Instanton size distribution: Repulsion or an infrared fixed point?

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(Received 3 April 1995)

We discuss available information about the instanton size distribution $d(\rho)$, which comes from lattice simulations and the interacting instanton liquid model. First, we demonstrate that they agree quite well. Second, we also show that an alternative idea, based on the infrared fixed point, can also reproduce the shape of $d(\rho)$. Third, we discuss how one can use lattice methods to clarify the nature of the suppression of the large-size instantons.

PACS number(s): 12.38.Lg, 11.15.Pg, 12.38.Gc

Instanton physics [1–3] was discovered nearly 20 years ago, but only now is their *quantitative* role in QCD being clarified. In particular, a large set ($\sim$ 40) of point-to-point correlation functions was calculated in the simplest "instanton liquid" model [4], both numerically [5] and (with certain approximations) analytically [6]. It was found that not only masses and coupling constants of many hadrons (including $\pi, \sigma, \rho, A_1, N, \Delta$, and others) are reproduced, but the whole correlation functions also agree with those extracted from phenomenology [7] and from lattice simulations [8].

Recently, glueballs were added to the list [9], with the conclusion that the instanton liquid model predictions (such as small mass and especially small *size* of the scalar glueball) are in *quantitative* agreement with the lattice data. This agreement is not accidental, as can be seen from the fact that the "instanton liquid" itself was "distilled" from lattice configurations by the "cooling" method [10–12]. Two parameters of the model (the average instanton size $\bar{\rho}$ and instanton spacing R) were found to be inside 10%, the same as those suggested in 1982 by the author [4]: namely, $\bar{\rho} \approx 1/3$ fm, $R \approx 1$ fm. Furthermore, in spite of absent perturbative effects and (nearly complete) loss of confinement, this operation does not change much the correlators and hadrons [11]. So the instanton-induced 't Hooft interaction [2] is truly the dominant part of the interquark forces.

With the mean instanton size being approximately measured, the obvious next step is the *shape* of the instanton (plus anti-instanton) size distribution function, $d(\rho)$. With all those advances at the phenomenological front, we are still lacking answers to many major questions related to it. One of them, to be discussed below, is: Why are *large-size* instantons absent in the QCD vacuum? Alternative explanations are (i) they are suppressed by *repulsive* interaction between instantons, (ii) the *higher-order effects* lead to charge renormalization so that their actions are large, and (iii) *confinement* effects screen their gluoelectric fields. In this work we will go through this list, and will show that the first two still remain strongly competitive.

The idea (i) is implemented in the "interacting instanton liquid model" (IILM), a statistical model [13,14] with the partition function (for $N/2$ instantons and $N/2$ anti-instantons)

$$Z = \int \prod_{I=1}^{N} d\Omega_I d_0(\rho_I) \exp\left[- \sum_{I<J} S_{IJ}^{\text{int}} \right] , \tag{1}$$

where Ω_I denote the orientation, position, and the size of pseudoparticle I, $d_0(\rho)$ here corresponds to *noninteracting* instantons, with the gluonic interactions S_{IJ}^{int}. In its previous applications the main effects are related with the quark-related interaction, but those are not considered in this work.

Let us briefly remind the history of the S_{IJ}^{int}. The simplest "hard core" model was introduced in [15]. Then it was shown that for the simplest "sum ansatz" [16] such a repulsion actually exists. Some defects of this ansatz were cured in [13], where the improved trial function known as "ratio ansatz" was proposed: this reproduces phenomenological parameters of the "instanton liquid." However, numerical solutions [17] of the "streamline" equation [18] have shown that the instanton–anti-instanton valley leads continuously to zero fields; so the original hopes to get repulsion at the purely classical level [16] are not satisfied. Presumably quantum effects (especially subtraction of perturbative contributions, relevant for close instanton–anti-instanton pairs with a strongly attractive interaction) will generate the effective repulsion (see also [19]). Meanwhile, in IILM-based recent studies [9,20], the streamline interaction is supplemented by a repulsive core, with the radius fitted to the value of the gluon and quark condensates.

We have simulated numerically the ensemble of interacting instantons (see details in [20]) for the pure gauge theories. The resulting $d(\rho)$ is shown in Fig. 1 (open points), for the SU(2) (a) and SU(3) (b) cases. Both have sharp maxima, supplemented by a tail toward large ρ, to be discussed below. Recently, the first lattice measurements [12] were made for the SU(2) case: those results (for two lattices, closed points) are compared to each other (and IILM) in Fig. 1(a). The comparison leads to our first conclusion: $d(\rho)$ obtained from both approaches are very similar. (A deviation at small ρ was expected: instantons with the radius $\rho \sim a$ "fall through the lattice" during "cooling.")

[In passing, let us mention here one practical aspect of the studies of $d(\rho)$: its measurements at *small* $\rho < 0.2$ fm is potentially the source of by far the most accurate measurements of Λ_{QCD}. As it is well known, here

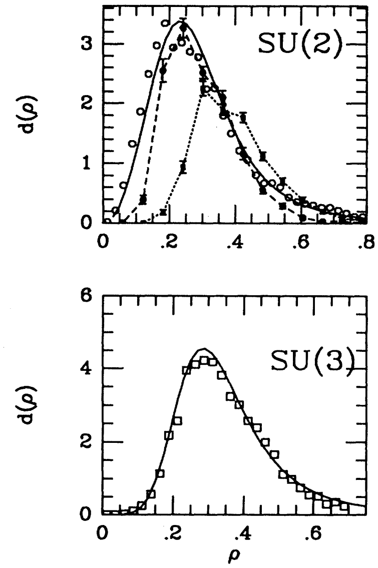

FIG. 1. The instanton size distribution in d=4 SU(2) (a) and SU(3) (b) gauge theories. The open points correspond to the "interacting instanton liquid," and the closed ones are lattice results by Michael and Spencer, [11] $16^4, 4/g^2 = 2.4$ for squares and $24^4, 4/g^2 = 2.5$ for dots (the dotted and dashed lines just guide the eye). Solid curves correspond to the parametrization discussed in the text. Units are in "femtometers," defined for lattice data by the scalar glueball mass defined as $m_{0^+} = 1.7$ "GeV"; and by $1/\Lambda_{\rm PV}$ for the instanton model.

$d(\rho) \sim \Lambda_{\rm QCD}^{b_0}$: thus even relatively poor accuracy in its measurements leads to $\Lambda_{\rm QCD}$ value with the accuracy better than measured today by other methods.]

However, it is well known that good agreement does not guarantee that the theory is right. To challange the "instanton liquid" model let us now consider an alternative idea (ii). If the action is large enough $S_{\rm eff} >> 1$, one can use the generic semiclassical expression

$$d(\rho) = \frac{2C_I}{\rho^{(d+1)}} S^{[N_{\rm ZM}/2]}(\rho) \exp[-S(\rho)] \ , \qquad (2)$$

where 2 accounts for instantons and anti-instantons, d is the space-time dimension, and $N_{\rm ZM}$ is the number of zero modes ($4N_c$ for Yang-Mills fields). The value of the (renormalization-dependent) constant was determined in the classical work [2] $C_I = 0.466/[(N_c-1)!(N_c-2)!]$. (In order to simplify our discussion of the large-N_c limit below, we have absorbed the extra N_c-dependent factor in C_I into a new lambda parameter $\Lambda_{\rm inst} = 0.632\Lambda_{\rm Pauli\text{-}Villars} = 0.657\Lambda_{\overline{\rm MS}}$, where $\overline{\rm MS}$ denotes the modified minimal subtraction scheme.)

Furthermore, various arguments in favor of existence of an *infrared fixed point* (or "freezing" of the coupling constant) have been many times made before, based on a variety of phenomenological observations [see, e.g., the recent paper [21], which deals with higher-loop corrections to $\sigma(e^+e^- \to \text{hadrons})$]. If so, at large ρ the action becomes ρ independent, and therefore $d(\rho) \sim \rho^{-(d+1)}$, where d is just the space-time dimension.

Surprisingly, the available data are actually in nice agreement with predictions following from this simple idea. In terms of statistical accuracy and the widest range studied, the best lattice measurements are those for d=2 O(3) σ model [22]. For *large* sizes the result is $d(\rho) \sim \rho^{-3}$ [12].

Furthermore, for the d=4 SU(2) gauge theory [see Fig. 1(a)] we have fitted the lattice-based distribution by semiclassical expression (2) with the following simple parametrization for the action:

$$\frac{8\pi^2}{g^2(\rho)} = b_0 L + \frac{b_1}{b_0} \ln L \ , \qquad (3)$$

a standard two-loop expression for the charge (where $b_0 = \frac{11}{3}N_c, b_1 = \frac{17}{3}N_c^2$) but with a "regularized" log

$$L = \frac{1}{p} \ln\left[\left(\frac{1}{\rho\Lambda}\right)^p + C^p\right] \ . \qquad (4)$$

It has two new parameters C and p, describing where and how rapidly the "freezing" occurs. The solid line in Fig. 1(a) shows such a fit, with $\Lambda_{\rm inst} = 0.66$ fm^{-1}, p=3.5, C=4.8. For the SU(3) gauge theory the data for $d(\rho)$ itself are still missing, but the total instanton density and *average* size were measured in [11]: $\int d\rho d(\rho) \approx 1.3$ fm^{-4}, $\bar\rho = 0.35$ fm. One may fix two parameters to reproduce those two values: the corresponding (solid) curve is shown in Fig. 1(b). (Parameters in this case are $\Lambda_{\rm inst} = 0.70$ fm^{-1}, p=3.5, C=5.0.) Although it is *not* a fit to open points (the interacting liquid) the curve agrees with the points. It shows that both the instanton repulsion (i) and the frozen coupling constant (ii) produce very similar distributions.

If one plots the action for those fits [Fig. 2(a)] one can see that it indeed means rather rapid "freezing" of the coupling constants. We do not know why such freezing

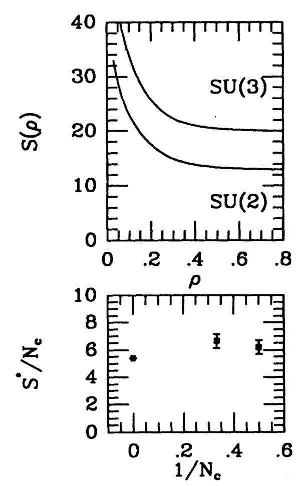

FIG. 2. The upper part shows the instanton action $S(\rho)$ versus ρ (in "fm") according to the parametrization used in Fig. 1. The lower part compares the fitted values of the "fixed point" actions S^* (divided by the number of colors $N_c = 2, 3$) with the critical point in the large N_c limit (star).

may occur: but it is remarkable that it should happen where the action itself is large, $S \sim 11, 20$ for $N_c = 2, 3$, so one should trust the semiclassical theory.

Let us now comment on alternative (iii) mentioned above, namely, that the cutoff of the large size instantons is related with *confinement*: an argument *against* it can be given. Recent studies of the instanton *sizes* at nonzero temperatures [23] have shown that up to the deconfinement transition the average instanton size does *not* change, and only above it the size slowly decreases. In contrast to that, the string tension decreases below T_c, and vanishes $K(T) \to 0$ at the transition point.

In connection with the "frozen coupling" idea, the author looked at the so-called nonperturbative β functions used in the lattice studies. Recall that the bare coupling $g(a)$ is fixed at the input of simulations, then one calculates some observables (hadronic masses, etc.) which allows to fix the *physical* magnitude of the lattice scale a. Reversing the function, one gets the charge $g(a)$. The procedure is subject for (a) universality test (its independence on the particular observable used) and (b) comparison with the expected perturbative behavior ("asymptotic scaling"). The nontrivial fact found (see, e.g., [24,25]) is that (a) extends beyond (b), so that nonperturbative β function makes sense in some window.

A sample of lattice data for the SU(3) lattice-gauge theory, without quarks [24] (open points) and with two massless quark flavors (closed points) [25] is shown in Fig. 3. Those are usually presented in form of the *derivative* of $g(a)$, the famous Gell-Mann–Low β function. In order to compare various theories, it is convenient to normalize the derivative to its asymptotic ($g \to 0$) perturbative value: thus we plot the ratio

$$R_\beta(g) = \left(\frac{d(1/g^2)}{d\ln(a)}\right) \Big/ \left(\frac{d(1/g^2)}{d\ln(a)}\right)\Bigg|_{g\to 0} \tag{5}$$

which tends to one at the right-hand side of Fig. 3. As one penetrates into the nonperturbative region the data

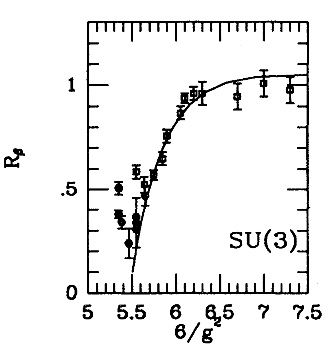

FIG. 3. The nonpertubative β function for the SU(3) lattice theory with Wilson action. Open points are taken from Gupta review (which is an analysis of the data set from Lepage and Mackenzie), the solid points are from Blum *et al.* The solid line is a fit discussed in the text.

display a deep *drop* of R_β at $6/g^2 \approx 6$, with subsequent turn upward. Both are very rapid: the latter turn is known in literature as a transition to a "strong coupling" regime. This drop may be an indication for an *infrared fixed point nearby.* (The turn upward is very similar to those commonly observed when a renormalization trajectory is going toward the fixed point, and then misses it closely.)

Another way to investigate charge renormalization (with the *standard* Wilson action) is related with the renormalization of *background fields.* This method is quite popular in the perturbative context [26] but (to our knowledge) has not yet been used on the lattice. It is clear which field is the best to try. First, to avoid complications with external current, one should better take a "self-supporting" classical field, such as $D_\mu G^c_{\mu\nu} = 0$. Second, topology adds additional stability: thus an *instanton* is the most natural classical background. Third, its renormalized action depends on one parameter ρ, $S_{\rm eff} = 8\pi^2/g^2(\rho)$, and thus this expression can be used as a definition of the nonperturbative charge renormalization.

One can put a classical instanton on the lattice and then "heat it up," performing standard updates: such studies have been made by Alles *et al.* [27], establishing renormalization of the topological susceptibility. However, for the proposed goal it is not enough to keep a topological charge (as done in Ref. [27]): one has to preserve the chosen value of ρ. This can be achieved in two ways: (i) while updating a link, one may keep the quantum field a_μ *orthogonal* to the *dilatational* zero mode $\delta A^c_\mu/\delta\rho$ [28]; or (ii) one may use a modified lattice action containing the two-plaquette operators with parameters tuned to make any given size the classical minimum of the lattice action. Either way, the main problem is to get high statistics measurements of the effective action, after subtraction of the usual "average plaquette" is made.

The last topic addressed in this work is the fate of instantons in the large N_c limit. Witten [29] has formulated the following dilemma: either (i) instantons are not dynamically relevant, while all observables have perturbative-type $1/N_c$ expansion, or (ii) instantons play a role in the real world, but are exponentially suppressed at large N_c. [Back in 1979, Witten argued in favor of (i), using analogies with some d=2 models, but today there is no doubt about the significance of the instantons at $N_c = 3$. However, it is hard to accept (ii) also, because then all the $1/N_c$ development is undermined.] This dilemma, however, can resolve itself [30,4]. More generally, while comparing worlds with different N_c, one has the freedom of selecting the right units. Reversing the argument, one may look for the particular definition, in which the large N_c limit looks smooth: e.g., *demand* that the instanton density is *not* exponential in N_c. In fact, we have selected above our "frozen coupling" parametrization with a desired smooth large N_c limit.

First of all, recall that the semiclassical formula (2) in the large N_c limit contains *factorial* terms in the denominator. The action, however, grows *linearly* with N_c, $S(\rho) = N_c s(\rho)$, and thus factorials are exactly canceled by the gauge zero modes. The main problem is with the *exponential* terms, which in our normalization are

$$d(\rho) \sim \exp\{N_c[2 - s(\rho) + 2\ln s(\rho)]\} \ . \tag{6}$$

The limit clearly depends on the sign of the bracket: if there exists a fixed point $s(\rho) \to s^*$, it is better to place it at the root of the bracket, $s_c^* \approx 5.4$. As shown in Fig. 2(b), the fitted values of the action limits for $N_c = 2, 3$ are indeed close to the critical value (shown by the star). Thus, this parametrization leads to finite $d(\rho)$ in the large N_c limit [which is, of course, nonsmooth: $d(\rho) \to 0$ below some critical value].

(One should not confuse the mere existence of smooth parametrization with the answer to a nontrivial *physical* question: Should the $N_c \to \infty$ limit be smooth? The answer, suggested in [16], is *negative*: in this limit an "instanton liquid" should become a solid, so that both color and translational symmetries are spontaneously broken. The reason is the growing action is analogous to a decreasing temperature. In [14] such phase transition was in fact found for $N_c = 3$, but at *nonphysical* instanton density, about 60 times the physical one. The critical N_c is unknown, and the work is in progress. Lattice studies of theories with $N_c > 3$ would certainly be of great value.)

In summary, we have presented results for the instanton size distributions in the interacting instanton liquid model, which were compared to those obtained on the lattice. We have concluded that they are very consistent. Furthermore, we have put forward another explanation, based on the existence of an infrared fixed point, which is also consistent with data. Better measurements of $d(\rho)$ on the lattice (especially at $N_c = 3$ and *larger*) are badly needed: among resolving several theoretical problems, they are potentially a promising way toward precision determination of $\Lambda_{\rm QCD}$. A new type of lattice measurement, especially of the renormalized action of the large-size instantons, is the only way to resolve the dilemma discussed in this work.

This work was partially supported by the DOE Grant No. DE-FG02-88ER40388.

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