sinh[\bar{\xi}/2] \Gamma(1 + i\zeta) P_{\nu'}^{-i\zeta}(\tanh[u]) \right], \quad (2.16)$$

which in terms of $C_1(\zeta)$ and $C_2(\zeta)$ may be rewritten as

$$\begin{aligned} F^{+i\zeta}(u) &= \frac{1}{4\pi\sqrt{|\zeta|}} \left[\sqrt{\frac{1 + \sqrt{1 - |C_2|^2/C_1^2}}{2}} \Gamma(1 - i\zeta) P_{\nu'}^{+i\zeta}(\tanh[u]) \right. \\ &\quad \left. - \frac{C_2}{|C_2|} \sqrt{\frac{1 - \sqrt{1 - |C_2|^2/C_1^2}}{2}} \Gamma(1 + i\zeta) P_{\nu'}^{-i\zeta}(\tanh[u]) \right]. \end{aligned} \quad (2.17)$$

It follows that the creation and annihilation operators associated with the modes

$$\mathcal{F}_{\zeta}^{(\pm)} = F^{+i\zeta}(u) \frac{e^{\mp i|\zeta|\tau}}{\cosh[\tau]}, \quad (2.18)$$

where

$$\hat{\phi}(\xi, \tau) = \int_{-\infty}^{+\infty} d\zeta \left[\mathcal{F}_{\zeta}^{(+)} \hat{a}^{(+)}(\zeta) + \mathcal{F}_{\zeta}^{(-)} \hat{a}^{(-)}(\zeta) \right] \quad (2.19)$$

obey the usual commutation relations

$$\begin{aligned} [\hat{a}^{(+)}(\zeta), \hat{a}^{(-)}(\zeta')] &= \delta(\zeta - \zeta'), \\ [\hat{a}^{(+)}(\zeta), \hat{a}^{(+)}(\zeta')] &= 0, \\ [\hat{a}^{(-)}(\zeta), \hat{a}^{(-)}(\zeta')] &= 0. \end{aligned} \quad (2.20)$$

Although these modes have the correct commutation relations, they are not useful for calculating expectation values with respect to the Bunch-Davies vacuum because the operators $\hat{a}^{(+)}(\zeta)$ do not annihilate the Bunch-Davies vacuum. The Bunch-Davies vacuum is related to the vacuum defined by the annihilation operators by a Bogolubov transformation, which we shall now calculate.

We form ‘‘positive frequency’’ modes (with respect to the Bunch-Davies vacuum) by considering linear combinations of the form

$$f_{\zeta}^{(+)} + c_{\zeta} f_{\zeta}^{(-)}, \quad (2.21)$$

where the coefficients c_{ζ} are to be determined.

We may determine the coefficient c_{ζ} , and verify the validity of the ansatz (2.21) as well, by requiring that

products of the form

$$(f_{\zeta}^{(+)} + c_{\zeta} f_{\zeta}^{(-)}, p) \quad (2.22)$$

vanish where p is a Bunch-Davies positive frequency mode, and where we define

$$(u, v) = (-i) \int_{\Sigma} d\Sigma^{\mu} \left\{ u(X) [\partial_{\mu} v(X)] - [\partial_{\mu} u(X)] v(X) \right\}, \quad (2.23)$$

where Σ is a Cauchy surface with unit normal n^{μ} , and $d\Sigma^{\mu} = d\Sigma n^{\mu}$ with $d\Sigma$ the volume element on Σ . If the product in Eq. (2.22) vanishes for all p , then $f_{\zeta}^{(+)} + c_{\zeta} f_{\zeta}^{(-)}$ is a positive frequency mode.

The Bunch-Davies positive frequency modes in terms of closed coordinates are known. They are

$$\phi_k^{\text{BD}} = \frac{\sin[k\sigma]}{\sin[\sigma]} T_k(\eta), \quad (2.24)$$

where

$$\begin{aligned} T_k(\eta) &= \sin^{1/2}[\eta] \left[P_{k-1/2}^{\nu}(-\cos[-\eta]) \right. \\ &\quad \left. - \frac{2i}{\pi} Q_{k-1/2}^{\nu}(-\cos[-\eta]) \right], \end{aligned} \quad (2.25)$$

where $\tanh[\eta/2] = e^{\tau}$. Here k is a positive integer. $\eta = \pi/2$ corresponds to $\tau = 0$ and at $\tau = 0$ one has $\partial_i = \partial_{\eta}$.

The condition

$$\begin{aligned} (f_{\zeta}^{(+)} + c_{\zeta} f_{\zeta}^{(-)}, \phi_k^{\text{BD}}) &= (-i4\pi) \int_0^{\pi} d\sigma \sin^2[\sigma] \left[\frac{P_{\nu'}^{i\zeta}(\cos[\sigma])}{\sin[\sigma]} \frac{e^{-i|\zeta|\tau} + c_{\zeta} e^{+i|\zeta|\tau}}{\cosh[\tau]} \right] \\ &\quad \times \left[\frac{\partial}{\partial \eta} - \frac{1}{\sin[\sigma]} \frac{\partial}{\partial \tau} \right] \times \left[\frac{\sin[k\sigma]}{\sin[\sigma]} T_k \left(\eta = \frac{\pi}{2} \right) \right] = 0 \end{aligned} \quad (2.26)$$

is equivalent to

$$\int_0^\pi d\sigma P_{\nu'}^{i\zeta}(\cos[\sigma]) \sin[k\sigma] \left[\frac{T'_k(\eta = \pi/2)}{T_k(\eta = \pi/2)} + \frac{i|\zeta|}{\sin[\sigma]} \left(\frac{1 - c_\zeta}{1 + c_\zeta} \right) \right] = 0. \quad (2.27)$$

We thus obtain an infinite number of equations, and the k th equation is solved by

$$\frac{1 - c_\zeta}{1 + c_\zeta} = \frac{i}{|\zeta|} \frac{T'_k(\eta = \pi/2)}{T_k(\eta = \pi/2)} \frac{\int_0^\pi d\sigma P_{\nu'}^{i\zeta}(\cos[\sigma]) \sin[k\sigma]}{\int_0^\pi d\sigma P_{\nu'}^{i\zeta}(\cos[\sigma]) \sin[k\sigma] / \sin[\sigma]}. \quad (2.28)$$

In the Appendix we prove that the right-hand side of Eq. (2.28) is independent of k , so that all of these equations are equivalent.

It readily follows from Eq. (2.25) that

$$\frac{T'_k(\eta = \pi/2)}{T_k(\eta = \pi/2)} = (-2i) \frac{\Gamma(k/2 - \nu'/2 + \frac{1}{2}) \Gamma(k/2 + \nu'/2 + 1)}{\Gamma(k/2 - \nu'/2) \Gamma(k/2 + \nu'/2 + \frac{1}{2})}, \quad (2.29)$$

where we have used $\nu = \nu' + \frac{1}{2}$. We first calculate the integrals in Eq. (2.28) for $k = 1$. We rewrite

$$\begin{aligned} I_1 &= \int_0^\pi d\sigma \sin[\sigma] P_{\nu'}^{i\zeta}(\cos[\sigma]) \\ &= \int_{-\infty}^{+\infty} du \operatorname{sech}^2[u] P_{\nu'}^{i\zeta}(\tanh[u]) \\ &= \frac{1}{\Gamma(1 - i\zeta)} \int_{-\infty}^{+\infty} \frac{du e^{i\zeta u}}{\cosh^2[u]} {}_2F_1\left(-\nu', \nu' + 1; 1 - i\zeta; \frac{1}{1 + e^{2u}}\right) \\ &= \frac{4}{\Gamma(1 - i\zeta)} \sum_{n=0}^{+\infty} \frac{(-\nu')_n (\nu' + 1)_n}{(1 - i\zeta)_n n!} \int_{-\infty}^{+\infty} \frac{du e^{i\zeta u}}{e^{-2u} (1 + e^{2u})^{n+2}} \\ &= \frac{2}{\Gamma(1 - i\zeta)} \sum_{n=0}^{+\infty} \frac{(-\nu')_n (\nu' + 1)_n}{(1 - i\zeta)_n n!} \int_0^{+\infty} \frac{dx x^{i\zeta/2}}{(1 + x)^{n+2}} \\ &= \frac{2}{\Gamma(1 - i\zeta)} \sum_{n=0}^{+\infty} \frac{(-\nu')_n (\nu' + 1)_n}{(1 - i\zeta)_n n!} \frac{\Gamma(+i\zeta/2 + 1) \Gamma(-i\zeta/2 + n + 1)}{\Gamma(n + 2)} \\ &= \frac{2\Gamma(+i\zeta/2 + 1) \Gamma(-i\zeta/2 + 1)}{\Gamma(1 - i\zeta)} {}_3F_2\left(-\nu', \nu' + 1, -\frac{i\zeta}{2} + 1; 1 - i\zeta, 2; 1\right). \end{aligned} \quad (2.30)$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^\pi d\sigma P_{\nu'}^{i\zeta}(\cos[\sigma]) \\ &= \frac{2}{\Gamma(1 - i\zeta)} \sum_{n=0}^{+\infty} \frac{(-\nu')_n (\nu' + 1)_n}{(1 - i\zeta)_n n!} \int_{-\infty}^{+\infty} \frac{du e^{i\zeta u}}{e^{-u} (1 + e^{2u})^{n+1}} \\ &= \frac{1}{\Gamma(1 - i\zeta)} \sum_{n=0}^{+\infty} \frac{(-\nu')_n (\nu' + 1)_n}{(1 - i\zeta)_n n!} \int_0^{+\infty} \frac{dx x^{i\zeta/2 - 1/2}}{(1 + x)^{n+1}} \\ &= \frac{1}{\Gamma(1 - i\zeta)} \sum_{n=0}^{+\infty} \frac{(-\nu')_n (\nu' + 1)_n}{(1 - i\zeta)_n n!} \frac{\Gamma(+i\zeta/2 + \frac{1}{2}) \Gamma(-i\zeta/2 + n + \frac{1}{2})}{\Gamma(n + 1)} \\ &= \frac{\Gamma(+i\zeta/2 + \frac{1}{2}) \Gamma(-i\zeta/2 + \frac{1}{2})}{\Gamma(1 - i\zeta)} {}_3F_2\left(-\nu', \nu' + 1, -\frac{i\zeta}{2} + \frac{1}{2}; 1 - i\zeta, 1; 1\right). \end{aligned} \quad (2.31)$$

We simplify the generalized hypergeometric functions using Whipple's theorem [11], which states that

$${}_3F_2(a, b, c; e, f; 1) = \frac{\pi \Gamma(e) \Gamma(f)}{\Gamma[(a + e)/2] \Gamma[(a + f)/2] \Gamma[(b + e)/2] \Gamma[(b + f)/2]} \quad (2.32)$$

whenever $a + b = 1$ and $e + f = 2c + 1$. The generalized hypergeometric functions in Eqs. (2.30) and (2.31) satisfy these conditions.

Consequently,

$$\begin{aligned} \frac{I_1}{I_2} &= \frac{2\Gamma(i\zeta/2 + 1)\Gamma(-i\zeta/2 + 1)}{\Gamma(i\zeta/2 + \frac{1}{2})\Gamma(-i\zeta/2 + \frac{1}{2})} \frac{{}_3F_2(-\nu', \nu' + 1, -i\zeta/2 + 1; 1 - i\zeta, 2; 1)}{{}_3F_2(-\nu', \nu' + 1, -i\zeta/2 + \frac{1}{2}; 1 - i\zeta, 1; 1)} \\ &= \frac{\Gamma(i\zeta/2 + 1)\Gamma(-i\zeta/2 + 1)}{\Gamma(i\zeta/2 + \frac{1}{2})\Gamma(-i\zeta/2 + \frac{1}{2})} \frac{\Gamma(-\nu'/2 + \frac{1}{2})\Gamma(\nu'/2 + 1)}{\Gamma(-\nu'/2 + 1)\Gamma(\nu'/2 + \frac{3}{2})} \end{aligned} \quad (2.33)$$

and

$$\begin{aligned} \left(\frac{1 - c_\zeta}{1 + c_\zeta} \right) &= \frac{i\zeta}{|\zeta|} \frac{\Gamma(+i\zeta/2)\Gamma(-i\zeta/2)}{\Gamma[\frac{1}{2} + (+i\zeta)/2]\Gamma[\frac{1}{2} + (-i\zeta)/2]} \\ &= \frac{\zeta}{|\zeta|} \coth[\pi\zeta/2] \\ &= \frac{1 + e^{-\pi|\zeta|}}{1 - e^{-\pi|\zeta|}}, \end{aligned} \quad (2.34)$$

so that $c_\zeta = -e^{-\pi|\zeta|}$. The result agrees with the previous calculation for the special case $\nu' = 0$ in [6], and with the results of [8] for general ν' obtained using different methods. Surprisingly, the result is independent of ν' .

Near the null surface the positive frequency part of the inflaton field operator is

$$\begin{aligned} \hat{\phi}^{(+)}(u, \tau) &= \frac{1}{4\pi} \int_0^{+\infty} \frac{d\zeta}{\sqrt{\zeta}} \frac{(e^{\pi\zeta/2} e^{-i\zeta\tau} - e^{-\pi\zeta/2} e^{+i\zeta\tau})}{(e^{\pi\zeta} - e^{-\pi\zeta})^{1/2}} \frac{1}{\cosh[\tau] \operatorname{sech}[u]} \\ &\times \left(\left\{ \sqrt{\frac{1 + \sqrt{1 - |C_2(\zeta)|^2/C_1^2(\zeta)}}{2}} e^{+i\zeta u} - e^{i\bar{\varphi}(\zeta)} \sqrt{\frac{1 - \sqrt{1 - |C_2(\zeta)|^2/C_1^2(\zeta)}}{2}} e^{-i\zeta u} \right\} \hat{a}^{(+)}(+\zeta) \right. \\ &\left. + \left\{ \sqrt{\frac{1 + \sqrt{1 - |C_2(\zeta)|^2/C_1^2(\zeta)}}{2}} e^{-i\zeta u} - e^{-i\bar{\varphi}(\zeta)} \sqrt{\frac{1 - \sqrt{1 - |C_2(\zeta)|^2/C_1^2(\zeta)}}{2}} e^{+i\zeta u} \right\} \hat{a}^{(+)}(-\zeta) \right), \end{aligned} \quad (2.35)$$

where $C_1(\zeta)$, $C_2(\zeta)$, and $\bar{\varphi}(\zeta)$ are defined in Eq. (2.15).

III. CONTINUATION INTO THE OPEN UNIVERSE

In the previous section we expanded the Bunch-Davies vacuum in region II in terms of the hyperbolic modes. In this section we continue these modes into region I, so that we can calculate the power spectrum of Gaussian adiabatic density perturbations today.

In the new thin-wall approximation explained in [6], the effective mass squared of the inflaton field (equal to $V''[\phi_b]$ where ϕ_b is the background value for the inflaton field), changes discontinuously from $m^2 = V''[\phi_{fv}]$ to zero as one passes across the forward light cone of the materialization center from region II into region I. This discontinuity is a result of the approximation, in which we assume that (1) the bubble radius (at materialization) is small compared to the Hubble radius H^{-1} during old inflation, and (2) the bubble radius (at materialization) and thickness are small compared to the co-moving wavelengths of interest. If these two conditions are not satisfied, the computation becomes more involved, and the

results would then depend on the detailed shape of the potential in the vicinity of the false vacuum.

In [6], we derived the following matching conditions across the light cone:

$$\begin{aligned} \frac{e^{-i\zeta u}}{\operatorname{sech}[u]} \frac{e^{+i\zeta\tau}}{\cosh[\tau]} &\rightarrow (+i) \frac{\sin[\zeta\xi]}{\sinh[\xi]} e^{(+i\zeta-1)\eta}, \\ \frac{e^{-i\zeta u}}{\operatorname{sech}[u]} \frac{e^{-i\zeta\tau}}{\cosh[\tau]} &\rightarrow 0, \\ \frac{e^{+i\zeta u}}{\operatorname{sech}[u]} \frac{e^{+i\zeta\tau}}{\cosh[\tau]} &\rightarrow 0, \\ \frac{e^{+i\zeta u}}{\operatorname{sech}[u]} \frac{e^{-i\zeta\tau}}{\cosh[\tau]} &\rightarrow (-i) \frac{\sin[\zeta\xi]}{\sinh[\xi]} e^{(-i\zeta-1)\eta}, \end{aligned} \quad (3.1)$$

where $\zeta > 0$. The left-hand side indicates asymptotic behavior in region II as $\sigma \rightarrow 0$ [$u \rightarrow +\infty$]; the right-hand side indicates asymptotic behavior in region I as $t \rightarrow 0$ [$\eta \rightarrow -\infty$]. We define region I conformal time with the relation $e^\eta = \tanh[t/2]$.

For small t the positive frequency part of the field operator is

$$\begin{aligned}
 \hat{\phi}^{(+)}(\xi, \eta) &= \frac{-i}{4\pi} \int_0^{+\infty} \frac{d\zeta}{\sqrt{\zeta}} \frac{1 \sin[\zeta\xi]}{e^\eta \sinh[\xi]} \\
 &\times \left\{ \left[\frac{e^{+\pi\zeta/2}}{(e^{\pi\zeta} - e^{-\pi\zeta})^{1/2}} \sqrt{\frac{1 + \sqrt{1 - |C_2|^2/C_1^2}}{2}} e^{-i\zeta\eta} \right. \right. \\
 &\quad \left. \left. - e^{i\bar{\varphi}(\zeta)} \frac{e^{-\pi\zeta/2}}{(e^{\pi\zeta} - e^{-\pi\zeta})^{1/2}} \sqrt{\frac{1 - \sqrt{1 - |C_2|^2/C_1^2}}{2}} e^{+i\zeta\eta} \right] \hat{a}^{(+)}(\zeta, +) \right. \\
 &\quad \left. + \left[\frac{e^{-\pi\zeta/2}}{(e^{\pi\zeta} - e^{-\pi\zeta})^{1/2}} \sqrt{\frac{1 + \sqrt{1 - |C_2|^2/C_1^2}}{2}} e^{+i\zeta\eta} \right. \right. \\
 &\quad \left. \left. - e^{-i\bar{\varphi}(\zeta)} \frac{e^{+\pi\zeta/2}}{(e^{\pi\zeta} - e^{-\pi\zeta})^{1/2}} \sqrt{\frac{1 - \sqrt{1 - |C_2|^2/C_1^2}}{2}} e^{-i\zeta\eta} \right] \hat{a}^{(+)}(\zeta, -) \right\}. \tag{3.2}
 \end{aligned}$$

In [6], it was shown that the asymptotic behavior near the null surface for the inflaton field $\phi \approx e^{\pm i\zeta\eta - \eta}$ corresponds to the asymptotic behavior

$$\Phi \approx \frac{4\pi G V_{,\phi}}{\pm i\zeta + 2} e^{\pm i\zeta\eta + \eta} \tag{3.3}$$

for the gauge invariant gravitational potential. It was further shown that the asymptotic behavior for Φ above matches onto the exact solution

$$\Phi = \frac{4\pi G V_{,\phi}}{\pm i\zeta + 2} e^{\pm i\zeta\eta + \eta} \left[1 - \frac{\zeta \pm i}{3(\zeta \mp i)} e^{2\eta} \right] \tag{3.4}$$

subject to the following assumptions: (1) H remains constant, and (2) the potential is linear. Consequently, to write the positive frequency part $\hat{\Phi}^{(+)}$, we modify Eq. (2.35) to become

$$\begin{aligned}
 \hat{\Phi}^{(+)}(\xi, \eta) &= \frac{-i}{4\pi} \frac{4\pi G V_{,\phi}}{H} e^\eta \int_0^{+\infty} \frac{d\zeta}{\sqrt{\zeta}} \frac{\sin[\zeta\xi]}{\sinh[\xi]} \frac{1}{(e^{\pi\zeta} - e^{-\pi\zeta})^{1/2}} \\
 &\times \left\{ \left[e^{+\pi\zeta/2} \sqrt{\frac{1 + \sqrt{1 - |C_2|^2/C_1^2}}{2}} \frac{e^{-i\zeta\eta}}{2 - i\zeta} \left\{ 1 - \frac{\zeta - i}{3(\zeta + i)} e^{2\eta} \right\} \right. \right. \\
 &\quad \left. \left. - e^{i\bar{\varphi}(\zeta)} e^{-\pi\zeta/2} \sqrt{\frac{1 - \sqrt{1 - |C_2|^2/C_1^2}}{2}} \frac{e^{+i\zeta\eta}}{2 + i\zeta} \left\{ 1 - \frac{\zeta + i}{3(\zeta - i)} e^{2\eta} \right\} \right] \hat{a}^{(+)}(+\zeta) \right. \\
 &\quad \left. + \left[e^{-\pi\zeta/2} \sqrt{\frac{1 + \sqrt{1 - |C_2|^2/C_1^2}}{2}} \frac{e^{+i\zeta\eta}}{2 + i\zeta} \left\{ 1 - \frac{\zeta + i}{3(\zeta - i)} e^{2\eta} \right\} \right. \right. \\
 &\quad \left. \left. - e^{-i\bar{\varphi}(\zeta)} e^{+\pi\zeta/2} \sqrt{\frac{1 - \sqrt{1 - |C_2|^2/C_1^2}}{2}} \frac{e^{-i\zeta\eta}}{2 - i\zeta} \left\{ 1 - \frac{\zeta - i}{3(\zeta + i)} e^{2\eta} \right\} \right] \hat{a}^{(+)}(-\zeta) \right\}. \tag{3.5}
 \end{aligned}$$

Even though we are working in units with $H = 1$, we insert the H^{-1} factor to facilitate later conversion to more conventional units in the final result. We define the power spectrum for Φ according to the following expression for the two-point function:

$$\langle \Phi(\xi = 0, t) \Phi(\xi, t) \rangle = \int_0^\infty d\zeta \zeta \frac{\sin[\zeta\xi]}{\sinh[\xi]} P_\Phi(\zeta, t). \tag{3.6}$$

Taking the limit $t \rightarrow \infty$ [$\eta \rightarrow 0^-$] isolates the growing mode. P_Φ shall denote the limit of $P_\Phi(\zeta, t)$ as $t \rightarrow \infty$. It follows from Eq. (3.5) that

$$\begin{aligned}
P_{\Phi}(\zeta) &= (GV_{,\phi})^2 \frac{4}{9H^2} \frac{1}{\zeta(\zeta^2+1)} \frac{1}{e^{\pi\zeta} - e^{-\pi\zeta}} \\
&\times \left[\left| e^{+\pi\zeta/2} \sqrt{\frac{1 + \sqrt{1 - |C_2|^2/C_1^2}}{2}} + e^{+i\bar{\varphi}(\zeta)} \left(\frac{\zeta+i}{\zeta-i} \right) e^{-\pi\zeta/2} \sqrt{\frac{1 - \sqrt{1 - |C_2|^2/C_1^2}}{2}} \right|^2 \right. \\
&\quad \left. + \left| e^{-\pi\zeta/2} \sqrt{\frac{1 + \sqrt{1 - |C_2|^2/C_1^2}}{2}} + e^{-i\bar{\varphi}(\zeta)} \left(\frac{\zeta-i}{\zeta+i} \right) e^{+\pi\zeta/2} \sqrt{\frac{1 - \sqrt{1 - |C_2|^2/C_1^2}}{2}} \right|^2 \right] \\
&= (GV_{,\phi})^2 \frac{4}{9} \frac{1}{\zeta(\zeta^2+1)} \frac{1}{e^{\pi\zeta} - e^{-\pi\zeta}} \\
&\times \left[e^{\pi\zeta} + e^{-\pi\zeta} + \frac{|C_2|}{C_1} \left(e^{i\bar{\varphi}} \frac{\zeta+i}{\zeta-i} + e^{-i\bar{\varphi}} \frac{\zeta-i}{\zeta+i} \right) \right]. \tag{3.7}
\end{aligned}$$

Using $\chi = 16\pi G(V^2/V_{,\phi}^2)\Phi$ (during inflation), one obtains that the power spectrum for χ (which has a normalization that relates more directly to the density perturbations seen after reheating) is

$$\begin{aligned}
P_{\chi}(\zeta) &= \left[16\pi G \left(\frac{V^2}{V_{,\phi}^2} \right) \right]^2 P_{\Phi}(\zeta) \\
&= \frac{9}{4\pi^2} \left(\frac{H^3}{V_{,\phi}} \right)^2 \frac{1}{\zeta(\zeta^2+1)} \left[\frac{e^{\pi\zeta} + e^{-\pi\zeta} + (|C_2|/C_1) \{ e^{i\bar{\varphi}} [(\zeta+i)/(\zeta-i)] + e^{-i\bar{\varphi}} [(\zeta-i)/(\zeta+i)] \}}{e^{\pi\zeta} - e^{-\pi\zeta}} \right], \tag{3.8}
\end{aligned}$$

where

$$\chi = \frac{2}{3} \frac{H^{-1}\dot{\Phi} + \Phi}{1+w} + \Phi. \tag{3.9}$$

Computing the power spectrum in the limit $t \rightarrow +\infty$, which formally assumes inflation without end, has the effect of completely eliminating the decaying mode. While Ω is close to one (which is true here except for the very early part of the new inflationary epoch) the variable χ is conserved on superhorizon scales irrespective of changes in H or in the slope of the potential. Note with the conventions used here $P \sim \zeta^{-3}$ corresponds to scale invariance.

Our result differs from that of Lyth and Stewart [12] and of Ratra and Peebles [13,14] (who assume different initial conditions for the quantum fields) only by the factor in square brackets in Eq. (3.8). Because $|C_2|/C_1 < 1$, the bracketed quantity lies between $\tanh[\pi\zeta/2]$ and $\coth[\pi\zeta/2]$. This fact severely limits the influence of mass at large wave numbers. Observations of the large-angle cosmic microwave background (CMB) anisotropy, for example, are only sensitive to values of $\zeta > 1$ or so. We conclude that the idea of open inflation is actually more predictive than it might appear at first sight. In Fig. 3, we plot the bracketed quantity versus co-moving wave number for various values of m^2/H^2 . The envelope of two dotted curves indicates the bounds $\tanh[\pi\zeta/2]$ and $\coth[\pi\zeta/2]$.

The phenomenology of the Ratra-Peebles spectrum has been explored in a number of recent papers, with the assumption that the inflaton potential $V(\phi)$ is linear. The conclusion of those papers was that for $\Omega_0 \sim 0.3-0.4$, adiabatic density perturbations of the form studied here were consistent with most current observational constraints [15,16].

We should point out, however, that in our scenario, which provides physically motivated initial conditions, there is no very strong reason to expect the potential to be linear over the range of ϕ of interest. By shortening the length of the slow-roll transition and assuming that the potential *does* have significant structure (i.e., a false vacuum) for the relevant range of ϕ , we are increasing the sensitivity of the final perturbation spectrum to the details of the potential. For example, it may be that within our framework a positively tilted spectrum (i.e., increasing power at shorter wavelengths) is quite likely, and this could restore the viability of models with even lower values of Ω_0 .

IV. CONCLUDING REMARKS

We conclude with the following comments.

(1) In [6] we calculated the power spectrum for the ‘‘conformal’’ mass case $m^2/H^2 = 2$. There was no physical motivation for choosing this mass; the choice was made solely for computational simplicity. Naively one might expect that allowing m^2/H^2 to be a free parameter would diminish the predictiveness of open inflation. However, this does not turn out to be the case, except for very small values of Ω_0 , which can be ruled out observationally based on lower bounds on the mean mass density of the Universe. For scales smaller than the curvature scale there is little freedom to alter the density perturbations by adjusting m^2/H^2 .

(2) The assumptions made here to calculate the power spectrum should be stressed. We assumed that at materialization the size of the bubble and the bubble wall thickness are both small compared to the Hubble radius H^{-1} during *old* inflation. Relaxing these assumptions would alter the power spectrum calculated here. If at material-

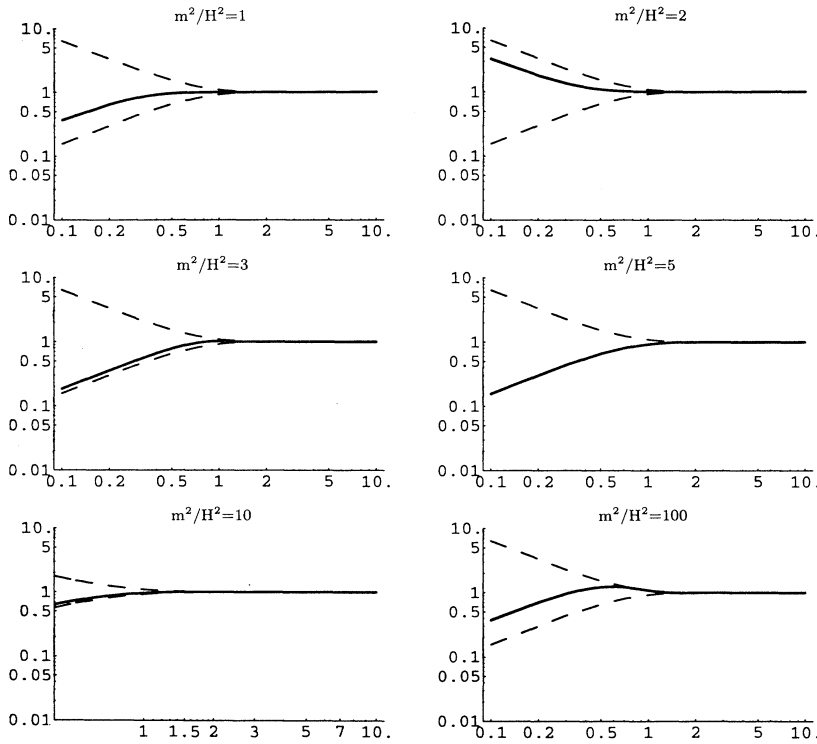


FIG. 3. Dependence of the power spectrum on m^2/H^2 . The bracketed quantity in Eq. (3.8) is plotted as a function of co-moving wave number ζ for various values of m^2/H^2 (solid curve). The dashed envelope indicates the lower and upper bounds $\tanh[\pi\zeta/2]$ and $\coth[\pi\zeta/2]$ for this quantity.

ization the bubble covers an appreciable fraction of the Hubble volume during old inflation, a precise calculation of the perturbations requires taking into account the effect of the Euclidean (classically forbidden) evolution of the background solution on the evolution of the inflaton field perturbations. Alternatively, if one considers perturbations on a co-moving scale not large compared to the bubble wall thickness (or bubble radius at materialization), details of the bubble wall profile, which are highly model dependent, become relevant. Our discontinuous treatment of the mass change across the bubble wall may be regarded as a sudden impulse approximation, which breaks down for sufficiently large co-moving wave numbers. Although the problem of calculating the perturbations when the bubble is not small remains an open problem, important progress in this direction has been made in [17,7].

In region I we assumed that the potential was exactly linear and that over the range of interest $V[\phi]$ does not change appreciably, so that H could be regarded as constant. These are precisely the same assumptions that give an *exact* scale invariant power spectrum for standard ($\Omega_0 = 1$) new inflation. In the recent literature, it has been stressed that because of the change in H during inflation and because of variation in the slope of the potential during inflation, deviations from exact scale invariance are to be expected. These considerations apply equally well to open inflation. The power spectrum calculated here should be regarded as the small- Ω_0 analogue of exact scale invariance for the flat ($\Omega_0 = 1$) case. It is the generic prediction from which small deviations that depend on the exact choice of potential are to be expected.

As mentioned at the end of the last section, in the open case there may be more reason to expect a tilted spectrum than in the flat case, because in open inflation the epoch of new inflation begins near a local maximum of the potential.

(3) It seems that for $m^2/H^2 < 2$ the approach taken here is problematic.¹ The problem is that in this case the two-point function for the Bunch-Davies vacuum in region I falls off so slowly on hyperbolic spatial slices that it cannot be represented as sum over the modes used here. For $m^2/H^2 \geq 2$ there is no such problem.

(4) For small values of m^2/H^2 one has to worry about whether the tunneling is described by an $SO(3,1)$ symmetric bounce (i.e., the Coleman-de Luccia instanton). When $|V''|/H^2 < 4$ at the top of the barrier (i.e., at the local maximum), the tunneling is described by the Hawking-Moss instanton [18] rather than the Coleman-de Luccia instanton [19]. In this regime there is no expanding bubble solution that is $SO(3,1)$ symmetric, and it seems doubtful that a sufficiently homogeneous and isotropic universe will result. Moreover, in the limit as the Coleman-de Luccia instanton approaches the Hawking-Moss instanton, the power spectrum on large scales diverges, because of the flatness of the potential near the local maximum. The physical interpretation of the Hawking-Moss instanton has been discussed in [20]. Generally, one would expect m^2/H^2 (at the local minimum) to be comparable to $|V''|/H^2$ at the local max-

¹We thank M. Sasaki for pointing this out to us. See [22] for a proposed solution to this problem. See also [23,24].

imum, so it would appear difficult to construct viable models with very small m^2/H^2 .

The implications of the power spectra calculated here for the cosmic microwave background will be discussed elsewhere [21].

Note added. After this work was completed, we received a preprint on related work by K. Yamamoto, M. Sasaki, and T. Tanaka on the CMB anisotropy from open inflation [22]. It should be noted that although these authors describe a scenario similar to the one described here, they assume that $m^2 = 0$ everywhere, both inside and outside the bubble. A. Linde [25] and A. Linde and A. Mezhlumian [26] have recently proposed various interesting scenarios for open inflation.

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APPENDIX

In this appendix we verify that the right-hand side of Eq. (2.28) is independent of k . Using Eq. (2.29), and the substitution $x = \cos[\sigma] = \tanh[u]$, $dx = \sin[\sigma]d\sigma$, we need to show that

$$\frac{\Gamma(k/2 - \nu'/2 + \frac{1}{2})\Gamma(k/2 + \nu'/2 + 1)}{\Gamma(k/2 - \nu'/2)\Gamma(k/2 + \nu'/2 + \frac{1}{2})} \frac{\int_{-\infty}^{\infty} du P_{\nu'}^{i\zeta}(\cos[\sigma])\sin[k\sigma]\sin[\sigma]}{\int_{-\infty}^{\infty} du P_{\nu'}^{i\zeta}(\cos[\sigma])\sin[k\sigma]} \quad (\text{A1})$$

is independent of k . We denote the integral in the numerator $A_{\nu',k}$ and that in the denominator $B_{\nu',k}$.

From the standard recursion relation for Legendre functions (e.g., [10], p. 161),

$$(1 + \nu' - i\zeta)P_{\nu'+1}^{i\zeta}(x) = (\nu' + 1)xP_{\nu'}^{i\zeta}(x) - \frac{d}{du}P_{\nu'}^{i\zeta}(x), \quad (\text{A2})$$

it follows upon substitution into the integrals, integrating by parts, and using trigonometric double angle formulas that

$$\begin{aligned} A_{\nu',k+1}(\nu' - k) &= 2(1 + \nu' - i\zeta)A_{\nu'+1,k} \\ &\quad - (\nu' + k)A_{\nu',k-1}, \\ B_{\nu',k+1}(1 + \nu' - k) &= 2(1 + \nu' - i\zeta)B_{\nu'+1,k} \\ &\quad - (1 + \nu' + k)B_{\nu',k-1}, \end{aligned} \quad (\text{A3})$$

which, since $A_{\nu',0} = B_{\nu',0} = 0$, give all the integrals we

need in terms of those for $k = 1$. Redefining

$$\begin{aligned} C_{\nu',k} &= \Gamma\left(\frac{k}{2} - \frac{\nu'}{2} + \frac{1}{2}\right)\Gamma\left(\frac{k}{2} + \frac{\nu'}{2} + 1\right)A_{\nu',k}, \\ D_{\nu',k} &= \Gamma\left(\frac{k}{2} - \frac{\nu'}{2}\right)\Gamma\left(\frac{k}{2} + \frac{\nu'}{2} + \frac{1}{2}\right)B_{\nu',k} \end{aligned} \quad (\text{A4})$$

(which are well defined for arbitrary positive mass), we find from Eq. (A3) that C and D obey identical recursion relations:

$$\begin{aligned} C_{\nu',k+1} &= -(1 + \nu' - i\zeta)C_{\nu'+1,k} \\ &\quad + \frac{1}{4}(k + \nu')(1 + k + \nu')C_{\nu',k-1}, \\ D_{\nu',k+1} &= -(1 + \nu' - i\zeta)D_{\nu'+1,k} \\ &\quad + \frac{1}{4}(k + \nu')(1 + k + \nu')D_{\nu',k-1}. \end{aligned} \quad (\text{A5})$$

Since both vanish for $k = 0$, it follows that $C_{\nu',k}/D_{\nu',k} = C_{\nu',1}/D_{\nu',1}$ and thus that Eq. (A1) is independent of k , as wanted.

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