Infrared behavior of the gluon propagator: Confining or confined?

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The possible infrared behavior of the gluon propagator is studied analytically, using the Schwinger-Dyson equations, in both the axial and the Landau gauge. The possibility of a gluon propagator less singular than $1/k^2$ when $k^2 \to 0$ is investigated and found to be inconsistent, despite claims to the contrary, whereas an infrared enhanced one is consistent. The implications for confinement are discussed.

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I. INTRODUCTION

The gluon propagator $\Delta_{\mu\nu}(k)$ is gauge dependent and, as such, is not experimentally observable. However, its infrared behavior has important implications for quark confinement. It can be shown that a gluon propagator, which is as singular as $1/k^4$ when $k^2 \to 0$, indicates that the interquark potential rises linearly with the separation. More formally, West [1] proved that if, in any gauge, $\Delta_{\mu\nu}$ is as singular as $1/k^4$, then the Wilson operator satisfies an area law, often regarded as a signal for confinement. Hence, quarks are confined through gluon interaction.

Another sufficient condition for confinement is that a propagator of a colored state should not have any singularities on the real, positive k^2 axis [2]. So, if gluons are confined, then they cannot propagate on shell and $\Delta_{\mu\nu}$ must be less singular than $1/k^2$ when $k^2 \to 0$. Such a behavior of the gluon propagator was assumed by Landshoff and Nachtmann [3] on purely phenomenological grounds, being needed to reproduce experiment with their model of the Pomeron.

Clearly (Fig. 1) the gluon propagator cannot both be more singular and less singular than $1/k^2$ as $k^2 \to 0$, but which is correct? The Schwinger-Dyson equations provide the natural starting point for a nonperturbative investigation of this infrared behavior of the gluon propagator. Extensive work has been previously performed in both the axial gauge [4–7] and the Landau gauge [8–10]. (For a comprehensive review see Roberts, and Williams [11].) A solution as singular as $1/k^4$ has been shown to exist in both gauges [4–6] and [8–10], whereas a *confined* solution for the gluon propagator, i.e., less singular than $1/k^2$, has only been claimed to exist in the axial gauge [7]. The purpose of this paper is to explore why these two different behaviors have been found. Fortunately, in studying just the infrared behavior, there is no need to solve the Schwinger-Dyson equation at all momenta. It is this that greatly simplifies our discussion and allows an analytic treatment.

In Sec. II we briefly describe the Schwinger-Dyson equation for the gluon propagator. The axial gauge studies are reviewed in Sec. III, and the possible, self-consistent solutions for the infrared behavior of the gluon propagator are reproduced analytically. In Sec. IV we repeat the discussion for the Landau gauge and find that a propagator less singular than $1/k^2$ when $k^2 \to 0$ is not a solution of the Schwinger-Dyson equation. In Sec. V we discuss the differing forms of the Schwinger-Dyson equations used to deduce these results.

II. SCHWINGER-DYSON EQUATION FOR THE GLUON PROPAGATOR

The Schwinger-Dyson equations are coupled integral equations, which interrelate the Green's functions of a field theory. Since they build an infinite tower of coupled equations, approximations and truncations are necessary to solve them. The Schwinger-Dyson equation for the gluon propagator yields a relation for $\Delta_{\mu\nu}$ in terms of the full three- and four-point vertex functions $\Gamma^3_{\mu\nu\rho}$ and $\Gamma^4_{\mu\nu\rho\sigma}$, the quark and the ghost propagators and couplings. The equation is displayed diagrammatically in Fig. 2. Here we only consider a pure gauge theory, i.e., a

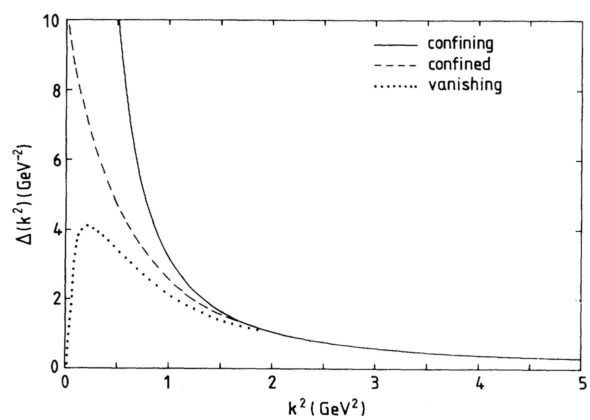

FIG. 1. Possible behavior of the gluon propagator $\Delta(k^2)$, which is the coefficient of the $g_{\mu\nu}$ or $\delta_{\mu\nu}$ component of $\Delta_{\mu\nu}(k)$. (a) *confining* gluon, $\Delta \sim (k^2)^{-2}$, (b) *confined* gluon, $\Delta \sim (k^2)^{-c}$ with c very small, and (c) infrared vanishing gluon $\Delta \sim k^2$. All are matched to the perturbative behavior for k larger than a few GeV.

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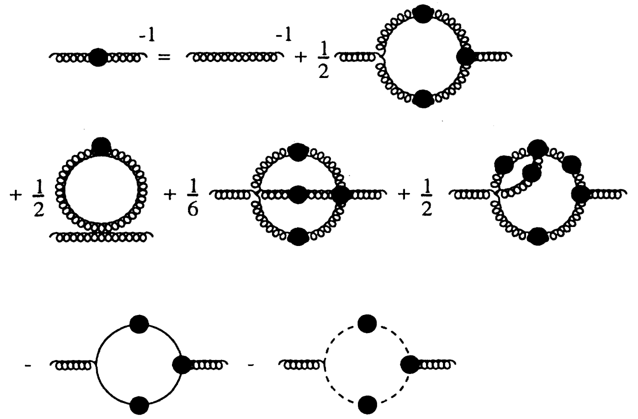

FIG. 2. The Schwinger-Dyson equation for the gluon propagator. Here the broken line represents the ghost propagator. The symmetry factors 1/2 and 1/6 and a negative sign for every ghost and fermion loop arise from the usual Feynman rules.

world without quarks. This is reasonable, since we expect it is the non-Abelian nature of QCD that is responsible for confinement.

III. AXIAL GAUGE STUDIES

In the axial gauge the gluon propagator is transverse to the gauge vector n_μ, so

$$\text{axial gauge formalism:} \quad n_\mu A^{a\mu} = 0 \ .$$

Studies of the axial gauge Schwinger-Dyson equation have the advantage that ghost fields are absent and the four-gluon vertex terms, Fig. 2, may be projected out of the Schwinger-Dyson equation. However, they have the drawback that the gluon propagator depends not only on p^2 but also on the unphysical gauge parameter γ, defined as

$$\gamma = \frac{(n\cdot p)^2}{n^2 p^2}$$

and in general must depend on two scalar functions F and H,

$$\Delta_{\mu\nu}(p^2,\gamma) = -\frac{i}{p^2}[F(p^2,\gamma)M_{\mu\nu} + H(p^2,\gamma)N_{\mu\nu}] \ , \quad (1)$$

with the tensors given by

$$M_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{n\cdot p} + n^2\frac{p_\mu p_\nu}{(n\cdot p)^2} \ ,$$

$$N_{\mu\nu} = g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \ .$$

The free propagator is obtained by substituting $F = 1$ and $H = 0$.

In all previous axial gauge studies it has been assumed that any infrared singular part of the propagator has the same tensor structure as the free one (though importantly this contradicts the result of West [12]). Thus, for $p^2 \to 0$ it is assumed that

$$\Delta_{\mu\nu}(p^2,\gamma) = -\frac{i}{p^2}F(p^2,\gamma)M_{\mu\nu} \ . \quad (2)$$

The gluon vacuum polarization tensor $\Pi_{\lambda\mu}(p^2,\gamma)$ is defined by

$$\Pi^{\lambda\mu}\Delta_{\mu\nu} = g^\lambda_\nu - \frac{n^\lambda p_\nu}{n\cdot p} \ .$$

Projecting the integral equation with $n_\mu n_\nu/n^2$, the loops involving the four-gluon vertex give an identically zero contribution because of the tensor structure of the bare four-gluon vertex and the fact that the gluon propagator is transverse to the axial gauge vector, that is

$$n_\mu \Delta^{\mu\nu} = 0 = \Delta^{\mu\nu} n_\nu \ .$$

Thus the relevant part of the Schwinger-Dyson equation of Fig. 2 becomes

$$\begin{aligned}\Pi_{\mu\nu} = \Pi^{(0)}_{\mu\nu} &- \frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\Gamma^{(0)}_{\mu\alpha\delta}(-p,k,q)\Delta^{\alpha\beta}(k)\\ &\times\Delta^{\gamma\delta}(q)\Gamma_{\beta\gamma\nu}(-k,p,-q)\\ &-\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\Gamma^0_{\mu\nu\alpha\beta}(p,k,-k,-p)\Delta^{\alpha\beta}(k) \ , \quad (3)\end{aligned}$$

where $q = p - k$, the last term is the tadpole contribution, and all color indices are implicitly included in the vertices. Once the full three-gluon vertex is known, we have a closed equation for the gluon vacuum polarization $\Pi_{\mu\nu}$.

The vertex is constrained by the Slavnov-Taylor identity in terms of this vacuum polarization:

$$q_\nu\Gamma^{\mu\nu\rho}(p,q,k) = \Pi^{\mu\rho}(k) - \Pi^{\mu\rho}(p) \ . \quad (4)$$

Separating $\Gamma^{\mu\nu\rho}$ into a transverse and a longitudinal part, where the transverse part is defined to vanish when contracted with any external momentum, the Slavnov-Taylor identity exactly determines the longitudinal part [13] if it is to be free of kinematic singularities. Thus,

$$\begin{aligned}\Gamma^L_{\mu\nu\rho}(p,k,q) = g_{\mu\nu}&\left(\frac{p_\rho}{F(q^2,\gamma)} - \frac{q_\rho}{F(q^2,\gamma)}\right)\\ &+\frac{1}{p^2-q^2}\left(\frac{1}{F(p^2,\gamma)} - \frac{1}{F(q^2,\gamma)}\right)\\ &\times(p_\nu q_\mu - g_{\mu\nu}p\cdot q)(p_\rho - q_\rho)\\ &+\text{cyclic permutations} \ . \quad (5)\end{aligned}$$

This longitudinal part is responsible for the dominant ultraviolet structure of the vertex. Moreover, it is assumed that it entirely embodies the infrared behavior, and so the transverse part can be neglected. The assumption is motivated by the fact that the transverse part (as defined) vanishes, when the external momenta approach zero.

Using the explicit expressions, Eq. (2) for Δ, Eqs. (4) and (5) for Γ, and multiplying with $n_\mu n_\nu/n^2$, we find, in Euclidean space,

$$-\frac{p^2}{F(p^2)}(1-\gamma) = -p^2(1-\gamma) + \frac{g^2 C_A}{2}\int \frac{d^4k}{(2\pi)^4}\frac{n\cdot(k-q)}{n^2}\Delta^{\lambda\rho}_{(0)}(k)\Delta^{\lambda\sigma}_{(0)}(q)$$
$$\times\left\{k\cdot n\left[\delta_{\rho\sigma}F(q^2) - \frac{F(q^2)-F(k^2)}{k^2-q^2}(\delta_{\rho\sigma}k\cdot q - k_\rho q_\sigma) + \frac{F(k^2)-F(p^2)}{p^2-k^2}\frac{F(q^2)}{F(p^2)}p_\sigma(p+k)_\rho\right]\right.$$
$$\left.-q\cdot n\left[\delta_{\rho\sigma}F(k^2) - \frac{F(q^2)-F(k^2)}{k^2-q^2}(\delta_{\rho\sigma}k\cdot q - k_\rho q_\sigma) + \frac{F(q^2)-F(p^2)}{p^2-q^2}\frac{F(k^2)}{F(p^2)}p_\rho(q+p)_\sigma\right]\right\}$$
$$+\frac{g^2 C_A}{2}\int\frac{d^4k}{(2\pi)^4}\frac{F(k^2)}{k^2}\left(2+\frac{k^2 n^2}{(n\cdot k)^2}\right) . \tag{6}$$

This is the equation found by Baker, Ball, and Zachariasen (BBZ) [4] who studied its solution numerically. They came to the conclusion that the only consistent infrared behavior for the function $F(p^2)$ is

$$F(p^2) \propto \frac{1}{p^2} \quad \text{as } p^2 \to 0 ,$$

and that this is independent of γ as a numerical approximation.

Schoenmaker [6] simplified the BBZ equation further by exactly setting $\gamma = 0$. Doing this, the contribution of the tadpole diagram vanishes. Moreover, approximating $F(q^2)$ by $F(p^2+k^2)$, which should be exact in the infrared limit, allows the angular integrals to be performed analytically. Consequently, Schoenmaker finds the simpler equation

$$p^2\left(\frac{1}{F(p^2)}-1\right) = \frac{g^2 C_A}{32\pi^2}\left\{\int_0^{p^2} dk^2\left[\left(-\frac{k^4}{12p^4}+\frac{5}{2}\frac{k^2}{p^2}-\frac{2}{3}\frac{k^2}{p^2-k^2}\right)F_1 + \left(\frac{k^6}{24p^6}-\frac{1}{4}\frac{k^4}{p^4}-\frac{1}{4}\frac{k^2}{p^2}\right)F_2\right.\right.$$
$$\left.+\left(-\frac{1}{6}\frac{k^2}{p^2}+\frac{2}{3}\frac{k^2}{p^2-k^2}\right)F_3\right] + \int_{p^2}^{\infty} dk^2\left[\left(-\frac{3}{4}-\frac{p^2}{4k^2}-\frac{2}{3}\frac{p^2}{p^2-k^2}\right)F_1\right.$$
$$\left.\left.+\left(-\frac{3}{4}\frac{k^2}{p^2}+\frac{5}{12}-\frac{1}{8}\frac{p^2}{k^2}\right)F_2 + \left(\frac{7}{6}\frac{p^2}{k^2}+\frac{2}{3}\frac{p^2}{p^2-k^2}\right)F_3\right]\right\} , \tag{7}$$

where

$$F_1 = F(p^2+k^2) ,$$
$$F_2 = F(p^2+k^2) - F(k^2) ,$$
$$F_3 = \frac{F(p^2+k^2)F(k^2)}{F(p^2)} .$$

In general, this equation has a quadratic ultraviolet divergence, which would give a mass to the gluon. Such terms have to be subtracted to ensure the masslessness condition

$$\lim_{p^2\to 0}\Pi_{\mu\nu} = 0 , \quad \text{i.e.,}\ \frac{p^2}{F(p^2)} = 0 \ \text{ for } p^2 \to 0 , \tag{8}$$

is satisfied. This property can be derived generally from the Slavnov-Taylor identity and always has to hold.

The complicated structure of the integral equation, Eq. (7), does not allow an exact analytic solution for the gluon renormalization function $F(p^2)$ to be found, and most previous studies [8,10,4,7] solve the equation numerically. However, the possible asymptotic behavior of $F(p^2)$ for both small and large p^2 can be investigated analytically.

We determine which infrared behavior of $F(p^2)$ can give a self-consistent solution to the integral equation by taking a trial input function $F_{\rm in}(p^2)$ and substituting it into the right-hand side of the equation. After performing the k^2 integration, we obtain an output function $1/F_{\rm out}(p^2)$ to be compared to the reciprocal of the input function. To do this, the gluon renormalization function is approximated in the infrared region by a Laurent expansion in powers of p^2 and at large momenta by its bare form: i.e.,

$$F(p^2) = \begin{cases}\sum_{n=0}^{\infty} a_n (p^2/\mu^2)^{n+\eta} & \text{for } p^2 < \mu^2 , \\ 1 \ \text{ for } p^2 > \mu^2 , & \end{cases} \tag{9}$$

where

$$\sum_{n=0}^{\infty} a_n = 1$$

to ensure continuity at $p^2 = \mu^2$. μ is the mass scale above which we assume perturbation theory applies. η can be negative to allow for an infrared enhancement. Equation (9) is a sufficiently general representation for finding the dominant self-consistent infrared behavior. Of course, the true renormalization function is modulated by powers of logarithms of momentum, characteristic of a gauge theory. However, these do not qualitatively affect the dominant infrared behavior and can be neglected. Indeed to make the presentation straightforward, we only need approximate $F(p^2)$ by its dominant infrared power $(p^2)^\eta$ for $p^2 < \mu^2$ to test whether consistency is possible, and this is what we describe below. However, as we shall see, if η is negative then potential mass terms arise, and these have to be subtracted. Only in this case do higher terms in Eq. (9) play a role too, and it is necessary to consider other than the leading term in the low-momentum input. Otherwise, higher powers make no qualitative difference, as we have checked. Consequently, we present only the results with the lowest powers in the representation, Eq. (9).

To illustrate the idea, let us take the trial infrared behavior to be just

$$F(p^2) \propto \left(\frac{p^2}{\mu^2}\right)^\eta \quad (\text{i.e., } a_n = 0 \text{ for } n \geq 1) \ . \tag{10}$$

Note that the masslessness condition, Eq. (8), restricts η to be less than 1. Furthermore, we demand that in the high-momentum region the solution of the integral equation matches the perturbative result, i.e., for $p^2 \to \infty$, we have $F(p^2) = 1$, modulo logarithms.

Taking $\eta = -1$, for example, i.e.,

$$F_{\text{in}}(p^2) = A\frac{\mu^2}{p^2} \ ,$$

in Schoenmaker's approximation Eq. (7) gives

$$p^2\left(\frac{1}{F(p^2)} - 1\right) = \text{const} \ .$$

This violates the masslessness condition of Eq. (8) and so has to be mass renormalized. As explained above, now terms in $F(p^2)$ of higher order in p^2 will generate a contribution to the right-hand side of the equation, making it possible to find a self-consistent solution by these terms canceling the explicit factor of 1. Consequently, we can approximate Eq. (9) by

$$F_{\text{in}}(p^2) = \begin{cases} A(\mu^2/p^2) + (p^2/\mu^2) & \text{if } p^2 < \mu^2 \ , \\ 1 & \text{if } p^2 > \mu^2 \ . \end{cases} \tag{11}$$

We then find, after mass renormalization,

$$\frac{1}{F_{\text{out}}(p^2)} = 1 + \frac{g^2 C_A}{32\pi^2}\left[\frac{67}{96}\frac{p^2}{\mu^2} - \frac{1}{12} - \frac{3}{8}\frac{p^2}{\mu^2}\ln\left(\frac{\mu^2}{p^2}\right) + \frac{5}{12}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right] \ , \tag{12}$$

where Λ is the ultraviolet cutoff introduced to make the integrals finite. The ultraviolet divergent constant can be arranged to cancel the 1, and we find self-consistency modulo logarithms. It is this result that Schoenmaker found [6] supporting the earlier result of BBZ [4]. However, importantly, self-consistency requires A, Eq. (11), to be negative, as also found by Schoenmaker.

More recently, Cudell and Ross [7] have taken Schoenmaker's equation, Eq. (7), and investigated whether one finds self-consistency for a gluon renormalization function, which is less singular than $1/k^2$ for $k^2 \to 0$, i.e., which corresponds to *confined* gluons. The trial input function they use in their investigation is

$$F_{\text{in}}(p^2) \propto (p^2)^{1-c} \ ,$$

where c is small and positive to ensure a massless gluon, Eq. (8). Once more we want the integral equation for $\Pi_{\mu\nu}$ to agree with perturbation theory in the ultraviolet region, but $F_{\text{in}}(p^2) \propto (p^2)^{1-c}$ grows for large momenta and hence spoils the ultraviolet behavior. So to check whether this input function gives self-consistency in the infrared, we input the trial form

$$F_{\text{in}}(p^2) = \begin{cases} (p^2/\mu^2)^{1-c} & \text{if } p^2 < \mu^2 \ , \\ 1 & \text{if } p^2 > \mu^2 \ . \end{cases} \tag{13}$$

Inserting this into Eq. (7), we find, after mass renormalization,

$$\frac{1}{F_{\text{out}}(p^2)} = 1 + \frac{g^2 C_A}{32\pi^2}\left[D_1 + D_2\left(\frac{\mu^2}{p^2}\right)^{1-c} + D_3\left(\frac{p^2}{\mu^2}\right)^{1-c} + D_4\left(\frac{p^2}{\mu^2}\right)^{c} + \cdots\right] \ , \tag{14}$$

where F_1, F_2, and F_3 have been expanded for small p^2, and only the first few terms have been collected in this equation so that

$$D_1 = \frac{5}{12(1-c)} - \frac{1}{3} + \frac{11}{12}\ln\frac{\Lambda^2}{\mu^2} \ ,$$

$$D_2 = \frac{1}{2(2-2c)} + \frac{1}{6}\ln\left(\frac{\Lambda^2}{\mu^2}\right) \ ,$$

$$D_3 = \frac{2407}{1440} - \frac{353c}{360} - \frac{3}{8c} - \frac{1}{24(5-c)} + \frac{1+2c}{12(4-c)} - \frac{7-8c}{12(3-c)} - \frac{7+c}{12(2-c)} - \frac{7}{12(1-c)} + \frac{1-4c}{6(2-2c)} + \frac{3-7c}{6(1-2c)} + \frac{2}{3}(2-c)\Psi(1) - \frac{4}{3}(2-c)\Psi(1-c) + \frac{2}{3}(2-c)\Psi(1-2c) \ ,$$

$$D_4 = -\frac{7-9c}{24c} \ ,$$

where Ψ is the logarithmic derivative of the Γ function. Again the 1 can be arranged to cancel with the constant term, and the dominant infrared behavior is indeed

$$\frac{1}{F_{\text{out}}(p^2)} \to \left(\frac{\mu^2}{p^2}\right)^{1-c} \quad \text{for } p^2 \to 0 \ . \tag{15}$$

Hence a gluon propagator less singular than $1/p^2$ for $p^2 \to 0$ can be derived from Schoenmaker's equation as Cudell and Ross [7] have found. Note once again that terms of higher order in Eq. (9) do not qualitatively alter the result. Thus, we see in the axial gauges that apparently both *confined* and *confining* solutions are possible for the gluon propagator. However, the singular *confining* behavior must be an artifact of the approximation that one of the gluon functions H vanishes, since West [12] has shown that in a gauge with only positive norm states a singular gluon renormalization function is not possible. Moreover, the approximation of setting $\gamma = 0$ in the BBZ equation, Eqs. (6) and (7), has been seriously questioned in Ref. [14]. It is therefore sensible to ask what the behavior in covariant gauges is, to which we now turn.

IV. LANDAU GAUGE STUDIES

The advantage of Landau gauge studies is the much simpler structure of the gluon propagator, which is defined by

$$\Delta_{\mu\nu} = -i\frac{G(p^2)}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \ . \tag{16}$$

However, other problems arise, and the following approximations have to be made.

In any covariant gauge, ghosts are necessary to keep the vacuum polarization transverse and, hence, are present in the Schwinger-Dyson equation of the gluon propagator, Fig. 2. However, in all previous studies [8,10] the ghost loop diagram is only included in as much as it ensures the transversality of the gluon propagator, assuming that otherwise it does not affect the infrared behavior of the propagator. This assumption is supported by the fact, that in a one-loop perturbative calculation the ghost loop makes a numerically small contribution to $G(p^2)$.

The four-gluon terms cannot be eliminated as in the axial gauge and are simply neglected. This can be regarded as a first step in a truncation of the Schwinger-Dyson equations.

With these assumptions, we again find a closed integral equation for the gluon vacuum polarization $\Pi_{\mu\nu}$ once the full three-gluon vertex is known. In the Landau gauge, the Slavnov-Taylor identity for the three-gluon vertex involves the ghost self-energy, which is simply set to zero, and the proper ghost-gluon vertex function $G_{\mu\nu}$. However, in the limit of vanishing ghost momentum the ghost-gluon vertex is approximately equal to the gluon propagator. With this simplification the Slavnov-Taylor identity has the same form as in the axial gauge and is given in Eq. (5). Once again neglecting the transverse part of the vertex, we obtain a closed integral equation

$$\Pi_{\mu\nu} = \Pi^{(0)}_{\mu\nu} - \frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\Gamma^{(0)}_{\mu\alpha\delta}(-p,k,q) \times\Delta^{\alpha\beta}(k)\Delta^{\gamma\delta}(q)\Gamma_{\beta\gamma\nu}(-k,p,-q) \ , \tag{17}$$

where once more the color indices are implicit and $q = p - k$.

A scalar equation is obtained by projecting with

$$P^{\mu\nu} = \frac{1}{3p^2}(4p^\mu p^\nu - p^2 g^{\mu\nu}) \ .$$

This projector has the advantage that the $g_{\mu\nu}$ term in Eq. (16), which is quadratically divergent in four dimensions, does not contribute. Thus we find

$$\frac{1}{G(p^2)} = 1 + \frac{g^2 C_A}{96\pi^4}\frac{1}{p^2}\int d^4k\left[G(q^2)A(k^2,p^2) + \frac{G(k^2)G(q^2)}{G(p^2)}B(k^2,p^2) + \frac{G(k^2)-G(p^2)}{k^2-p^2}\frac{G(q^2)}{G(p^2)}C(k^2,p^2) + \frac{G(q^2)-G(k^2)}{q^2-k^2}D(k^2,p^2)\right] , \tag{18}$$

where

$$A(k^2,p^2) = 48\frac{(k\cdot p)^2}{k^2p^2q^2} - 64\frac{(k\cdot p)}{k^2q^2} + 16\frac{(k\cdot p)^3}{k^2p^2q^4} - 12\frac{1}{k^2} + 22\frac{p^2}{k^2q^2} - 42\frac{(k\cdot p)^2}{k^2q^4} - 10\frac{p^4}{k^2q^4} + 36\frac{p^2(k\cdot p)}{k^2q^4} \ ,$$

$$B(k^2,p^2) = -13\frac{p^2}{q^4} + 18\frac{p^2(k\cdot p)}{k^2q^4} - 2\frac{(k\cdot p)^2}{k^2q^4} - 4\frac{p^4}{k^2q^4} + \frac{p^2(k\cdot p)^2}{k^4q^4} \ ,$$

$$C(k^2,p^2) = 4\frac{(k\cdot p)^2}{k^2q^2} + 6\frac{p^2(k\cdot p)}{k^2q^2} + 6\frac{(k\cdot p)}{q^2} + 8\frac{p^2}{q^2} \ ,$$

$$D(k^2,p^2) = 12\frac{k^2}{q^2} - 48\frac{(k\cdot p)^2}{p^2q^2} + 48\frac{(k\cdot p)^3}{k^2p^2q^2} + 24\frac{(k\cdot p)}{q^2} - 5\frac{p^2}{q^2} - 40\frac{(k\cdot p)^2}{k^2q^2} + 9\frac{p^2(k\cdot p)}{k^2q^2} \ .$$

Brown and Pennington [10] studied this equation numerically and found

$$G(p^2) = A\frac{\mu^2}{p^2} \quad \text{for } p^2 \to 0$$

to be a consistent solution. This result is in agreement with Mandelstam's study of the gluon propagator [8].

Again, approximating $G(q^2)$ by $G(p^2+k^2)$ allows us to perform the angular integrations analytically in Eq. (18), giving

$$\begin{aligned}\frac{1}{G(p^2)} = 1 + \frac{g^2 C_A}{48\pi^2}\frac{1}{p^2}\Bigg\{ \int_0^{p^2} dk^2 \Bigg[G_1\Bigg(-1 - 10\frac{k^2}{p^2} + 6\frac{k^4}{p^4} + \frac{k^2}{p^2-k^2}\left(\frac{75}{4} - \frac{39}{4}\frac{k^2}{p^2} + 4\frac{k^4}{p^4} - 5\frac{p^2}{k^2}\right)\Bigg) \\ + G_2\left(-\frac{21}{4}\frac{k^2}{p^2} + 7\frac{k^4}{p^4} - 3\frac{k^6}{p^6}\right) + G_3\Bigg(\frac{k^2}{p^2-k^2}\left(-\frac{27}{8} - \frac{11}{4}\frac{k^2}{p^2} - \frac{15}{8}\frac{p^2}{k^2}\right)\Bigg)\Bigg] \\ + \int_{p^2}^{\infty} dk^2 \Bigg[G_1\Bigg(\frac{p^2}{k^2} - 6 + \frac{p^2}{p^2-k^2}\left(\frac{29}{4} + \frac{3}{4}\frac{p^2}{k^2}\right)\Bigg) \\ + G_2\left(-\frac{3}{2} + \frac{1}{4}\frac{p^2}{k^2}\right) + G_3\Bigg(\frac{p^2}{p^2-k^2}\left(\frac{3}{4} - \frac{67}{8}\frac{p^2}{k^2} - \frac{3}{8}\frac{p^4}{k^4}\right)\Bigg)\Bigg]\Bigg\} ,\end{aligned} \tag{19}$$

where

$$G_1 = G(p^2+k^2) ,$$
$$G_2 = G(p^2+k^2) - G(k^2) ,$$
$$G_3 = \frac{G(k^2)G(p^2+k^2)}{G(p^2)} .$$

Note that the integral equation has the usual ultraviolet divergences, but infrared divergences are also possible. The ultraviolet divergences can be handled in the standard way to give a renormalized function $G_R(p^2)$—this will not be discussed here. However, we have to make the potentially infrared divergent integrals finite in order to calculate the integrals.[1] The infrared regularization procedure proposed by Brown and Pennington [10] is to use the plus prescription of the theory of distributions, which is defined as

$$\left(\frac{\mu^2}{k^2}\right)_+ = \frac{\mu^2}{k^2} \quad \text{for } \infty > k^2 > 0 ,$$

and in the neighborhood of $k^2 = 0$ it is a distribution that satisfies

$$\int_0^{\infty} dk^2 \left(\frac{\mu^2}{k^2}\right)_+ S(k^2,p^2) = \int_0^{p^2} dk^2 \frac{\mu^2}{k^2}[S(k^2,p^2) - S(0,p^2)] + \int_{p^2}^{\infty} dk^2 \frac{\mu^2}{k^2} S(k^2,p^2). \tag{20}$$

Simply taking

$$G_{\rm in}(k^2) = A\left(\frac{\mu^2}{k^2}\right)_+$$

as an input function once again leads to a mass term, and higher terms in the expansion Eq. (8) are necessary. Then we do have the chance of finding self-consistency for a gluon propagator as singular as $1/k^4$ and, hence *confining* quarks. With

$$G_{\rm in}(p^2) = \begin{cases} A(\mu^2/p^2)_+ + (p^2/\mu^2) & \text{if } p^2 < \mu^2 , \\ 1 \ \text{ if } p^2 > \mu^2 , \end{cases} \tag{21}$$

we find, after mass renormalization,

$$\frac{1}{G_{\rm out}(p^2)} = 1 + \frac{g^2 C_A}{48\pi^2}\left[-\frac{479}{24}\frac{p^2}{\mu^2} + \frac{13}{8}\frac{p^2}{\mu^2}\ln\left(\frac{\mu^2}{p^2}\right) - \frac{81}{4} - \frac{25}{4}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right] . \tag{22}$$

The ultraviolet divergent constant can be arranged to cancel the 1, and again we find self-consistency. This is the result found numerically by Brown and Pennington [10] with a positive infrared enhancement to the gluon renormalization function, i.e., $A > 0$.

[1] These divergences do not arise in an axial gauge when γ is set equal to zero as Schoenmaker does, Eq. (7).

Now we check whether it is possible in the Landau gauge to find the behavior Cudell and Ross discovered using Schoenmaker's approximation in the axial gauge. With

$$G_{\rm in}(p^2) = \begin{cases} (p^2/\mu^2)^{1-c} & \text{if } p^2 < \mu^2 , \\ 1 & \text{if } p^2 > \mu^2 , \end{cases} \tag{23}$$

we find, again after mass renormalization,

$$\frac{1}{G_{\rm out}(p^2)} = 1 + \frac{g^2 C_A}{48\pi^2}\left[D_1 + D_2\left(\frac{\mu^2}{p^2}\right)^{1-c} + D_3\left(\frac{p^2}{\mu^2}\right)^{1-c} + D_4\left(\frac{p^2}{\mu^2}\right)^{c} + \cdots\right] , \tag{24}$$

where

$$\begin{aligned} D_1 &= -\left[\frac{3}{2} + \frac{5+6c}{1-c} + \frac{25}{4}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right] , \\ D_2 &= -\left[\frac{3}{4(2-2c)} + \frac{3}{4}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right] , \\ D_3 &= -\frac{1971}{60} + \frac{29c}{2} + \frac{37}{20c} + \frac{6-13c}{2(1-c)} + \frac{59-32c}{4(2-c)} + \frac{155-64c}{8(3-c)} + \frac{127-49c}{8(4-c)} + \frac{23-11c}{4(5-c)} - \frac{125+61c}{8(1-2c)} \\ &\quad - \frac{55+6c}{8(2-2c)} + \frac{3}{4(3-2c)} - 8(2-c)\Psi(-2c) - 8(2-c)\Psi(1) , \\ D_4 &= \frac{61+6c}{8(1-2c)} . \end{aligned}$$

Thus the dominant infrared behavior is

$$\frac{1}{G_{\rm out}(p^2)} \to -\left(\frac{\mu^2}{p^2}\right)^{1-c} ,$$

and self-consistency is spoiled by a negative sign, since c is small and positive.

V. CONFINED GLUONS

A gluon propagator, which is less singular than $1/k^2$ for $k^2 \to 0$ and, hence, describes *confined* gluons, appears to be a self-consistent solution only of the axial gauge Schwinger-Dyson equation using Schoenmaker's approximate integral, Eq. (7). In the Landau gauge this behavior of the gluon propagator is not possible: A minus sign spoils self-consistency. We should therefore comment on the origin of this crucial minus sign.

Starting from BBZ's integral, Eq. (6), there is no difference in sign between the two gauges. Equation (6) is Bose symmetric (as it should be) and can therefore be rewritten as

$$\frac{p^2}{F(p^2)}(1-\gamma) = p^2(1-\gamma) - \frac{g^2 C_A}{2}\int \frac{d^4k}{(2\pi)^4}\frac{n\cdot(k-q)}{n^2}\Delta^{\lambda\rho}_{(0)}(k)\Delta^{\lambda\sigma}_{(0)}(q)2k\cdot nK_{\rho\sigma}(k,p) , \tag{25}$$

where

$$K_{\rho\sigma}(k,p) = \delta_{\rho\sigma}F(q^2) - \frac{F(q^2)-F(k^2)}{k^2-q^2}(\delta_{\rho\sigma}k\cdot q - k_\rho q_\sigma) + \frac{F(k^2)-F(p^2)}{p^2-k^2}\frac{F(q^2)}{F(p^2)}(p+k)_\rho p_\sigma$$

whereas, taking the starting equation of Schoenmaker's paper [Eq. (3.5) of Ref. [6]] we find

$$\frac{p^2}{F(p^2)}(1-\gamma) = p^2(1-\gamma) - \frac{i}{2}g^2 C_A\int \frac{d^4k}{(2\pi)^4}\frac{n\cdot(k-q)}{n^2}\Sigma_{\rho\sigma}(k,q)[-2k\cdot nK_{\rho\sigma}(k,p)] , \tag{26}$$

where

$$\Sigma_{\rho\sigma}(k,q) = (i)^2\Delta^{\lambda\rho}_{(0)}(k)\Delta^{\lambda\sigma}_{(0)}(q) .$$

Schoenmaker formulates his equation in Minkowski space. Performing a Wick rotation to transform to Euclidean space by $d^4k_M \to id^4k_E$, we find that Schoenmaker's equation, Eq. (26), becomes

$$\frac{p^2}{F(p^2)}(1-\gamma) = p^2(1-\gamma) + \frac{g^2 C_A}{2}\int \frac{d^4k}{(2\pi)^4}\frac{n\cdot(k-q)}{n^2}\Delta^{\lambda\sigma}_{(0)}(k)\Delta^{\lambda\rho}_{(0)}(q)2k\cdot nK_{\sigma\rho}(k,p)\ , \tag{27}$$

which differs from Eq. (25) by a crucial minus sign.

We therefore see that in the axial gauge using BBZ's integral equation for the gluon propagator, and simplifying the angular dependence in the way Schoenmaker does in order to make an analytical discussion of the infrared behavior of the propagator possible, yields an integral equation very similar to the one found by Brown and Pennington [10] in the Landau gauge. These equations lead to the correct perturbative behavior at large momenta. In contrast, a self-consistent solution of the gluon propagator less singular than $1/k^2$ for $k^2 \to 0$ cannot be found in either gauge. Schoenmaker's own equation, which is the starting point for the study of Cudell and Ross [7], has an incorrect additional minus sign. This should have been heralded by the self-consistent enhanced gluon of Eq. (11) having a negative sign, using Schoenmaker's equation. In an axial gauge this sign should have been a little worrying for a wave function renormalization of a state with positive definite norm.

VI. SUMMARY AND CONCLUSION

We have studied the Schwinger-Dyson equation of the gluon propagator to determine analytically the possible infrared solutions for the gluon renormalization function $G(p^2)$. In both the axial and Landau gauges, one can find a self-consistent solution, which behaves as $1/k^2$ for $k^2 \to 0$ and, hence, a propagator that is as singular as $1/k^4$ for $k^2 \to 0$. This form of the gluon propagator is consistent with area law behavior of the Wilson loop, which is regarded as a signal for confinement.[2] Numerical studies [4,10] have shown that a gluon propagator with such an enhanced behavior in the infrared region, which connects to the perturbative regime at a finite momentum (as indicated by experiment), can indeed be found as a self-consistent solution to the boson Schwinger-Dyson equation. Such a behavior of the boson propagator has been shown to give quark propagators with no physical poles [15]. Furthermore, extending these nonperturbative methods to hadron physics, it has been found that a regularized, infrared, singular gluon propagator together with the Schwinger-Dyson equation for the quark self-energy gives rise to a good description of dynamical chiral symmetry breaking. For instance, one obtains values for quantities such as the pion decay constant that agree with experimental results [11].

A gluon propagator, which is less singular than $1/k^2$ for $k^2 \to 0$ and, hence, describes *confined* gluons, cannot be found in either the axial or the Landau gauge. Solutions of this type have only been found using approximations to the gluon Schwinger-Dyson equation with an incorrect sign. Possible consequences for models of the Pomeron are discussed elsewhere [16].

We should also mention the related work of the group of Häbel *et al.* [17]. They too start from an approximate, but larger, set of Schwinger-Dyson equations, which is then to be solved self-consistently. However, the method employed is completely different. The philosophy [17] is to obtain the solution of these equations as power series in the coupling, as in perturbation theory, and to include nonperturbative effects by letting each Green's function depend upon a spontaneously generated mass scale $b(g^2)$. The gluon propagator is assumed to be of the form (see Fig. 1)

$$\Delta(k^2) \equiv \frac{G(k^2)}{k^2} = \frac{k^2}{k^4+b^4}\ , \tag{28}$$

representing *confined* gluons. This grossly violates the masslessness condition of Eq. (8). In general, gluon masses can only arise in four dimensions if the vertex functions have singularities themselves corresponding to colored massless scalar states; otherwise the Slavnov-Taylor identities sufficiently constrain the vertex functions to require the inverse of the gluon propagator to vanish at $k \to 0$.[3] Not only do the vertices of Häbel *et al.* have these massless singularities, but self-consistency can only be found if the three-gluon vertex is complex, when conventional understanding of its singularity structure would lead us to expect it to be real for momenta, which in Minkowski space are spacelike. Subsequently Hawes, Roberts, and Williams [20] and Bender and Alkofer [21] have shown that with this gluon propagator, Eq. (28), the solution of the quark Schwinger-Dyson equation does not describe a confined particle. They therefore also conclude that the full gluon propagator in QCD cannot vanish in the infrared region.

To summarize, at first sight there appears to be a distinction between a *confining* and a *confined* gluon. A *confining* gluon is one whose interactions lead to quark confinement. $\Delta(k^2) \sim 1/k^4$ behavior is of this *confining* type. In contrast, it is sometimes argued that $\Delta(k^2)$ must be less singular than $1/k^2$ to ensure that gluons themselves do not propagate over large distances. However, whether gluons are *confining* or *confined* are not real alternatives. Gluons must be both. They confine quarks by having very strong long-range interactions. They themselves are confined by not having a Lehman representa-

[2]However, as remarked at the end of Sec. IV, axial gauge studies are seriously marred by the simplifying assumption that $H \equiv 0$ in Eq. (1).

[3]This remark equally applies to the work of Zwanziger [18] and the lattice studies in [19].

tion, which any physical asymptotic state must have [11].

While infrared singular gluons satisfy both criteria, softened gluons, though *confined*, do not generate quark confinement or dynamical chiral symmetry breaking, which are features of our world. Remarkably, a study of the field equations of QCD reveals that this theory naturally exhibits these aspects with an infrared enhanced gluon propagator.

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[1] G. B. West, Phys. Lett. **115B**, 468 (1982).
[2] C. D. Roberts, A. G. Williams, and G. Krein, Int. J. Mod. Phys. A **7**, 5607 (1992).
[3] P. V. Landshoff and O. Nachtmann, Z. Phys. C **39**, 405 (1989).
[4] M. Baker, J. S. Ball, and F. Zachariasen, Nucl. Phys. **B186**, 531 (1981); **B186**, 560 (1981).
[5] A. I. Alekseev, Yad. Fiz. **33**, 516 (1981) [Sov. J. Nucl. Phys. **33**, 269 (1981)]; F. Paccanoni, Nuovo Cimento A **74**, 267 (1983).
[6] W. J. Schoenmaker, Nucl. Phys. **B194**, 535 (1982).
[7] J. R. Cudell and D. A. Ross, Nucl. Phys. **B359**, 247 (1991).
[8] S. Mandelstam, Phys. Rev. D **20**, 3223 (1979).
[9] U. Bar-Gadda, Nucl. Phys. **B163**, 312 (1980).
[10] N. Brown and M. R. Pennington, Phys. Lett. B **202**, 257 (1988); Phys. Rev. D **39**, 2723 (1989).
[11] C. D. Roberts and A. G. Williams, Prog. Part. Nucl. Phys. **33**, 477 (1994).
[12] G. B. West, Phys. Rev. D **27**, 1878 (1983).
[13] J. S. Ball and T. W. Chiu, Phys. Rev. D **22**, 2550 (1980).
[14] D. Atkinson, P. W. Johnson, W. J. Schoenmaker, and H. A. Slim, Nuovo Cimento A **77**, 197 (1983); E. L. Heck and H. A. Slim, *ibid.* **88**, 407 (1985).
[15] For example, R. L. Stuller, Phys. Rev. D **13**, 513 (1976); H. Pagels, *ibid.* **15**, 2991 (1977).
[16] K. Büttner and M. R. Pennington, Phys. Lett. B **356**, 354 (1995).
[17] U. Häbel, R. Könning, H. G. Reusch, M. Stingl, and S. Wigard, Z. Phys. A **336**, 435 (1990).
[18] D. Zwanziger, Nucl. Phys. **B364**, 127 (1991).
[19] J. E. Mandula and M. Ogilvie, in *Nonperturbative Methods in Field Theory*, Proceedings of the Conference, Irvine, California, 1987, edited by H. Hamber [Nucl. Phys. B (Proc. Suppl.) **1A**, 117 (1987)]; Phys. Lett. B **185**, 127 (1987); C. Bernard, C. Parrinello, and A. Soni, Phys. Rev. D **49**, 1585 (1994).
[20] F. T. Hawes, C. D. Roberts, and A. G. Williams, Phys. Rev. D **49**, 4683 (1994).
[21] A. Bender and R. Alkofer, Universität Tübingen Report No. UNITU-THEP-15/1994.