Gravitational waves in Bianchi type-I universes: The classical theory

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The propagation of classical gravitational waves in Bianchi type-I universes is studied. We find that gravitational waves in Bianchi type-I universes are not equivalent to two minimally coupled massless scalar fields as in the Robertson-Walker universe. Because of its tensorial nature, the gravitational wave is much more sensitive to the anisotropy of the spacetime than the scalar field is and it gains an effective mass term. Moreover, we find a coupling between the two polarization states of the gravitational wave which is also not present in the Robertson-Walker universe.

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I. INTRODUCTION AND SUMMARY

A Bianchi type-I (B-I) universe, being the staightforward generalization of the flat Robertson-Walker (RW) universe, is one of the simplest models of an anisotropic universe. Unlike a RW universe which has the same scale factor for each of the three spatial directions, a B-I universe has a different scale factor in each direction, thereby introducing an anisotropy to the system. It moreover has the agreeable property that near the singularity it behaves like a Kasner universe even in the presence of matter and consequently falls within the general (classical) analysis of the singularity given in [1]. And in a universe filled with matter for which $p = \gamma\rho$, $\gamma < 1$, it has been shown [2] that any initial anisotropy in a B-I universe quickly dies away and a B-I universe will eventually evolve into a RW universe. Since the present-day universe is surprisingly isotropic, this feature of the B-I universe makes it a prime candidate for studying the possible effects of an anisotropy in the early universe on present-day observations.

Curiously, in light of the importance of B-I cosmologies the general behavior of gravitational waves (GW's) in a B-I universe has not been fully analyzed. The propagation of a specific GW in a B-I universe has been studied before in [3]. This analysis, however, was only done for a single wave propagating along the symmetry axis of the 2/3, 2/3, $-1/3$ axial symmetric Kasner spacetime. More recently GW's in B-I universes were studied by Miedema and van Leeuwen [4] within the context of a general perturbative analysis of the B-I universe. They found, however, certain subtleties with gauge freedom and gauge fixing which we have not encountered. In particular, they found that an initially transverse GW will, as it evolves with time, become longitudinal. This we did not find and we believe that their result is due to their gauge fixing. We shall comment further on this after our analysis.

The purpose of this paper, therefore, is to begin the study of GW's in general B-I spacetimes. The approach we shall take follows that given in [5] which analyzes the propagation of GW in the RW universe. There are, however, certain subtleties in the propagation of GW in B-I universes which do not occur in RW universes that makes this analysis much more difficult. These involve, in part, the choice of the appropriate gauge for the GW as well as the definition of the polarization tensors for the wave. We find that the usual transverse-traceless and synchronous gauge conditions that are valid in the RW universe are inconsistant in the B-I universe. As it is the gauge conditions which determine the properties of the polarization tensor, modifications to the gauge conditions were needed to produce a polarization tensor which has the usual properties expected of a GW. (Indeed, if we take the usual transverse and synchronous gauges we would end up with a GW which is not traceless.) Because, however, the usual gauge choice is no longer valid in a B-I universe, additional gauge-dependent terms are introduced into the Lagrangian which will, in general, cause a coupling between the two polarization states of the GW.

The second subtlety we have encountered in defining a GW is in the definition of the polarization tensors themselves even after a gauge choice has been made. As usual, the polarization tensor can be expressed in terms of the appropriate tensor product of two polarization vectors which are transverse to the propagation direction of the GW. There is, however, a certain amount of freedom in the choice of these polarization vectors even after requiring that they be orthonormal to one another. Namely, one can always do a rotation of the polarization vectors, which can be time dependent, in the plane perpendicular to the propagation direction of the GW and the resultant vectors would still be a valid choice of polarization vectors for the GW. While this rotational freedom does not play a role in the isotropic RW universe, the B-I universe is anisotropic and we have found that an arbitrary choice of polarization vectors will cause a coupling between the two polarization vectors. This coupling is fictitious, however, and vanishes once an appropriate choice of polarization

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vectors has been made. In fact something quite similar happens even with GW's propagating in RW universes if we choose polarization vectors which are not orthogonal to one another. This introduces a coupling between the two polarization states which vanishes under an appropriate rotation of polarization vectors.

In general, we find the propagation of GW's in B-I universes to be very much different than in RW universes. In the RW universe the two polarization states of the GW decouple from one another and the Lagrangian for GW is equivalent to two minimally coupled massless scalar fields. Neither is true for GW in B-I universes. Because of its tensorial nature, the GW is much more sensitive to the anisotropy of the B-I universe than a scalar field is and it gains an effective, time-dependent *negative* masslike term. Moreover, the two polarizations of the GW now couple to one another. As this coupling comes in part from the gauge dependent piece of the Lagrangian, its physical relevance is questionable. In particular, it is *possible* that with a more clever choice of gauge this term will disappear. After our analysis we shall present arguments for the physical relevance of the coupling terms.

The rest of the paper is divided into five parts. In Sec. II we shall review the basic properties of a B-I cosmology paying particular attention to the Kasner and Zel'dovich universes. In Sec. III we shall address the question of gauge fixing and the choice of polarization vectors for the GW. In Sec. IV we shall derive the equation of motion for the GW by expanding the GW in plane waves and writing the action for the GW in momentum space. Then in Sec. V we shall solve the evolution equations for the special case of a GW propagating along an asymmetry axis in a Kasner universe or a Zel'dovich universe. This is the only case in which closed form solutions of the equations of motion can be found. Fortunately, they are the most physically relevant. Concluding remarks can then be found in Sec. VI. Finally, in the Appendix we shall repeat the analysis in Sec. IV using a different gauge choice for the gravitational wave and show that coupling terms similar to those found in Sec. IV appear even in this gauge choice.

II. REVIEW OF B-I COSMOLOGY

In this section we shall present a brief review of B-I cosmologies in the absence of perturbations. In doing so we shall follow the notational and sign conventions found in [6]. In particular, one may always choose coordinates for the B-I spacetime such that the metric has the form

$$ds^2 = dt^2 - \sum_{i=1}^{3} a_i^2(t)(dx^i)^2 , \tag{1}$$

where as usual Greek indices will run over the four spacetime directions while Roman indices will run only over the spatial directions. Although we shall use the summation convention for Greek indices, for clarity we shall *not* use it for Roman indices.

It is convenient to define a pseudoconformal time variable η through

$$\frac{d\eta}{dt} = \frac{1}{a(\eta)} , \tag{2}$$

where $a \equiv (a_1a_2a_3)^{1/3}$ is the geometric average of the three scale factors along each direction. Then the only nonvanishing components of the Riemann tensor are

$$R_{0i,0i} = -\frac{C_j}{2}\left(d_i' + \frac{d_i^2}{2} - \frac{d_iD}{2}\right) , \tag{3}$$

while, for $i \neq j$,

$$R_{ij,ij} = \frac{1}{4}\frac{d_id_jC_iC_j}{C} , \tag{4}$$

where $C = a^2(\eta)$, $C_j = a_j^2(\eta)$,

$$d_j \equiv \frac{C_j'}{C_j} , \qquad D \equiv \frac{C'}{C} = \frac{1}{3}\sum_{i=1}^{3} d_i , \tag{5}$$

and the prime denotes a derivative with respect to η. Einstein's equations in the presence of matter are then

$$\begin{aligned} \tfrac{3}{2}D' + 6Q &= -4\pi C(\rho + 3p) , \\ d_j' + d_jD &= 8\pi C(\rho - p) , \end{aligned} \tag{6}$$

where

$$Q \equiv \frac{1}{72}\sum_{i>j}^{3}(d_i - d_j)^2 , \tag{7}$$

and

$$\left(\frac{1}{a}\frac{da}{d\eta}\right)^2 = \frac{8\pi a^2}{3}\rho + Q . \tag{8}$$

This, aside from the term dependent on Q, is identical to the RW case. Q is thus the physical measure of the anisotropy of the spacetime and is called the anisotropy factor. Taking then the difference between two different directions in Eq. (6) we find that

$$d_i - d_j = (d_i^0 - d_j^0)/a^2 , \tag{9}$$

where d_i^0 is the value of d_i at some initial time η_0 for all ρ and p and we have chosen the overall scale such that $a(\eta_0) = 1$. Consequently, $Q = Q_0/a^4$ where $Q_0 = Q(\eta_0)$.

Equation (6) has been solved in the presence of various types of matter [2]. We shall, however, only be concerned with two special cases in this paper: the Kasner universe, which is free of matter, and the Zel'dovich universe where $\rho = p$. In both cases,

$$a^2 = y/y_0 , \qquad a_j^2 = a_{0j}^2(y/y_0)^{3p_j} , \tag{10}$$

where $y = 2\eta\sqrt{Q_0}$ for the Kasner universe while $y = 2\eta\sqrt{8\pi\rho_0/3 + Q_0}$ for the Zel'dovich universe. ρ_0 is the energy density of the universe at η_0 while $a_{0j} = a_j(0)$ are the initial scale factors in the various directions which, without loss of generality, can be set to unity.

p_j are parameters of the B-I spacetime which measure the *relative* anisotropy between any two asymmetry axis. As they must satisfy the constraints

$$1=\sum_{i=1}^{3}p_i\,, \qquad 1=\sum_{i=1}^{3}p_i^2\,, \tag{11}$$

out of the three parameters, only one is arbitrary. Since Eq. (11) describes the intersection of a sphere with a plane in the parameter space (p_1,p_2,p_3), we may parametrize the allowed values of p_j by an angle on the unit circle [7]. One particular choice of parametrization is

$$p_1=\tfrac{1}{3}\left(1+\cos\theta+\sqrt{3}\sin\theta\right)\,,$$
$$p_2=\tfrac{1}{3}\left(1+\cos\theta-\sqrt{3}\sin\theta\right)\,,$$
$$p_3=\tfrac{1}{3}\left(1-2\cos\theta\right)\,. \tag{12}$$

Although *a priori* θ ranges over the unit circle, note that the labeling of each p_j is quite arbitrary. Thus the unit circle can be divided into six equal parts each of which span 60°, and the choice of p_j is unique within each section separately. Notice that when $\theta=0$, $p_1=p_2=2/3$ while $p_3=-1/3$. This is the spacetime considered by Hu in [3]. When $\theta=\pi/3$, on the other hand, $p_1=1$ while $p_2=p_3=0$. It can be shown that this spacetime is equivalent to the Minkowski spacetime up to a coordinate transformation.

III. GAUGE CONDITIONS AND POLARIZATION STATES

In this section we address the problem of gauge fixing for the GW and the subsequent definition of polarization states. As usual, we consider the GW as a perturbation off the background metric by writing $g_{\mu\nu}=g^0_{\mu\nu}+h_{\mu\nu}$, where $g^0_{\mu\nu}$ is the unperturbed B-I metric given in Eq. (1) while $h_{\mu\nu}$ is the perturbation which has a gauge freedom to be taken care of. The usual gauge choice, which works for both flat Minkowski and RW spacetimes, is

$$\nabla^\mu h_{\mu\nu}=0,\quad u^\mu h_{\mu\nu}=0,\quad h^\mu_\mu=0\,, \tag{13}$$

and are called the transverse, synchronous, and traceless conditions, respectively. u_μ is a timelike Killing vector which, without lose of generality, can be taken to be $(1,0,0,0)$. In the RW spacetime, the transverse and synchronous conditions actually imply that the GW is traceless and this condition is redundant. Unfortunately, for a B-I universe this is no longer true.

Let us express $h^{(s)}_{\mu\nu}$ as a plane wave

$$h^{(s)}_{\mu\nu}=\varpi^{(s)}_{\mu\nu}(\eta)h_{\vec{q}s}(\eta)e^{i\vec{q}\cdot\vec{x}}\,, \tag{14}$$

where $\vec{q}\cdot\vec{x}=\sum_j q_jx^j$ and s labels the polarization state. Because $g^0_{\mu\nu}$ is a function of η only, we can always make this choice. Our convention is that the q_j are independent of η while $q^j=g^{jj}q_j$. $\varpi^{(s)}_{\mu\nu}$ is the polarization tensor for the GW and must, by definition, be independent of the amplitude $h_{\vec{q}s}(\eta)$ of the GW. The usual transversality condition then gives

$$0=g^{\mu\nu}\partial_\mu\varpi^{(s)}_{\nu0}+\varpi^{(s)\mu}{}_0\partial_\mu\ln h_{\vec{q}s}+i\sum_j q^j\varpi^{(s)}_{j0}+\sum_j\frac{a_j'}{a_j}\varpi^{(s)0}{}_0-\sum_j\frac{a_j'}{a_j}\varpi^{(s)j}{}_j\,, \tag{15}$$

as well as

$$0=g^{\mu\nu}\partial_\mu\varpi^{(s)}_{\nu k}+\varpi^{(s)\mu}{}_k\partial_\mu\ln h_{\vec{q}s}+i\sum_j q^j\varpi^{(s)}_{jk}+2\frac{a'}{a}\varpi^{(s)0}{}_k\,. \tag{16}$$

From this, we see that for the polarization tensor to be independent of $h_{\vec{q}s}$, we *must* choose the synchronous gauge. After doing so, the transversality condition reduces to

$$0=\sum_j\frac{a_j'}{a_j}\varpi^{(s)j}{}_j,\quad 0=\sum_j q^j\varpi^{(s)}_{j\nu}, \tag{17}$$

and becomes the standard transverse-traceless condition in a RW universe. For a B-I universe, on the other hand, we see that the GW is no longer traceless if we use the usual transverse and synchonous gauges. The usual gauge choices Eq. (13) are no longer self-consistant in a B-I universe and must be modified.

There are two straightforward modifications that we can make. The first is to use the usual transverse and synchronous gauges and live with a GW which is not traceless. This, however, introduces a great deal of complexity to the problem and we shall not do so (see the Appendix). The second is to require that the GW be traceless, but to modify the transversality condition, namely, to choose

$$0=\nabla^\mu h_{\mu j},\quad 0=h_{0\mu},\quad 0=h^\mu_\mu, \tag{18}$$

which gives the following constraints on the polarization tensor

$$0=\sum_j q^j\varpi^{(s)}_{jk},\quad 0=\varpi^{(s)}_{0\nu},\quad 0=\sum_j\varpi^{(s)j}{}_j. \tag{19}$$

Notice that these equations determining $\varpi^{(s)}_{\mu\nu}$ have precisely the same form as the those in either a RW or Minkowski spacetime which is the reason we shall work primarily in this gauge.

Instead of the polarization tensor $\varpi^{(s)}_{ij}$, we shall find it conceptually easier to work with the polarization vectors $\epsilon^{(s)}_\mu$ with

$$\varpi^{(+)}_{jk}=\epsilon^{(1)}_j\epsilon^{(1)}_k-\epsilon^{(2)}_j\epsilon^{(2)}_k,\quad \varpi^{(\times)}_{jk}=\epsilon^{(1)}_j\epsilon^{(2)}_k+\epsilon^{(2)}_j\epsilon^{(1)}_k, \tag{20}$$

as long as

$$0=\sum_j q^j\epsilon^{(s)}_j,\quad \sum_j\epsilon^{(s)}_j\epsilon^{(s')j}=-\delta_{ss'}\,. \tag{21}$$

Implicit in this definition is the added requirement that the two polarization states be "orthonormal" to one another: $\varpi^{(s)}_{\mu\nu}\varpi^{(s')\mu\nu}=2\delta^{ss'}$. Notice, however, that this orthonormality condition is *not* required by the gauge choice Eq. (19) but is rather made separately as a minimal requirement for the two polarization states to decouple from one another. While such a choice is sufficient for a RW or Minkowski spacetime, we shall see in the next section that additional requirements are needed for a B-I spacetime.

Equation (21) does not determine $\epsilon_j^{(s)}$ uniquely, however. There is still a rotational freedom in the plane perpendicular to q_j. Namely, we can rotate

$$\hat{\epsilon}_j^{(1)} = \epsilon_j^{(1)}\cos\phi + \epsilon_j^{(2)}\sin\phi\,,$$
$$\hat{\epsilon}_j^{(2)} = -\epsilon_j^{(1)}\sin\phi + \epsilon_j^{(2)}\cos\phi\,, \tag{22}$$

and $\hat{\epsilon}_j^{(s)}$ will still satisfy Eq. (21). In particular, ϕ may also be η dependent. Moreover, once a specific choice of polarization vectors has been made rotational symmetry will be broken. While this breaking of rotational symmetry is of no consequence in the RW universe, which is isotropic to begin with, it is of great consequence in the anisotropic B-I universe. In particular, any specific choice of polarization vectors will tend to break the expected exchange symmetry 1-2-3 among the labels of the axis of the spacetime. We shall also find that the equations of motion for the GW will be greatly simplified if we make a special choice of polarization vectors. For these reasons we shall work as much as possible with a general set of polarization vectors and shall delay making a specific choice of polarization vectors until forced to.

To find a specific representation of $\epsilon_j^{(s)}$ we first define a local coordinate system using the vierbeins e^i_μ,

$$e^i_\mu e^{\mu j} = -\delta^{ij}\,. \tag{23}$$

Then, taking $e^1_\mu = (0, a_1, 0, 0)$, $e^2_\mu = (0, 0, a_2, 0)$, $e^3_\mu = (0, 0, 0, a_3)$, one specific choice of polarization vectors is

$$\epsilon_j^{(1)} = \frac{\hat{q}_1\hat{q}_3}{\sqrt{\hat{q}_1^2+\hat{q}_2^2}}e^1_j + \frac{\hat{q}_2\hat{q}_3}{\sqrt{\hat{q}_1^2+\hat{q}_2^2}}e^2_j - \sqrt{\hat{q}_1^2+\hat{q}_2^2}\,e^3_j\,,$$
$$\epsilon_j^{(2)} = -\frac{\hat{q}_2}{\sqrt{\hat{q}_1^2+\hat{q}_2^2}}e^1_j + \frac{\hat{q}_1}{\sqrt{\hat{q}_1^2+\hat{q}_2^2}}e^2_j\,, \tag{24}$$

where

$$\hat{q}_j = -\frac{q_j/a_j}{\sqrt{\sum_l q_l^2/a_l^2}}\,. \tag{25}$$

Notice in particular that $\epsilon_j^{(s)}(-\vec{q}) = (-1)^{s+1}\epsilon_j^{(s)}(\vec{q})$ so that $\varpi_{jk}^{(+)}(-\vec{q}) = \varpi_{jk}^{(+)}(\vec{q})$ while $\varpi_{jk}^{(\times)}(-\vec{q}) = -\varpi_{jk}^{(\times)}(\vec{q})$.

The polarization tensors defined in Eq. (24) are for linearly polarized GW. We shall also consider circularly polarized GW defined through

$$\varpi_{jk}^{(L)} = \frac{1}{\sqrt{2}}\left(\varpi_{jk}^{(+)} + i\varpi_{jk}^{(\times)}\right),$$
$$\varpi_{jk}^{(R)} = \frac{1}{\sqrt{2}}\left(\varpi_{jk}^{(+)} - i\varpi_{jk}^{(\times)}\right)\,, \tag{26}$$

with $\varpi^{(L)}(-\vec{q}) = \varpi^{(R)}(\vec{q}) = \overline{\varpi}^{(L)}(\vec{q})$ and $\varpi^{(R)}(-\vec{q}) = \varpi^{(L)}(\vec{q}) = \overline{\varpi}^{(R)}(\vec{q})$ and the overbar denotes complex conjugation. We can, of course, also define circularly polarized polarization vectors in precisely the same way

$$\epsilon_j^{(L)} = \frac{1}{\sqrt{2}}\left(\epsilon_j^{(1)} + i\epsilon_j^{(2)}\right),$$
$$\epsilon_j^{(R)} = \frac{1}{\sqrt{2}}\left(\epsilon_j^{(1)} - i\epsilon_j^{(2)}\right)\,, \tag{27}$$

and there is a particularly simple relationship between ϖ and ϵ,

$$\varpi_{jk}^{(L)} = \sqrt{2}\epsilon_j^{(L)}\epsilon_k^{(L)}, \quad \varpi_{jk}^{(R)} = \sqrt{2}\epsilon_j^{(R)}\epsilon_k^{(R)}. \tag{28}$$

Finally, we note that under a rotation of polarization vectors as in Eq. (22), $\hat{\epsilon}_j^{(L)} = e^{-i\phi}\epsilon_j^{(L)}$, $\hat{\epsilon}_j^{(R)} = e^{i\phi}\epsilon_j^{(R)}$.

IV. THE ACTION

In this section we shall derive the equation of motion for the GW. The approach we shall follow was first used by Ford and Parker [5] in their analysis of GW's in RW universes and involves the expansion of the GW in plane waves. To lowest order the action for the GW is

$$I = \frac{1}{4}\int\sqrt{-g}\,d^4x\Bigg(\nabla_\mu h^\beta_\alpha\nabla^\mu h^\alpha_\beta + 8\pi(\rho-p)h^\beta_\alpha h^\alpha_\beta$$
$$-2R^\mu_\beta h^\alpha_\mu h^\beta_\alpha - 2R^{\mu\beta}{}_{\alpha\nu}h^\nu_\mu h^\alpha_\beta - 2\nabla_\rho h^{\mu\rho}\nabla_\alpha h^\alpha_\mu\Bigg)\,, \tag{29}$$

which differs from the one used in [5] for GW's in RW spacetimes by the addition of a kinetic term. This is because our gauge choice Eq. (18) is different from the one they used as in Eq. (13). We should also mention that even if we did use the usual transverse and synchronous gauges, our action will have additional terms dependent on h^μ_μ which no longer vanishes in this gauge for a B-I universe. Indeed, *any* choice of gauge for the GW in a B-I universe will introduce additional terms to the Lagrangian. The form that these additional terms take is dependent on the choice of gauge one makes, however, and at first glance the physical relevance of these terms, since they are seemingly gauge dependent, is questionable. We shall see, however, that although the explicit form of these terms changes with the choice of gauge, they all have the same root physical cause. Namely, they are due to the fact that in a B-I universe one cannot define a polarization state for the GW which will not change with time. Consequently, these terms are physically relevant.

It is straightforward to show that due to Einstein's equations, the second and third terms in Eq. (29) cancel one another. We are then left with three terms to consider:

$$I_K = \frac{1}{4}\int a^4d^4x\nabla_\mu h^\beta_\alpha\nabla^\mu h^\alpha_\beta\,,$$
$$I_R = -\frac{1}{2}\int a^4d^4xR^{\mu\beta}{}_{\alpha\nu}h^\nu_\mu h^\alpha_\beta\,,$$
$$I_g = -\frac{1}{2}\int a^4d^4x\nabla_\rho h^{\mu\rho}\nabla_\alpha h^\alpha_\mu\,, \tag{30}$$

where $I = I_K + I_R + I_g$. We next expand $h_{\mu\nu}$ in plane waves,

$$h_{\mu\nu} = \sum_{\vec{q}} \sum_{s=+,\times} \varpi^{(s)}_{\mu\nu} h_s(\vec{q}, x) , \tag{31}$$

where $h_s(\vec{q}, x) = h_{\vec{q}s} e^{i\vec{q}\cdot\vec{x}}$ with the reality condition given by

$$\bar{h}_+(\vec{q}, x) = h_+(-\vec{q}, x), \quad \bar{h}_\times(\vec{q}, x) = -h_\times(-\vec{q}, x) . \tag{32}$$

Turning our attention first to I_R, we use Eq. (31) and find that

$$I_R = -\frac{1}{2} \sum_{\vec{q}} \sum_{ss'} \int a^4 d^4x T_{ss'}(\epsilon) \bar{h}_{s'}(\vec{q}, x) h_s(\vec{q}, x) , \tag{33}$$

where

$$T_{ss'}(\epsilon) = R^{\mu\rho\alpha\nu} \varpi^{(s)}_{\mu\nu} \varpi^{(s')}_{\rho\alpha} . \tag{34}$$

In terms of polarization vectors,

$$T_{++}(\epsilon) = T_{\times\times}(\epsilon) = -2 \sum_{ijkl} R^{ijkl} \epsilon^{(1)}_i \epsilon^{(1)}_l \epsilon^{(2)}_j \epsilon^{(2)}_k , \tag{35}$$

while

$$T_{+\times}(\epsilon) = T_{\times +}(\epsilon) = 0 , \tag{36}$$

due to the antisymmetry properties of the curvature tensor. Next, using Eq. (4),

$$T_{++}(\epsilon) = \frac{2}{a^2} \left\{ \left(\sum_i \frac{a'_i}{a_i} \epsilon^{(1)}_i \epsilon^{(1)^i} \right) \left(\sum_j \frac{a'_j}{a_j} \epsilon^{(2)}_j \epsilon^{(2)^j} \right) - \left(\sum_j \frac{a'_j}{a_j} \epsilon^{(1)}_j \epsilon^{(2)^j} \right)^2 \right\}, \tag{37}$$

where it is important to note that we have not as yet chosen a specific polarization vector $\epsilon^{(s)}_j$.

We next consider the gauge term which becomes

$$I_g = -\frac{1}{2} \sum_{\vec{q}} \sum_{ss'} \int a^4 d^4x \Big\{ \varpi^{(s')}_{\mu\rho} \varpi^{(s)\mu\alpha} \nabla^\rho \bar{h}_{s'} \nabla_\alpha h_s + 2\varpi^{(s')}_{\mu\rho} \nabla_\alpha \varpi^{(s)\mu\alpha} \nabla^\rho \bar{h}_{s'} h_s + \nabla^\rho \varpi^{(s')}_{\mu\rho} \nabla_\alpha \varpi^{(s)\mu\alpha} \bar{h}_{s'} h_s \Big\}, \tag{38}$$

for plane waves. The first two terms involve the term $\varpi^{(s)}{}_{\mu\alpha} \nabla^\alpha h_s(\vec{q}, \eta)$ which vanishes identically due to Eq. (19). The last term does not, however, and if we define

$$N^2_{ss'} = -\nabla^\rho \varpi^{(s')}_{\mu\rho} \nabla^\alpha \varpi^{(s)\mu}_{\ \ \alpha} \tag{39}$$

we find that

$$\begin{aligned}
N^2_{++} &= -\sum_i \left(\epsilon^{(1)}_i \nabla^i \epsilon^{(1)}_\mu - \epsilon^{(2)}_i \nabla^i \epsilon^{(2)}_\mu \right) \sum_j \left(\epsilon^{(1)}_j \nabla^j \epsilon^{(1)\mu} - \epsilon^{(2)}_j \nabla^j \epsilon^{(2)\mu} \right) = -\frac{1}{a^2} \left(\sum_j \frac{a'_j}{a_j} \varpi^{(+)j}_{\ \ j} \right)^2 , \\
N^2_{\times\times} &= -\sum_i \left(\epsilon^{(1)}_i \nabla^i \epsilon^{(2)}_\mu + \epsilon^{(2)}_i \nabla^i \epsilon^{(1)}_\mu \right) \sum_j \left(\epsilon^{(1)}_j \nabla^j \epsilon^{(2)\mu} + \epsilon^{(2)}_j \nabla^j \epsilon^{(1)\mu} \right) = -\frac{1}{a^2} \left(\sum_j \frac{a'_j}{a_j} \varpi^{(\times)j}_{\ \ j} \right)^2 , \\
N^2_{+\times} &= -\sum_i \left(\epsilon^{(1)}_i \nabla^i \epsilon^{(1)}_\mu - \epsilon^{(2)}_i \nabla^i \epsilon^{(2)}_\mu \right) \sum_j \left(\epsilon^{(1)}_j \nabla^j \epsilon^{(2)\mu} + \epsilon^{(2)}_j \nabla^j \epsilon^{(1)\mu} \right) = -\frac{1}{a^2} \left(\sum_i \frac{a'_i}{a_i} \varpi^{(+)i}_{\ \ i} \right) \left(\sum_j \frac{a'_j}{a_j} \varpi^{(\times)j}_{\ \ j} \right) ,
\end{aligned} \tag{40}$$

while $N^2_{+\times} = N^2_{\times +}$ and we have used $\nabla^\mu \epsilon^{(s)}_\mu = 0$. Notice, in particular, that in general $N^2_{+\times} \neq 0$ and this gauge term introduces a coupling between the two polarization states. In addition, $N^2_{++} \neq N^2_{\times\times}$. Notice also that each of the N^2 are related in a simple manner to the trace of the polarization vectors which would vanish under the usual choice of gauge Eq. (17).

Finally, we turn our attention to the kinetic piece, the most difficult one to work with. Proceeding exactly as before, we find that

$$I_K = \frac{1}{2} \sum_{\vec{q}} \int a^4 d^4x \Big\{ \sum_s \nabla_\rho \bar{h}_s \nabla^\rho h_s - \sum_{ss'} M^2_{ss'} \bar{h}_{s'} h_s - 2 \sum_{ss'} D_{\mu_{ss'}} \bar{h}_{s'} \nabla^\mu h_s \Big\}, \tag{41}$$

where

$$D_{\mu_{ss'}} = -\tfrac{1}{2} \varpi^{(s)\alpha}_{\ \ \beta} \nabla_\mu \varpi^{(s')\beta}_{\ \ \alpha},$$

$$M^2_{ss'} = -\tfrac{1}{2} \nabla_\rho \varpi^{(s)}_{\mu\nu} \nabla^\rho \varpi^{(s')\mu\nu} . \tag{42}$$

This can be reduced to manageable form by using polarization vectors

$$M^2_{++} = M^2_{\times\times} = \sum_s \nabla_\rho \epsilon^{(s)}_\mu \nabla^\rho \epsilon^{(s)\mu} - 2(\epsilon^{(1)}_\mu \nabla_\rho \epsilon^{(2)\mu})(\epsilon^{(1)}_\nu \nabla^\rho \epsilon^{(2)\nu}) , \tag{43}$$

where we have used $\epsilon^{(s')}_\mu \epsilon^{(s)\mu} = -\delta_{ss'}$, while $M_{+\times} \equiv 0$.

Similarly,

$$D_{\mu_{++}} = D_{\mu_{\times\times}} = \epsilon^{(1)}_\alpha \nabla_\mu \epsilon^{(1)^\alpha} + \epsilon^{(2)}_\alpha \nabla_\mu \epsilon^{(2)^\alpha} , \quad (44)$$

which vanishes identically while

$$D_{\mu_{+\times}} = -D_{\mu_{\times+}} = \epsilon^{(1)}_\alpha \nabla_\mu \epsilon^{(2)^\alpha} - \epsilon^{(2)}_\alpha \nabla_\mu \epsilon^{(1)^\alpha} . \quad (45)$$

This term will not vanish in general, however, and we see that the kinetic piece of the Lagrangian also introduces a coupling between the two polarizations. Notice also that the only nonvanishing term in $D_{\mu_{+\times}}$ is $D_{0+\times} \equiv D_{+\times}$.

Combining these three pieces together, we finally arrive at

$$I = \sum_{\vec{q}} \int a^4 d^4x \Bigg\{ \frac{1}{2} \sum_s \left(\nabla_\mu \bar{h}_s \nabla^\mu h_s - \frac{m_s^2}{a^2} \bar{h}_s h_s \right) - \frac{1}{a^2} D_{\times+} \bar{h}_+ \frac{dh_\times}{d\eta} - \frac{1}{a^2} D_{+\times} \bar{h}_\times \frac{dh_+}{d\eta} + \frac{N^2_{+\times}}{2} \left(\bar{h}_+ h_\times + \bar{h}_\times h_+ \right) \Bigg\} , \quad (46)$$

where

$$m_s^2(\epsilon) = a^2 \left(M^2_{++} + T_{++} - N^2_{ss} \right) . \quad (47)$$

Once again it is important to note that we have not, as yet, made any specific choice of polarization vectors.

At this point we can see the differences between the propagation of GW's in a B-I universe versus a RW universe. Ford and Parker has shown that in a RW universe the two polarizations of the GW decouple from one another and each, separately, is equivalent to a minimally coupled massless scalar field. Neither is true for the GW propagating in a B-I universe, however. Here the GW picks up an effective mass term due to it being a spin-2 field and is much more sensitive to the anisotropy than a scalar field is. What is even more surprising is the apparent coupling which is present between the two polarization states of the GW.

The physical relevance of this coupling term is somewhat questionable at this point, however. They could have arisen from an inappropriate choice of polarization vectors or, since the second such term comes from the gauge piece of the action, because we made an inconvenient gauge choice for the GW. For example even in a RW universe a coupling term between the polarizations would appear if we had chosen polarization tensors which were not orthogonal to one another; an "inappropriate" choice of polarization tensors. To begin to separate the truly physical effects of the anisotropy on the GW from the arbitrariness in defining the GW, we have to develop a better understanding of the effects of the anisotropy on the GW. This is best done by considering the behavior of the polarization vectors instead of the full tensors.

Consider the triad $(\hat{q}_j, \epsilon^{(1)}_j, \epsilon^{(2)}_j)$ which form a local orthonormal coordinate system. Because we are in the B-I universe, these three vectors are not fixed, but rather changes with η. Since they are constrained to lie on the unit sphere, their motion consists of two rotations; one in the plane perpendicular to $\hat{q}$ which is spanned by $\epsilon^{(1)}_j$ and $\epsilon^{(2)}_j$, the other parallel to $\hat{q}$. Let us consider the two rotations separately. Rotations parallel to $\hat{q}$ occur because the direction of propagation of the GW, $\hat{q}_j$, is always changing with time [see Eq. (25)]. This is because the medium through which the GW is propagating, the B-I universe, is anisotropic and always changing with η. In particular, notice that in a RW universe, where such coupling between polarization states is not present, $\hat{q}_j$ does not change with time and there are no rotations along this direction. Rotations in this direction are therefore caused by a physical effect of the anisotropy on the GW.

Rotations in the $\epsilon^{(1)}$-$\epsilon^{(2)}$ plane, however, are not physical. They can occur even when $\hat{q}_j$ does not change directions as in a RW universe. Remember that the definition of $\epsilon^{(s)}_j$ is somewhat arbitrary. One still has the freedom to do a rotation as in Eq. (22) of the polarization vectors in this plane even if this rotation is η dependent. Returning to Eq. (45) we see that $D_{+\times} = 2\epsilon^{(1)}_\mu \nabla_0 \epsilon^{(2)^\mu} = -2\epsilon^{(2)}_\mu \nabla_0 \epsilon^{(1)^\mu}$ depends explicitly on the velocity $\nabla_0 \epsilon^{(s)}_j$ of the polarization vectors *in this plane.* Consequently, the presence of the $D_{+\times}$ coupling term is due solely to the rotation of the polarization vectors in the plane perpendicular to $\hat{q}_j$. A $D_{+\times} \neq 0$ therefore only means that we have not chosen the "correct" polarization vectors: one in which the polarization vectors do not rotate about $\hat{q}_j$.

To demonstrate that such a choice of polarization vectors always exists, we preform the η-dependent rotation of the polarization vectors given in Eq. (22). Under this rotation we find that

$$D_{+\times}(\hat{\epsilon}) = 2\phi' + D_{+\times}(\epsilon) , \quad (48)$$

while

$$\begin{aligned} m^2_+(\hat{\epsilon}) &= m^2_+(\epsilon) - 4(\phi')^2 - 8\phi' \sum_j \epsilon^{(1)}_j \nabla_0 \epsilon^{(2)^j} \\ &\quad - a^2 \left\{ N^2_{++}(\epsilon)(\cos^2 2\phi - 1) + N^2_{+\times}(\epsilon) \sin 4\phi + N^2_{\times\times}(\epsilon) \sin^2 2\phi \right\} , \\ m^2_\times(\hat{\epsilon}) &= m^2_\times(\epsilon) - 4(\phi')^2 - 8\phi' \sum_j \epsilon^{(1)}_j \nabla_0 \epsilon^{(2)^j} \\ &\quad - a^2 \left\{ N^2_{++}(\epsilon) \sin^2 2\phi - N^2_{+\times}(\epsilon) \sin 4\phi + N^2_{\times\times}(\epsilon)(\cos^2 2\phi - 1) \right\} , \\ N^2_{+\times}(\hat{\epsilon}) &= \tfrac{1}{2} \left[N^2_{\times\times}(\epsilon) - N^2_{++}(\epsilon) \right] \sin 4\phi + N^2_{+\times}(\epsilon) \cos 4\phi . \end{aligned} \quad (49)$$

Clearly, one needs only to choose, for any given $\epsilon_j^{(s)}$,

$$\phi = -\frac{1}{2}\int^{\eta} D_{+\times}(\epsilon)d\eta' + \phi_0 , \tag{50}$$

and the coupling term proportional to $D_{+\times}(\hat{\epsilon})$ vanishes. ϕ_0 is an arbitrary, *constant* angle and its presence means that we still have a degree of freedom to choose our polarization vectors. For the specific choice of polarization vectors given in Eq. (24),

$$\phi = -\int^{\eta} \frac{\hat{q}_1\hat{q}_2\hat{q}_3}{\hat{q}_1^2+\hat{q}_2^2}\left(\frac{a_1'}{a_1}-\frac{a_2'}{a_2}\right)d\eta' + \phi_0 . \tag{51}$$

Notice also that Eq. (51) does not have a symmetry under exchange of 1-2-3 which is a reflection of the fact that once a specific choice of polarization vectors has been made rotational symmetry is broken. Other choices of $\epsilon_j^{(s)}$ will only result in different forms for ϕ, but $D_{+\times}$ will still vanish.

While $D_{+\times}$ vanishes after an appropriate choice of polarization vectors is made, this still leaves the coupling term proportional to $N^2_{+\times}$. This term came from the gauge dependent term of the original action, however, and is different for different choices of gauge. It is possible that under other, more clever gauge choices this term will no longer be present. Its physical relevance is therefore questionable at this point. There are two ways to address this problem. The first is to redo this calculation with a different gauge choice and see if the coupling terms are still present. This we have done in the Appendix. This method, however, has the shortcoming in that one can never be sure there is still some other gauge choice for which the coupling terms vanish identically. The second way is to develop a physical argument for or against the relevance of the physical terms. The final conclusion of this argument should therefore be valid with any gauge choice. This is the approach we shall now take.

Consider the function

$$f_{ss'}(\eta) \equiv \sum_j \epsilon^{(s)j}(\eta_1)\epsilon^{(s')}{}_j(\eta) , \tag{52}$$

where η_1 is some *fixed* time and $\epsilon^{(s)j}(\eta_1) \equiv g^{jj}(\eta_1)\epsilon^{(s)}{}_j(\eta_1)$. Clearly, $f_{ss'}(\eta_1) = -\delta_{ss'}$ and thus $f_{ss'}$ is a measure of how the orthonormality of the polarization vector changes with time. Next, considering $f_{ss'}$ as a function of η only, we expand $\epsilon_j(\eta)$ in a Taylor series in η about η_1 and find that, to first order,

$$f_{ss'}(\eta) = -\delta_{ss'} + \left(\sum_j \epsilon^{(s)j}\frac{d\epsilon_j^{(s')}}{d\eta}\right)\Bigg|_{\eta=\eta_1}(\eta-\eta_1) + \cdots . \tag{53}$$

Since η_1 is arbitrary, we see that the infinitesimal change in the normality condition is measured by

$$\sum_j \epsilon^{(1)j}\frac{d\epsilon_j^{(1)}}{d\eta} = \sum_j \frac{a_j'}{a_j}\epsilon^{(1)j}\epsilon_j^{(1)},$$

$$\sum_j \epsilon^{(2)j}\frac{d\epsilon_j^{(2)}}{d\eta} = \sum_j \frac{a_j'}{a_j}\epsilon^{(2)j}\epsilon_j^{(2)} . \tag{54}$$

The difference of these two terms is just $\sqrt{-a^2N^2_{++}} = \sum \varpi^{(+)j}{}_j a_j'/a_j$ while their sum is just the $\sum \tau_j^j a_j'/a_j$ defined in Eq. (A2) of the Appendix. For the choice of gauge Eq. (18) it plays no role. The infinitesimal change to the orthogonality condition is measured by the linear combinations

$$\sum_j\left(\epsilon^{(1)j}\frac{d\epsilon_j^{(2)}}{d\eta} + \epsilon^{(2)j}\frac{d\epsilon_j^{(1)}}{d\eta}\right) = 2\sum_j \frac{a_j'}{a_j}\epsilon^{(1)j}\epsilon_j^{(2)} , \tag{55}$$

which is just $\sqrt{-a^2N^2_{\times\times}} = \sum \varpi^{(\times)j}{}_j a_j'/a_j$, and

$$\sum_j\left(\epsilon^{(1)j}\frac{d\epsilon_j^{(2)}}{d\eta} - \epsilon^{(2)j}\frac{d\epsilon_j^{(1)}}{d\eta}\right) = D_{+\times} . \tag{56}$$

We thus see that each of the terms which may cause a coupling between the two polarization states, N^2_{++}, $N^2_{\times\times}$, $N^2_{+\times}$, and $D_{+\times}$, have their roots in the time rate of change of the polarization vectors. Since the polarization vectors change with time due to the anisotropic rate of expansion of a B-I universe, we therefore conclude that the presence of these coupling terms is a natural consequence of the physical properties of the spacetime. In particular, we see that a coupling between the two polarization states will be present so long as we are not able to consistently define a polarization state which is valid at all times. They will be present in some form no matter what gauge conditions one chooses and the coupling terms are physically relevant.

To further illustrate the connection between the rate of change in the direction of the polarization vectors and the coupling term, let us look at the conditions under which theses coupling terms will vanish. For the special choice of polarization vectors given in Eq. (24),

$$\begin{aligned} D_{+\times} &= -2\frac{\hat{q}_1\hat{q}_2\hat{q}_3}{\hat{q}_1^2+\hat{q}_2^2}\left(\frac{a_1'}{a_1}-\frac{a_2'}{a_2}\right) , \\ N^2_{+\times} &= \frac{2}{a^2}\frac{\hat{q}_1\hat{q}_2\hat{q}_3}{\hat{q}_1^2+\hat{q}_2^2}\left(\frac{a_1'}{a_1}-\frac{a_2'}{a_2}\right) \\ &\quad \left[-\hat{q}_1^2\left(\frac{a_2'}{a_2}-\frac{a_3'}{a_3}\right)+\hat{q}_2^2\left(\frac{a_3'}{a_3}-\frac{a_1'}{a_1}\right)\right. \\ &\quad \left.+\frac{(\hat{q}_1^2-\hat{q}_2^2)\hat{q}_3^2}{\hat{q}_1^2+\hat{q}_2^2}\left(\frac{a_1'}{a_1}-\frac{a_2'}{a_2}\right)\right]. \end{aligned} \tag{57}$$

We caution the reader that while $D_{+\times}$ is invariant under *constant* rotations of the polarization vectors, $N^2_{+\times}$ is not and our arguments are valid only within this choice $\epsilon_j^{(s)}$. We see that all coupling between the two linear polarizations will vanish if either $a_1 = a_2$ or if $q_j = 0$ along some direction j. Although the first condition seems to break the 1-2-3 exchange symmetry in the labeling of the anisotropy axis, this simply means that we were fortunate enough to choose the "canonical" polarization vectors for the GW which decouples the two polarizations. Any other choice of polarization vectors will produce an $N^2_{+\times} \neq 0$, but can always be made to vanish with a constant rotation of the polarization vectors. $D_{+\times}$, on the other hand, vanishes for all such choices of polarization vectors.

Referring to Eq. (24) we see that when $a_1 = a_2$, $\epsilon^{(2)}_j$ lies in the 1-2 plane and, more importantly, its direction does not change with time; $\nabla_0 \epsilon^{(2)}_j$ is once again parallel to $\epsilon^{(2)}$. Thus any direction we choose for this polarization will always lie in this direction and will not change with time.

Suppose, now that $a_1 \neq a_2$, but $q_3 = 0$, then all the coupling terms between the two polarization states vanish. In this case $\hat{q}_j$ lies in the 1-2 plane but more importantly we may always choose one of the polarization vectors to lie in the three-direction perpendicular to this plane. Once again we see that the direction of this polarization vector does not change and we can once again have a consistant definition of polarization vectors for all time.

We now see explicitly that the coupling terms vanish as long as the direction of one of the polarization vectors does not change with time. This also can be seen in the definition of $N^2_{+\times}$ which depends on the directional derivative of the polarization vectors. Physically, it means that when this happens we can consistantly define the direction of at least one of the polarization vectors at all times. The second polarization can be found by taking the cross product of this polarization vector with the direction of propagation of the GW. Although this can be done in certain special cases, it is not true in general and there will in general be a coupling between polarization states of the GW no matter which gauge one picks.

Finally, we note that, for circularly polarized GW,

$$I = \sum_{\vec{q}} \int a^4 d^4x \left\{ \sum_{s=R,L} \frac{1}{2} \left(\nabla_\rho \bar{h}_s \nabla_\rho h_s - \frac{m^2(\hat{\epsilon})}{a^2} \bar{h}_s h_s \right) + \frac{\beta^2(\hat{\epsilon})}{a^2} \bar{h}_R h_L + \frac{\bar{\beta}^2(\hat{\epsilon})}{a^2} h_R \bar{h}_L \right\}, \tag{58}$$

where $h_R = (h_+ - ih_\times)/\sqrt{2}$, $h_L = (h_+ + ih_\times)/\sqrt{2}$,

$$\beta(\hat{\epsilon}) = e^{2i\phi} \beta(\epsilon) , \tag{59}$$

with

$$\beta(\epsilon) = \tfrac{1}{2} l [\epsilon^{(1)}_\mu \nabla^\mu \epsilon^{(2)}_0 + \epsilon^{(2)}_\mu \nabla^\mu \epsilon^{(1)}_0 + i(\epsilon^{(1)}_\mu \nabla^\mu \epsilon^{(1)}_0 - \epsilon^{(2)}_\mu \nabla^\mu \epsilon^{(2)}_0)] , \tag{60}$$

and ϕ defined through Eq. (50). We have thus eliminated the $D_{+\times}$ with this choice of polarization vectors. Notice that for circularly polarized GW the mass terms for both polarizations are equal,

$$m^2(\hat{\epsilon}) = \tfrac{1}{2}[m^2_+(\hat{\epsilon}) + m^2_-(\hat{\epsilon})] = -\sum_{cyclic} \left(\frac{a'_j}{a_j} - \frac{a'_k}{a_k} \right)^2 \hat{q}^2_l - \sum_{j>k} \left(\frac{a'_j}{a_j} - \frac{a'_k}{a_k} \right)^2 \hat{q}^2_j \hat{q}^2_k + \frac{a^2}{2}[N^2_{++}(\hat{\epsilon}) + N^2_{\times\times}(\hat{\epsilon})] , \tag{61}$$

but we pay for this symmetry through the addition of a complex coupling term between the two polarization states. This coupling term will only vanish when both the real and imaginary parts of $\beta(\epsilon)$ vanish. If, however, they vanish for one choice of polarization vectors, they will vanish for any other choice from Eq. (59). Consequently, we may use the polarization vectors given in Eq. (25), and find that

$$0 = \epsilon^{(1)}_\mu \nabla^\mu \epsilon^{(2)}_0 + \epsilon^{(2)}_\mu \nabla^\mu \epsilon^{(1)}_0 = \frac{\hat{q}_1 \hat{q}_2 \hat{q}_3}{\sqrt{\hat{q}^2_1 + \hat{q}^2_2}} \left(\frac{a'_1}{a_1} - \frac{a'_2}{a_2} \right) , \tag{62}$$

as well as

$$0 = \epsilon^{(1)}_\mu \nabla^\mu \epsilon^{(1)}_0 - \epsilon^{(2)}_\mu \nabla^\mu \epsilon^{(2)}_0 = -\hat{q}^2_1 \left(\frac{a'_2}{a_2} - \frac{a'_3}{a_3} \right) + \hat{q}^2_2 \left(\frac{a'_3}{a_3} - \frac{a'_1}{a_1} \right) + \frac{(\hat{q}^2_1 - \hat{q}^2_2)\hat{q}^2_3}{\hat{q}^2_1 + \hat{q}^2_2} \left(\frac{a'_1}{a_1} - \frac{a'_2}{a_2} \right) , \tag{63}$$

must both vanish separately. This can only happen when the GW propagates along an asymmetry axis and the spacetime is axially symmetric in the plane perpendicular to this axis. For example, the coupling term will vanish if $\hat{q}_3 = -1$ and $a_1 = a_2$. This is a much more stringent condition than for linearly polarized GW's and is because although the two linearly polarized GW's may decouple from one another, they will still have different mass terms and this introduces an additional coupling. Thus we have the peculiar situation that while the linear polarized states may decouple from one another, the circularly polarized states need not. This once again underscores the fact that in a general B-I universe one cannot consistently define a polarization state at all times.

V. SOLUTIONS OF EQUATION OF MOTION

The evolution equation for GW in an arbitrary B-I spacetime cannot be solved in terms of known functions in general, especially when there is a coupling between the two polarization states. Consequently, we shall only consider GW's propagating in either the Kasner or the Zel'dovich spacetimes where solutions *can* be found in certain special cases.

From the action equation (46) the evolution equation for a linearly polarized GW propagating in a Kasner or Zel'dovich spacetime is

$$0=\frac{d^2h_{\vec q+}}{dy^2}+2\frac{1}{a}\frac{da}{dy}\frac{dh_{\vec q+}}{dy}+\left\{\sum_{j=1}^{3}y^{1-3p_j}\widetilde{q}_j^2+\widetilde{m}_+^2\right\}h_{\vec q+}$$
$$+2\widetilde{D}_{+\times}\frac{dh_{\vec q\times}}{dy}-a^2\widetilde{N}_{+\times}^2h_{\vec q\times}\,,$$
$$0=\frac{d^2h_{\vec q\times}}{dy^2}+2\frac{1}{a}\frac{da}{dy}\frac{dh_{\vec q\times}}{dy}+\left\{\sum_{j=1}^{3}y^{1-3p_j}\widetilde{q}_j^2+\widetilde{m}_\times^2\right\}h_{\vec q\times}$$
$$+2\widetilde{D}_{+\times}\frac{dh_{\vec q+}}{dy}-a^2\widetilde{N}_{+\times}^2h_{\vec q+}\,, \qquad (64)$$

where we have not chosen a $D_{+\times}=0$ and we have taken $y_0=1$ for convenience. In the above, the tilde denotes the fact that we have scaled the corresponding quantity by either $4Q_0$ or $4(8\pi\rho_0/3+Q_0)$ depending on whether the spacetime is a Kasner or a Zel'dovich universe to make the resulting quantity dimensionless. For example, $\widetilde{q}_j^2\equiv q_j^2/4/Q_0$ for the Kasner universe while $\widetilde{q}_j^2\equiv q_j^2/4/(8\pi\rho_0/3+Q_0)$ for the Zel'dovich universe. We see that due to the coupling term between the two polarizations, even if the GW is initially plus polarized, a cross polarization will be generated. Unfortunately, even in the case of the Kasner universe Eq. (64) cannot be solved generally. We shall instead have to look at limiting conditions.

Suppose, for convenience, that $p_1>p_2>p_3$. Let us now consider a general $\vec q$ for which each of the components of $\vec q$ do not vanish. If we then take the limit $y\to 0$ we find that $\hat q_1=-1$ while $\hat q_2=\hat q_3=0$. With the choice of polarization vectors given in Eq. (22),

$$\widetilde{m}_+^2=0,\;\;\widetilde{m}_\times^2=-\left(\frac{a_2'}{a_2}-\frac{a_3'}{a_3}\right)^2, \qquad (65)$$

where the prime now denotes the derivative with respect to y. $N_{+\times}=D_{+\times}=0$ and there is no coupling between the two polarizations. Similarly, when $y\to\infty$, $\hat q_1=\hat q_2=0$ while $\hat q_3=-1$. Now

$$\widetilde{m}_+^2=0,\;\;\widetilde{m}_\times^2=-\left(\frac{a_1'}{a_1}-\frac{a_2'}{a_2}\right)^2. \qquad (66)$$

In these two limits the behavior of the GW simplifies dramatically and can be solved in closed form. More importantly, these two limits, which corresponds to the behavior of the GW near the initial singularity as well as the large time behavior of the GW, are the most physically significant.

It is therefore sufficient to consider a GW propagating along one of the symmetry axis; say in the lth direction. Then in Eq. (64) $h_{\vec q+}$ and $h_{\vec q\times}$ decouple into two equations:

$$0=\frac{d^2h_{\vec q+}}{dy^2}+2\frac{1}{a}\frac{da}{dy}\frac{dh_{\vec q+}}{dy}+y^{1-3p_l}\widetilde{q}_l^2h_{\vec q+}\,,$$
$$0=\frac{d^2h_{\vec q\times}}{dy^2}+2\frac{1}{a}\frac{da}{dy}\frac{dh_{\vec q\times}}{dy}$$
$$+\left\{y^{1-3p_l}\widetilde{q}_l^2-\left(\frac{a_j'}{a_j}-\frac{a_k'}{a_k}\right)^2\right\}h_{\vec q\times}\,, \qquad (67)$$

where $l\neq j\neq k$. Notice that in general the behavior of $h_{\vec q+}$ will be different from that of $h_{\vec q\times}$ due to the additional masslike term in its equation of motion. It is only when we are propagating along the asymmetry axis of an axially symmetric B-I universe will the two polarizations have the same behavior. Since the solution for $h_{\vec q+}$ can be obtained from that of $h_{\vec q\times}$ by setting $a_j=a_k$ in the mass term we shall concentrate our attention on the solution of $h_{\vec q\times}$.

It is straightforward to see that the solution to Eq. (67) are Bessel functions J_ν and Y_ν. Using the parametrization of the p_j given in Eq. (12) we find that

$$h_\times^{(1)}(\vec q,x)=e^{iq_1x^1}\left\{A_\times^{(1)}J_{\nu_+}\left(\frac{\widetilde{q}_1y^{1-\frac12\cos\theta-\frac{\sqrt3}{2}\sin\theta}}{1-\frac12\cos\theta-\frac{\sqrt3}{2}\sin\theta}\right)\right.$$
$$\left.+B_\times^{(1)}Y_{\nu_+}\left(\frac{\widetilde{q}_1y^{1-\frac12\cos\theta-\frac{\sqrt3}{2}\sin\theta}}{1-\frac12\cos\theta-\frac{\sqrt3}{2}\sin\theta}\right)\right\},$$
$$h_\times^{(2)}(\vec q,x)=e^{iq_2x^2}\left\{A_\times^{(2)}J_{\nu_-}\left(\frac{\widetilde{q}_2y^{1-\frac12\cos\theta+\frac{\sqrt3}{2}\sin\theta}}{1-\frac12\cos\theta+\frac{\sqrt3}{2}\sin\theta}\right)\right.$$
$$\left.+B_\times^{(2)}Y_{\nu_-}\left(\frac{\widetilde{q}_2y^{1-\frac12\cos\theta+\frac{\sqrt3}{2}\sin\theta}}{1-\frac12\cos\theta+\frac{\sqrt3}{2}\sin\theta}\right)\right\},$$
$$h_\times^{(3)}(\vec q,x)=e^{iq_3x^3}\left\{A_\times^{(3)}J_{\nu_3}\left(\frac{\widetilde{q}_3y^{1+\cos\theta}}{1+\cos\theta}\right)\right.$$
$$\left.+B_\times^{(3)}Y_{\nu_3}\left(\frac{\widetilde{q}_3y^{1+\cos\theta}}{1+\cos\theta}\right)\right\}, \qquad (68)$$

where

$$\nu_\pm\equiv\frac{|3\cos\theta\mp\sqrt3\sin\theta|}{2-\cos\theta\mp\sqrt3\sin\theta}\,,$$
$$\nu_3\equiv\sqrt3\frac{|\sin\theta|}{1+\cos\theta}\,, \qquad (69)$$

while $A_\times^{(j)}$ and $B_\times^{(j)}$ are integration constants. The superscript $h_{\vec q\times}^{(j)}$ signifies that this is a GW which propagates along the jth direction. The $h_{\vec q+}^{(j)}$ solutions are obtained from the above by replacing the Bessel functions of various orders with Bessel functions of order zero but with the same arguments.

Of particular interest are the small and large y limits to Eq. (68). The large y limit determines whether or not the GW (which is a perturbation of the metric) increases or decreases in magnitude. It thereby establishes whether or not the Kasner and Zel'dovich universe are stable under small perturbations. The small y limit, on the other hand, determines the behavior of the GW near the initial singularity. To do so, however, we shall have to find a way of comparing matrices. We therefore define the norm

$$\|T_{\mu\nu}\|\equiv\max|T_{\mu\nu}|\,, \qquad (70)$$

where the max is taken over all components of the tensor. This norm has the advantage of having all the nice properties associated with a norm [8].

Taking first the $y \to \infty$ limit, we find that

$$h_{\times}^{(1)}(\vec{q},x) \approx e^{iq_1x^1}\sqrt{\frac{2-\cos\theta-\sqrt{3}\sin\theta}{\widetilde{q}_1\pi}}y^{-(2-\cos\theta-\sqrt{3}\sin\theta)/4}\left\{A_{\times}^{(1)}\cos\left[\frac{2\widetilde{q}_1y^{(2-\cos\theta-\sqrt{3}\sin\theta)/2}}{2-\cos\theta-\sqrt{3}\sin\theta}-\frac{\pi}{2}\left(\nu_{+}+\frac{1}{2}\right)\right]+B_{\times}^{(1)}\sin\left[\frac{2\widetilde{q}_1y^{(2-\cos\theta-\sqrt{3}\sin\theta)/2}}{2-\cos\theta-\sqrt{3}\sin\theta}-\frac{\pi}{2}\left(\nu_{+}+\frac{1}{2}\right)\right]\right\}, \quad (71)$$

while

$$h_{\times}^{(3)}(\vec{q},x) \approx e^{iq_3x^3}\sqrt{\frac{2(1+\cos\theta)}{\widetilde{q}_1\pi}}y^{-(1+\cos\theta)/2}\left\{A_{\times}^{(3)}\cos\left[\frac{\widetilde{q}_3y^{1+\cos\theta}}{1+\cos\theta}-\frac{\pi}{2}\left(\nu_3+\frac{1}{2}\right)\right]+B_{\times}^{(3)}\sin\left[\frac{2\widetilde{q}_3y^{1+\cos\theta}}{1+\cos\theta}-\frac{\pi}{2}\left(\nu_3+\frac{1}{2}\right)\right]\right\}, \quad (72)$$

with $h_{\vec{q}\times}^{(2)}$ being obtained from $h_{\vec{q}\times}^{(1)}$ by taking $\theta \to -\theta$ and replacing (1) with (2) everywhere. It is then straightforward to show that the amplitude of $\|h_{\mu\nu}^{(s)}\|/\|g_{\mu\nu}\|$ decreases with increasing y. Consequently, we see that the perturbation $h_{\mu\nu}$ of the metric decreases with increasing y.

Taking now the small y limit, we find that

$$h_{\vec{q}\times}^{(1)} \sim y^{-\frac{3}{2}|\cos\theta-\frac{\sqrt{3}}{3}\sin\theta|},$$

$$h_{\vec{q}\times}^{(3)} \sim y^{-\sqrt{3}\sin\theta}, \quad (73)$$

where once again $h_{\vec{q}\times}^{(2)}$ is obtained from $h_{\vec{q}\times}^{(1)}$ by taking $\theta \to -\theta$. Equation (73) holds as long as the degree of the divergence in $h_{\vec{q}\times}^{(j)}$ does not vanish. When it does, that $h_{\vec{q}\times}^{(j)}$ will have a logarithmic divergence. We therefore find that as long as $p_2 \neq p_3$, $\|h_{\mu\nu}^{(\times)}\|/\|g_{\mu\nu}\| \sim 1$ from GW's propagating along the one- or three-direction while $\|h_{\mu\nu}^{(\times)}\|/\|g_{\mu\nu}\| \sim 0$ for a GW propagating along the two-direction as $y \to 0$. Similarly, $\|h_{\mu\nu}^{(+)}\|/\|g_{\mu\nu}\| \sim \ln y$ for a GW propagating along the one- or two-direction and $\|h_{\mu\nu}^{(+)}\|/\|g_{\mu\nu}\| \to 0$ for a GW propagating along the three-direction. We thus see that along the one- and two-directions the perturbation of the metric becomes unboundly large near the singularity.

For the special case of $p_2 = p_3$, we find that $\|h_{\mu\nu}^{(\times)}\|/\|g_{\mu\nu}\| \sim 1$ for GW's propagating along the two- or three-direction while $\|h_{\mu\nu}^{(\times)}\|/\|g_{\mu\nu}\| \sim \ln y$ for a GW propagating along the one-direction. Similarly, we find that $\|h_{\mu\nu}^{(+)}\|/\|g_{\mu\nu}\| \sim \ln y$ for a GW propagating along the one- or two-direction, while $\|h_{\mu\nu}^{(\times)}\|/\|g_{\mu\nu}\| \to 0$ for a GW propagating along the three-direction. Thus, in this special case the perturbation of the metric becomes unboundly large along all three directions. The case considered by Hu, which corresponds to $\theta = 0$, belongs to this category and the results we have obtained agree with his.

The case where $\theta = \pi/3$ is quite special since this corresponds to $p_2 = p_3 = 0$ while $p_1 = 1$. Supposedly, this case also falls within the above analysis and we would still obtain the same small y behavior. When $\rho_0 = 0$ this is the Rindler spacetime, however, which is known to be equivalent to Minkowski space and supposedly the propagation of GW's in a Minkowski spacetime does not have a singularity. To address this problem, we now perform a more detailed analysis of the solution in this special case.

Going back to the equation of motion, we see that, when $\theta = \pi/3$,

$$\begin{aligned}
h_{\mu\nu}^{(+)}(q_1,x) &\equiv \varpi_{\mu\nu}^{(+)}(q_1)h_{+}^{(1)} \\
&= \left(e_\mu^3e_\nu^3-e_\nu^2e_\mu^2\right)\left(A_{+}^{(1)}e^{iq_1(x^1+t_0\ln t/t_0)}+B_{+}^{(1)}e^{iq_1(x^1-t_0\ln t/t_0)}\right), \\
h_{\mu\nu}^{(+)}(q_2,x) &\equiv \varpi_{\mu\nu}^{(+)}(q_2)h_{+}^{(2)} \\
&= \left(e_\mu^3e_\nu^3-e_\nu^1e_\mu^1\right)\left[A_{+}^{(2)}J_0(q_2t)+B_{+}^{(2)}Y_0(q_2t)\right]e^{iq_2x^2}, \\
h_{\mu\nu}^{(+)}(q_3,x) &\equiv \varpi_{\mu\nu}^{(+)}(q_3)h_{+}^{(3)} \\
&= \left(e_\mu^2e_\nu^2-e_\nu^1e_\mu^1\right)\left[A_{+}^{(3)}J_0(q_3t)+B_{+}^{(3)}Y_0(q_3t)\right]e^{iq_3x^3}, \quad (74)
\end{aligned}$$

where we have put back in the polarization tensors explicitly, $t_0 = 2\eta_0/3$ and we have used $y = (t/t_0)^{2/3}$ for the Rindler spacetime. Similarly,

$$\begin{aligned} h_{\mu\nu}^{(\times)}(q_1,x) &\equiv \varpi_{\mu\nu}^{(\times)}(q_1)h_\times^{(1)} \\ &= \left(e_\mu^2 e_\nu^3 + e_\nu^2 e_\mu^3\right)\left(A_\times^{(1)} e^{iq_1(x^1+t_0\ln t/t_0)} + B_\times^{(1)} e^{iq_1(x^1-t_0\ln t/t_0)}\right) , \\ h_{\mu\nu}^{(\times)}(q_2,x) &\equiv \varpi_{\mu\nu}^{(\times)}(q_2)h_\times^{(2)} \\ &= -\left(e_\mu^1 e_\nu^3 + e_\nu^1 e_\mu^3\right)\left[A_\times^{(2)} J_1(q_2 t) + B_\times^{(2)} Y_1(q_2 t)\right] e^{iq_2 x^2} , \\ h_{\mu\nu}^{(\times)}(q_3,x) &\equiv \varpi_{\mu\nu}^{(\times)}(q_3)h_\times^{(3)} \\ &= -\left(e_\mu^1 e_\nu^2 + e_\nu^1 e_\mu^2\right)\left[A_\times^{(3)} J_1(q_3 t) + B_\times^{(3)} Y_1(q_3 t)\right] e^{iq_3 x^3} , \end{aligned} \tag{75}$$

and we can see explicitly the singularity in the solutions when $t \to 0$.

Let us now do a coordinate transformation into Minkowski spacetime. When $\theta = \pi/3$, the metric is

$$ds^2 = (dt)^2 - \left(\frac{t}{t_0}\right)^2 (dx^1)^2 - (dx^2)^2 - (dx^3)^2 . \tag{76}$$

To map this into the Minkowski spacetime, we make the coordinate transformation

$$T = t\cosh(x^1/t_0) , \quad X^1 = t\sinh(x^1/t_0) ,$$
$$X^2 = x^2, \quad X^3 = x^3 . \tag{77}$$

Notice, however, that because $h_{\mu\nu}$ is a tensor, it transforms as

$$_M h_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu} h_{\alpha\beta} , \tag{78}$$

where $_M h_{\mu\nu}$ is the transformed GW. Note, however, that $\partial x^\alpha/\partial X^\mu$ is block diagonal and mixes the 0-1 components of the GW while leaving the 2-3 components alone. For GW propagation in the two- or three-direction, this means that the transformed $_M h_{0\nu} \neq 0$, although $q_M^j h_{ij} = 0$ still and it will still be traceless. Consequently, for GW's propagating in these two directions, the transformed GW will not satisfy the usual synchronous gauge and is not what one usually calls a GW in a Minkowski spacetime.

For GW's propagating along the one-axis, however, the situation is a little different. Notice first that the polarization matrices will not be changed by the transformation and that they are, in fact, precisely the polarization matrices for a GW propagating in this direction in a Minkowski spacetime. Next, using the coordinate transeformation Eq. (77), we see that

$$h_{\mu\nu}^+(q_1,x) = \varpi_{\mu\nu}^+(q_1)\left\{A_+^{(1)}\left(\frac{T+X^1}{t_0}\right)^{iq_1 t_0} + B_+^{(1)}\left(\frac{T-X^1}{t_0}\right)^{-iq_1 t_0}\right\} , \tag{79}$$

and the equation for the $\times$ mode has the same form. Notice that Eq. (79) is a solution of the wave equation in Minkowski spacetime, as a GW in a Minkowski spacetime should. In this sense it is a GW in the Minkowski spacetime, but it is not plane wave and satisfies different boundary conditions. Consequently, we see that after a coordinate transformation into a Minkowski spacetime, the GW in the Rindler spacetime will not be transformed into what we usually refer to as a GW in the Minkowski spacetime.

VI. CONCLUDING REMARKS

To conclude, we have begun the analysis of the propagation of GW's in B-I universes using the method developed by Ford and Parker. We find the behavior of a GW in a B-I universe to be very much different than a GW in a RW universe. The two polarization states are *not* equivalent to two minimally coupled, massless scalar fields. Rather, *each* polarization state gains what is effectively a time-dependent mass term due to the tensorial nature of the GW. Namely, the GW is a spin-2 particle instead of a scalar particle and is consequently much more sensitive to any anisotropy in the universe. We have also found the two polarization states to be coupled to one another. This coupling term depends explicitly on the gauge one picks for the GW and varies as one varies the choice of gauge. No matter what gauge choice one makes, however, a coupling between the two polarization states will always be present and the coupling is not a gauge artifact.

The reason for this is fairly straightforward. We can consider the propagation of a GW in a B-I universe as the propagation of a wave through an anisotropic medium. Although the polarization and direction of propagation of this GW is initially arbitrary, as it propagates through the anisotropic medium the medium itself will tend to change the direction of propagation, and thus the direction of polarization, of the wave. This can be seen explicitly in a Kasner universe where the anisotropic medium gradually forces the wave to propagate only along one of the asymmetry axis. The anisotropy of the medium is constantly changing with time, however, and thus the polarization state that the medium wishes the GW to adopt also constantly changes. This is done through the coupling term between the polarization states and is caused by changing of the polarization vectors with time.

We would therefore expect a coupling between the two modes of the GW to be present in any anisotropic expanding universe. Indeed, such a coupling has also been found by Ezawa and Soda [9] who analyzed the effects of the topology on the propagation of GW's. They considered GW's propagating on plain symmetric spacetimes with two of the spacetime directions compactified into a torus and also found a coupling between the plus and cross modes of the GW.

Hu considered a B-I spacetime for which $p_1 = p_2 = 2/3$ while $p_3 = -1/3$ (we have changed slightly the notation used in [3]). Moreover, he only considered a GW propagating along the cylindrical axis. We have seen, however, that the behavior of GW's in this spacetime is quite special and atypical. In fact, it is only for a GW propagating in this manner in this spacetime that the two linear as well as circular polarizations decouple from one another. Propagation along any other direction in the spacetime will introduce a coupling between the two polarization states.

Miedema and van Leeuwen have also analyzed the propagation of GW's in B-I universes within a general analysis of perturbations of a B-I universe. Looking at the GW component of the perturbations, they found that while a GW can be defined as being transversal at any given time, as the wave evolves in time nontransversal components of the wave will begin to appear. They thereby conclude that GW's in B-I universes are in general nontransversal. The longitudinal components of the wave, however, were considered to have no physical meaning since they can be gauged away at any given time. We did not encounter any such subtleties in our analysis. We have instead found that *with the correct gauge choice* the GW will always be transverse.

The difference between our two results can be found in the gauge conditions that we have taken. Miedema and van Leeuwen have chosen to work in the synchronous and traceless gauges (see Eqs. (38) and (108) of [4]). They then attempted to enforce the usual transverse condition on the GW (Eq. (114) of [4]). As we have shown, these three gauge conditions are incompatible and one of them must be modified. By imposing the synchronous and traceless conditions on the evolution equations for the GW (Eq. (135) of [4]), they cannot then choose the usual transverse condition. This inconsistency may be the root cause for the generation of the longitudinal components as the wave propagates that they observed.

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APPENDIX

In this appendix we shall repeat the analysis found in Sec. IV using the usual transverse-synchronous gauge in order to show that coupling terms between the polarization states are still present in this gauge. Indeed, we shall see that the problem gets worse, not better, with this gauge choice.

Let us denote the polarization tensor for the GW in this gauge by $\widehat{\varpi}^{(s)}_{jk}$. Then, so as to keep a basis with which we can compare $\widehat{\varpi}^{(s)}_{jk}$ with $\varpi^{(s)}_{jk}$, we shall not change the definition of the polarization vectors. Rather, we shall once again express $\widehat{\varpi}^{(s)}_{jk}$ in terms of $\epsilon^{(s)}_j$ by taking

$$\widehat{\varpi}^{(s)}_{jk} \equiv \alpha^{(s)}\varpi^{(+)}_{jk} + \beta^{(s)}\tau_{jk} + \gamma^{(s)}\varpi^{(\times)}_{jk} , \tag{A1}$$

where

$$\tau_{jk} = \epsilon^{(1)}_j\epsilon^{(1)}_k + \epsilon^{(2)}_j\epsilon^{(2)}_k , \tag{A2}$$

and $\alpha^{(s)}$, $\beta^{(s)}$, $\gamma^{(s)}$ are coefficients which are to be determined. Clearly $\widehat{\varpi}^{(s)}_{jk}$ still satisfies the spatial components of the transverse condition. We still require that the two polarization tensors for the two polarization states of the GW be orthonormal to one another and this gives the constraint

$$1 = (\alpha^{(s)})^2 + (\beta^{(s)})^2 + (\gamma^{(s)})^2 . \tag{A3}$$

The 0-component of the transversality condition in Eq. (17) can now be used with Eq. (A3) to determine $\widehat{\varpi}^{(s)}_{jk}$ uniquely:

$$\begin{aligned}\widehat{\varpi}^{(+)}_{jk} &= \cos\varphi\,\varpi^{(+)}_{jk} - \sin\varphi\,\tau_{jk} ,\\ \widehat{\varpi}^{(\times)}_{jk} &= -\sin\varphi\cos\vartheta\,\varpi^{(+)}_{jk} - \cos\varphi\cos\vartheta\,\tau_{jk} + \sin\vartheta\,\varpi^{(\times)}_{jk} ,\end{aligned} \tag{A4}$$

where

$$\begin{aligned}\sum_j \frac{a'_j}{a_j}\tau^j_j &\equiv \varrho\cos\varphi\sin\vartheta ,\\ \sum_j \frac{a'_j}{a_j}\varpi^{(+)j}_{\ \ j} &\equiv \varrho\sin\varphi\sin\vartheta ,\\ \sum_j \frac{a'_j}{a_j}\varpi^{(\times)j}_{\ \ j} &\equiv \varrho\cos\vartheta ,\end{aligned} \tag{A5}$$

and

$$\varrho^2 = \left(\sum_j \frac{a'_j}{a_j}\varpi^{(+)j}_{\ \ j}\right)^2 + \left(\sum_j \frac{a'_j}{a_j}\tau^j_j\right)^2 + \left(\sum_j \frac{a'_j}{a_j}\varpi^{(\times)j}_{\ \ j}\right)^2 . \tag{A6}$$

For a RW universe, $\vartheta = \pi/2$ while $\varphi = 0$.

At this point, we make a few observations. First, the angles ϑ and φ are determined precisely by linear combinations of the terms found in Eqs. (54) and (55). Second, these terms, which measure the infinitesimal change in the orthonormality conditions for the polarization vectors, did not disappear when we changed to this gauge choice. This underscores the fact that they are physically relevant. Third, the trace of the polarization tensors no longer vanishes,

$$\sum_j \widehat{\varpi}^{(+)j}_{\ \ j} = 2\sin\varphi , \quad \sum_j \widehat{\varpi}^{(\times)j}_{\ \ j} = 2\cos\varphi\cos\vartheta , \tag{A7}$$

which will introduce additional terms to the GW Lagrangian.

In this gauge, the Lagrangian for the GW now becomes

$$\widehat{I} = \frac{1}{4}\int \sqrt{-g}d^4x \bigg(\nabla_\mu h^\beta_\alpha \nabla^\mu h^\alpha_\beta + 8\pi(\rho - p) h^\beta_\alpha h^\alpha_\beta -2R^\mu_\beta h^\alpha{}_\mu h^\beta_\alpha - 2R^{\mu\beta}{}_{\alpha\nu} h^\nu_\mu h^\alpha_\beta -\nabla_\rho h \nabla^\rho h - hR^{\mu\nu} h_{\mu\nu} \bigg), \quad \text{(A8)}$$

where $h = h^\mu_\mu$ is the trace of $h_{\mu\nu}$ which does not vanish in this gauge and we are denoting the different choice of gauge with a caret. Once again we can divide this action up into three terms, the only difference being that the form of the gauge dependent piece is now different:

$$\widehat{I}_g \equiv \frac{1}{4}\int \sqrt{-g}d^4x \left(-\nabla_\rho h \nabla^\rho h - hR^{\mu\nu} h_{\mu\nu}\right) . \quad \text{(A9)}$$

Proceeding in exactly the same way as before, we find that

$$\widehat{I}_K = \frac{1}{2}\sum_{\vec{q}} \int a^4 d^4x \bigg(\sum_s \nabla_\rho \bar{h}_s \nabla^\rho h_s - \sum_{ss'} \widehat{M}_{ss'} h_s \bar{h}_{s'} -2\sum_{ss'} \widehat{D}_{\rho s s'} \bar{h}_{s'} \nabla^\rho h_s \bigg) , \quad \text{(A10)}$$

where $\widehat{M}_{ss'}$ and $\widehat{D}_{\rho ss'}$ are defined in exactly the same way as $M_{ss'}$ and $D_{\rho ss'}$ but with $\widehat{\varpi}^{(s)}_{\mu\nu}$ replacing $\varpi^{(s)}_{\mu\nu}$ in Eq. (42).

It is straightforward to show that

$$\begin{aligned}
\tau^\mu_\nu \nabla_\rho \varpi^{(+)}{}^\nu_\mu &= -\nabla_\rho \tau^\mu_\nu \varpi^{(+)}{}^\nu_\mu = 0 , \\
\nabla_\rho \tau^\mu_\nu \nabla^\rho \tau^\nu_\mu &= -2M^2_{++} - 2D^2_{+\times} , \\
\nabla_\rho \varpi^{(\times)}{}^\mu_\nu \nabla^\rho \tau^\nu_\mu &\equiv -2G = -4\nabla_\mu \epsilon^{(1)}_\nu \nabla^\mu \epsilon^{(2)\nu} . \\
\nabla_\rho \varpi^{(+)}{}^\mu_\nu \nabla^\rho \tau^\nu_\mu &\equiv -2H \\
&= -2\left(\nabla_\mu \epsilon^{(1)}_\nu \nabla^\mu \epsilon^{(1)\nu} - \nabla_\mu \epsilon^{(2)}_\nu \nabla^\mu \epsilon^{(2)\nu}\right) .
\end{aligned} \quad \text{(A11)}$$

Then

$$\begin{aligned}
\widehat{M}^2_{++} &= M^2_{++} - \nabla_\rho \varphi \nabla^\rho \varphi - \sin 2\varphi H + \sin^2 \varphi D^2_{+\times} , \\
\widehat{M}^2_{\times\times} &= M^2_{++} - \nabla_\rho \vartheta \nabla^\rho \vartheta - \cos^2 \vartheta \nabla_\rho \varphi \nabla^\rho \varphi \\
&\quad + \cos^2 \vartheta \sin 2\varphi H - \sin 2\vartheta \cos \varphi G \\
&\quad + \cos^2 \vartheta \cos^2 \varphi D^2_{+\times} \\
&\quad +(2\sin\varphi \nabla_\rho \vartheta - \cos\varphi \sin 2\vartheta \nabla_\rho \varphi) D^\rho_{+\times} , \\
\widehat{M}^2_{+\times} &= \sin\vartheta \nabla_\rho \vartheta \nabla^\rho \varphi - \cos 2\varphi \cos\vartheta H - \sin\vartheta \sin\varphi G \\
&\quad - (\sin\varphi \sin\vartheta \nabla^\rho \varphi + \cos\varphi \cos\vartheta \nabla^\rho \vartheta)\, D_{\rho+\times} \\
&\quad + \tfrac{1}{2} \sin 2\varphi \cos\vartheta D^2_{+\times} .
\end{aligned} \quad \text{(A12)}$$

While $\widehat{D}_{\rho++} = \widehat{D}_{\rho\times\times} = 0$ still,

$$\widehat{D}_{\rho+\times} = \sin\vartheta \cos\varphi D_{\rho+\times} + \cos\vartheta \nabla_\rho \varphi . \quad \text{(A13)}$$

In a similar fashion, we find that

$$\widehat{I}_R = -\frac{1}{2}\sum_{\vec{q}} \sum_{ss'} \int a^4 d^4x \widehat{T}_{ss'} \bar{h}_{s'} h_s , \quad \text{(A14)}$$

with

$$\begin{aligned}
\widehat{T}_{++} &= \cos 2\varphi\, T_{++} , \\
\widehat{T}_{\times\times} &= \left(\sin^2 \vartheta - \cos^2 \vartheta \cos 2\varphi\right) T_{++} , \\
\widehat{T}_{+\times} &= -\cos\vartheta \sin 2\varphi\, T_{++} ,
\end{aligned} \quad \text{(A15)}$$

where we have used

$$\sum_{ijkm} R^{ij}{}_{km} \varpi^{(+)}{}^m_i \tau^k_j = 0 , \quad \sum_{ijkm} R^{ij}{}_{km} \varpi^{(\times)}{}^m_i \tau^k_j = 0 ,$$

$$\sum_{ijkm} R^{ij}{}_{km} \tau^m_i \tau^k_j = -T_{++} . \quad \text{(A16)}$$

Finally, we have the gauge term

$$\widehat{I}_g = -\frac{1}{2}\sum_{\vec{q}ss'} \int a^4 d^4x \bigg\{ \widehat{\varpi}^{(s)} \widehat{\varpi}^{(s')} \nabla_\rho h_s \nabla^\rho \bar{h}_{s'} + 2\widehat{\varpi}^{(s)} \nabla_\rho \widehat{\varpi}^{(s')} \bar{h}_{s'} \nabla^\rho h_s + \left[\nabla_\rho \widehat{\varpi}^{(s)} \nabla^\rho \widehat{\varpi}^{(s')} + \frac{1}{a^2} \widehat{\varpi}^{(s)} \sum_j \frac{d}{d\eta}\left(\frac{a'_j}{a_j}\right) \widehat{\varpi}^{(s')}{}^j_j \right] h_s \bar{h}_{s'} \bigg\} , \quad \text{(A17)}$$

where $\widehat{\varpi}^{(s)} \equiv \widehat{\varpi}^{(s)}{}^\mu_\mu/\sqrt{2}$ and is given in Eq. (A7). Combining the three terms together we obtain

$$\widehat{I} = \frac{1}{2}\sum_{\vec{q}ss'} \int a^4 d^4x \bigg\{ \left(\delta_{ss'} - \widehat{\varpi}^{(s)} \widehat{\varpi}^{(s')}\right) \nabla_\rho \bar{h}_{s'} \nabla^\rho h_s -2\widehat{\varpi}^{(s)} \nabla_\rho \widehat{\varpi}^{(s')} \bar{h}_{s'} \nabla^\rho h_s - 2\widehat{D}_{\rho ss'} \bar{h}_{s'} \nabla^\rho h_s - \bigg[\widehat{M}_{ss'} + \widehat{T}_{ss'} + \nabla_\rho \widehat{\varpi}^{(s)} \nabla^\rho \widehat{\varpi}^{(s')} + \frac{1}{a^2} \widehat{\varpi}^{(s)} \sum_j \frac{d}{d\eta}\left(\frac{a'_j}{a_j}\right) \widehat{\varpi}^{(s')}{}^j_j \bigg] h_s \bar{h}_{s'} \bigg\}. \quad \text{(A18)}$$

We can now see explicitly that instead of eliminating the coupling term, this choice of gauge merely makes things worse. In fact, the kinetic term for the two modes are now quite different than what we would expect and additional coupling terms now appear. It is, however, important to note that these additional terms have the same origin as those obtained by using the previous gauge. Namely, they all come from linear combinations of Eqs. (54)–(56) and arise from the fact that the directions of the polarization vectors are continually changing with time.

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