$U(2)_L \times U(2)_R$ chiral theory of mesons

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(Received 13 April 1995)

A $U(2)_L \times U(2)_R$ chiral theory of pseudoscalar, vector, and axial-vector mesons is proposed and studied at the tree level. VMD is a natural result of this theory. The physical processes of normal parity and abnormal parity have been studied by using the same Lagrangian leading to a universality of coupling. Two new mass relations between vector and axial-vector mesons have been found. Weinberg's first sum rule and new relations about the amplitude of a_1 decay are satisfied. The KSFR sum rule is satisfied reasonably well. The ρ pole in the pion form factor has been achieved. The theoretical results for the processes $\rho \to \pi\pi$, $\omega \to \pi\pi$, $a_1 \to \rho\pi$, and $\pi\gamma$, $\tau \to \rho\nu$, $\tau \to a_1\nu$, $\pi^0 \to \gamma\gamma$, $\omega \to \pi\gamma$, $\rho \to \pi\gamma$, $f_1 \to \rho\pi\pi$, $f_1 \to \eta\pi\pi$, $\rho \to \eta\gamma$, $\omega \to \eta\gamma$ are in good agreement with the data. The $\pi\pi$ scattering lengths and slopes have been found to be the same as obtained by Weinberg. In particular, the ρ resonance in the amplitude T_1^1 of $\pi\pi$ scattering has been obtained from this theory. Two coefficients of chiral perturbation theory have been determined and they are close to the values used in chiral perturbation theory. This theory has dynamical chiral symmetry breaking.

PACS number(s): 11.30.Rd, 13.25.−k, 13.75.Lb, 14.40.−n

Chiral symmetry is one of the most important features revealed from quantum chromodynamics (QCD). Chiral perturbation theory (CPT) is successful in describing many aspects of pseudoscalar meson physics [1,2]. It has been well known for a long time that vector meson dominance (VMD) [3] provides a fruitful mechanism in understanding the electromagnetic properties of hadrons. A chiral Lagrangian has been used to study physics of vector and axial-vector mesons before the advent of QCD [4]. Weinberg's sum rules [5] of ρ and a_1 mesons, and the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) sum rule [6] of the ρ meson are also among earlier works in this field. On the other hand, in Ref. [7] in terms of large N_c expansion 't Hooft argues that QCD is equivalent to a meson theory at low energies. In principle, all mesons should be included in this meson theory. Of course, in chiral perturbation theory the effects of other mesons have been included in the coefficients of the Lagrangian up to $O(p^4)$. As a matter of fact, in Ref. [8,9] the authors have found that the vector meson dominates the structure of the phenomenological chiral Lagrangian. Various effective theories including ρ and a_1 mesons have been studied in the past decade [10]. In Ref. [11] a new realization of chiral $SU(3)_L \times SU(3)_R$ symmetry has been proposed, in which the light vector mesons, as well as the pseudoscalar mesons, are intimately involved. The Wess-Zumino Lagrangian [12] is an important part of the effective meson theory. Witten [13] and other authors [14,15] have generalized the Wess-Zumino Lagrangian to include vector and axial-vector mesons by requiring gauge invariance. In Refs. [14,16] the generalized Wess-Zumino terms have been used to study meson physics, where abnormal parity is involved. In this paper a $U(2)_L \times U(2)_R$ chiral theory of pseudoscalar, vector, and axial-vector mesons has been studied and we try to unify the phenomenology of mesons within this theory in the chiral limit. The paper is organized as follows: (1) formalism of the theory; (2) definitions of the physical fields; (3) new mass relations between ρ, a_1, and ω, $f_1(1285)$; (4) VMD; (5) decays of $\rho \to 2\pi$ and $\omega \to 2\pi$; (6) pion form factor; (7) decays of $a_1 \to \rho\pi$ and $a_1 \to \gamma\pi$; (8) reexamination of Weinberg's sum rules; (9) decays of $\tau \to \rho\nu$ and $\tau \to a_1\nu$; (9) $\pi\pi$ scattering and determination of the coefficients of CPT; (10) decays of $\omega \to \rho\pi$, $\omega \to \gamma\pi$, $\rho \to \gamma\pi$, and $\pi^0 \to \gamma\gamma$; (11) decays of $f_1(1285)$; (12) decays of $\rho \to \eta\gamma$ and $\omega \to \eta\gamma$; (13) large N_c expansion; (14) dynamical chiral symmetry breaking; (15) derivative expansion; (16) summary of the results.

THE FORMALISM OF $U(2)_L \times U(2)_R$ CHIRAL THEORY OF MESONS

In this paper only two flavors are taken into account and we do not need to worry about the processes forbidden by the Okubo-Zweig-Iizuka (OZI) rule. The η' meson will not be discussed. Therefore, the $U(1)$ problem is not an issue of this paper. In flavor space the mesons are coupled to quarks only. The background field method is a convenient way of deriving an effective Lagrangian of mesons. The ingredients of this effective meson theory are pseudoscalar mesons (pions and u and d quark components of η), vector mesons (ρ and ω), axial-vector mesons [a_1 and $f_1(1285)$], quarks, lepton, photon, and W bosons. Using $U(2)_L \times U(2)_R$ chiral symmetry and the minimum coupling principle, the Lagrangian is constructed as

$$\begin{aligned}\mathcal{L} = &\bar{\psi}(x)[i\gamma\cdot\partial + \gamma\cdot v + e_0 Q\gamma\cdot A \\ &+\gamma\cdot a\gamma_5 - mu(x)]\psi(x) \\ &+\frac{1}{2}m_0^2(\rho_i^\mu\rho_{\mu i} + \omega^\mu\omega_\mu + a_i^\mu a_{\mu i} + f^\mu f_\mu) \\ &+\bar{\psi}(x)_L g_w \gamma\cdot W\psi(x)_L + \mathcal{L}_{\text{EM}} + \mathcal{L}_W + \mathcal{L}_{\text{lepton}}, \quad (1)\end{aligned}$$

where $a_\mu = \tau_i a_\mu^i + f_\mu$, $v_\mu = \tau_i \rho_\mu^i + \omega_\mu$, A_μ is the photon field, $Q = \frac{\tau_3}{2} + \frac{1}{6}$ is the electric charge operator of u and d

quarks, W_μ^i is the W boson, and $u = \exp\{i\gamma_5(\tau_i\pi_i + \eta)\}$; m is a parameter. In Eq. (1) u can be written as

$$u = \frac{1}{2}(1+\gamma_5)U + \frac{1}{2}(1+\gamma_5)U^\dagger, \tag{2}$$

where $U = \exp\{i(\tau_i\pi_i + \eta)\}$. Since mesons are bound states solutions of QCD they are not independent degrees of freedom. Therefore, in Eq. (1) there are no kinetic terms for meson fields. The kinetic terms of meson fields are generated from quark loops. Using the method of path integration to integrate out the quark fields, the effective Lagrangian of mesons (indicated by M) is obtained:

$$\exp\left\{i\int d^4x\mathcal{L}^M\right\} = \int[d\psi][d\bar{\psi}]\exp\left\{i\int d^4x\mathcal{L}\right\}. \tag{3}$$

The functional integral is used and the quark fields are regulated by the proper time method [17]. A review of this method has been given by Ball [18]. This integration can be done in Euclidean space (leaving out the photon and W boson first):

$$\mathcal{L}_E^M = \ln\det\mathcal{D}, \tag{4}$$

where

$$\mathcal{D} = \gamma\cdot\partial - i\gamma\cdot v - i\gamma\cdot a\gamma_5 + mu. \tag{5}$$

Equation (4) can be written in two parts:

$$\mathcal{L}_E^M = \mathcal{L}_{\rm Re} + \mathcal{L}_{\rm Im},$$
$$\mathcal{L}_{\rm Re} = \frac{1}{2}\ln\det(\mathcal{D}^\dagger\mathcal{D}), \quad \mathcal{L}_{\rm Im} = \frac{1}{2}\ln\det(\mathcal{D}/\mathcal{D}^\dagger), \tag{6}$$

where

$$\mathcal{D}^\dagger = -\gamma\cdot\partial + i\gamma\cdot v - i\gamma\cdot a\gamma_5 + m\hat{u},$$
$$\hat{u} = \exp(-i)\gamma_5(\tau_i\pi_i + \eta). \tag{7}$$

From this effective Lagrangian it can be seen that the physical processes with normal parity are described by $\mathcal{L}_{\rm Re}$ and the ones with abnormal parity are described by $\mathcal{L}_{\rm Im}$. In terms of Schwinger's proper time method [17] we have

$$\mathcal{L}_{\rm Re} = \frac{1}{2}\int d^4x\,{\rm Tr}\int_0^\infty \frac{d\tau}{\tau}e^{-\mathcal{D}^\dagger\mathcal{D}}, \tag{8}$$

where the trace is taken in color, flavor, and Lorentz space. Inserting a complete set of plane wave and subtracting the divergence at $\tau = 0$, we obtain

$$\mathcal{L}_{\rm Re} = \frac{1}{2}\frac{1}{\delta^D(0)}\int d^Dx\frac{d^Dp}{(2\pi)^D}{\rm Tr}\int_0^\infty\frac{d\tau}{\tau} \times(e^{-\tau\mathcal{D}'^\dagger\mathcal{D}'} - e^{-\Delta_0})\delta^D(x-y)|_{y\to x}, \tag{9}$$

where

$$\mathcal{D}' = \gamma\cdot\partial + i\gamma\cdot p - i\gamma\cdot v - i\gamma\cdot a\gamma_5 + mu,$$
$$\mathcal{D}'^\dagger = -\gamma\cdot\partial - i\gamma\cdot p + i\gamma\cdot v - i\gamma\cdot a\gamma_5 + m\hat{u},$$
$$\Delta_0 = p^2 + m^2. \tag{10}$$

In Ref. [18], the Seeley-DeWitt coefficients have been used to evaluate the expansion series of Eq. (9). In this paper we use dimensional regularization. After completing the integration over τ, the Lagrangian $\mathcal{L}_{\rm Re}$ reads

$$\mathcal{L}_{\rm Re} = \frac{1}{2}\int d^Dx\frac{d^Dp}{(2\pi)^D}\sum_{n=1}^\infty\frac{1}{n}\frac{1}{(p^2+m^2)^n}$$
$$\times{\rm Tr}\{(\gamma\cdot\partial - i\gamma\cdot v + i\gamma\cdot a\gamma_5)(\gamma\cdot\partial - i\gamma\cdot v - i\gamma\cdot a\gamma_5) + 2ip\cdot(\partial - iv - ia\gamma_5) + m\gamma\cdot Du\}^n, \tag{11}$$

where $\gamma\cdot Du = \gamma^\mu D_\mu u$ and

$$D_\mu u = \partial_\mu u - i[v_\mu, u] + i\{a_\mu, u\}. \tag{12}$$

To the fourth order in covariant derivatives in Minkowski space the Lagrangian takes the form

$$\begin{aligned}\mathcal{L}_{\rm Re} = {}& \frac{N_c}{(4\pi)^2}m^2\frac{D}{4}\Gamma\left(2-\frac{D}{2}\right){\rm Tr}D_\mu UD^\mu U^\dagger - \frac{1}{3}\frac{N_c}{(4\pi)^2}\frac{D}{4}\Gamma\left(2-\frac{D}{2}\right)\{2\omega_{\mu\nu}\omega^{\mu\nu} + {\rm Tr}\rho_{\mu\nu}\rho^{\mu\nu} + 2f_{\mu\nu}f^{\mu\nu} + {\rm Tr}a_{\mu\nu}a^{\mu\nu}\}\\
&+\frac{i}{2}\frac{N_c}{(4\pi)^2}{\rm Tr}\{D_\mu UD_\nu U^\dagger + D_\mu U^\dagger D_\nu U\}\rho^{\nu\mu}\\
&+\frac{i}{2}\frac{N_c}{(4\pi)^2}{\rm Tr}\{D_\mu U^\dagger D_\nu U - D_\mu UD_\nu U^\dagger\}a^{\nu\mu} + \frac{N_c}{6(4\pi)^2}{\rm Tr}D_\mu D_\nu UD^\mu D^\nu U^\dagger\\
&-\frac{N_c}{12(4\pi)^2}{\rm Tr}\{D_\mu UD^\mu U^\dagger D_\nu UD^\nu U^\dagger + D_\mu U^\dagger D^\mu UD_\nu U^\dagger D^\nu U - D_\mu UD_\nu U^\dagger D^\mu UD^\nu U^\dagger\}\\
&+\frac{1}{2}m_0^2(\omega_\mu\omega_\mu + \rho_\mu^i\rho^{i\mu} + a_\mu^i a^{i\mu} + f_\mu f^\mu),\end{aligned} \tag{13}$$

where

$$\begin{aligned}
D_\mu U &= \partial_\mu U - i[\rho_\mu, U] + i\{a_\mu, U\},\\
D_\mu U^\dagger &= \partial_\mu U^\dagger - i[\rho_\mu, U^\dagger] - i\{a_\mu, U^\dagger\},\\
\omega_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu,\\
f_{\mu\nu} &= \partial_\mu f_\nu - \partial_\nu f_\mu,\\
\rho_{\mu\nu} &= \partial_\mu \rho_\nu - \partial_\nu \rho_\mu - i[\rho_\mu, \rho_\nu] - i[a_\mu, a_\nu],\\
a_{\mu\nu} &= \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, \rho_\nu] - i[\rho_\mu, a_\nu],\\
D_\nu D_\mu U &= \partial_\nu (D_\mu U) - i[\rho_\nu, D_\mu U] + i\{a_\nu, D_\mu U\},\\
D_\nu D_\mu U^\dagger &= \partial_\nu (D_\mu U^\dagger) - i[\rho_\nu, D_\mu U^\dagger] - i\{a_\nu, D_\mu U^\dagger\}.
\end{aligned}$$

There is a correspondence between the schemes of regularization used in this paper and in Ref. [18]. Using this correspondence and transforming the formalism of Ref. [18] to Minkowski space, it can be found that except for the mass terms this formalism [Eq. (13)] is the same as the one presented in Ref. [18].

The imaginary Lagrangian [Eq. (6)] describes the physical processes with abnormal parity. The generalized Wess-Zumino Lagrangian should be derived from $\mathcal{L}_{\rm Im}$. The mass terms of the vector and axial-vector fields are part of $\mathcal{L}_{\rm Re}$ [Eq. (13)] therefore, if we take the vector and axial-vector fields as gauge fields technically $\mathcal{L}_{\rm Im}$ would be locally $U(2)_L \times U(2)_R$ gauge invariant. We can compute $\mathcal{L}_{\rm Im}$ without the vector and axial-vector fields first, then add these fields into $\mathcal{L}_{\rm Im}$ by requiring gauge invariance as in Ref. [13]. Differentiating $\mathcal{L}_{\rm Im}$ [Eq. (6)] and inserting a complete set of plane waves we obtain

$$\delta\mathcal{L}_{\rm Im} = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}{\rm Tr}\{[p^2+m^2-\partial^2-2ip\cdot\partial - m(\gamma\cdot\partial u)]^{-1}D^\dagger\delta D - [p^2+m^2-\partial^2-2ip\cdot\partial + m(\gamma\cdot\partial\hat{u})]^{-1}D\delta D^\dagger\},$$

where D and $D^\dagger$ are given in Eqs. (10) without the vector and axial-vector fields. From this formula an expansion which is similar to Eq. (11) is obtained as

$$\delta\mathcal{L}_{\rm Im} = \frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\sum_{n=0}^{\infty}\frac{1}{(p^2+m^2)^{n+1}}{\rm Tr}\{[\partial^2+2ip\cdot\partial + m(\gamma\cdot\partial u)]^n(-\gamma\cdot\partial - i\gamma\cdot p + m\hat{u})m\delta u - [\partial^2+2ip\cdot\partial - m(\gamma\cdot\partial\hat{u})]^n(\gamma\cdot\partial + i\gamma\cdot p - mu)m\delta\hat{u}\}.$$

Nonzero terms come from $n = 4$ and in Minkowski space we have

$$\delta\mathcal{L}_{\rm Im} = \frac{-N_c}{48\pi^2}\varepsilon^{\mu\nu\alpha\beta}{\rm Tr}\partial_\mu UU^\dagger\partial_\nu UU^\dagger\partial_\alpha UU^\dagger\partial_\beta UU^\dagger\delta UU^\dagger.$$

In order to finish this variation a new parameter τ has to be introduced and U is a function of x, y, z, t, and τ. When $\tau = 1$, U becomes physical. The expression of $\mathcal{L}_{\rm Im}$ at $n = 4$ is just the form of the Wess-Zumino Lagrangian given by Witten [13]:

$$\mathcal{L}_{\rm Im} = \frac{iN_c}{240\pi^2}\int d^5x\varepsilon^{\mu\nu\alpha\beta\lambda}{\rm Tr}\partial_\mu UU^\dagger\partial_\nu UU^\dagger\partial_\alpha UU^\dagger\partial_\beta UU^\dagger\partial_\lambda UU^\dagger.$$

As pointed out by Witten [13], the boundary of the integral is a five-dimensional disk and the Minkowski space is the boundary of the disk. Following Ref. [13] the vector and axial-vector fields can be added to the Lagrangian by trial and error. In Ref. [14] the Bardeen form of the anomaly [19] has been accepted and an arbitrary constant in their formula has been chosen to be one. The authors of Ref. [14] claim that their Wess-Zumino Lagrangian with spin-1 fields agrees with Witten's expression [13] except for an inadvertently omitted term. Therefore, the Wess-Zumino Lagrangian with spin-1 fields is obtained from the Lagrangian [Eq. (1)] and it is the leading term of $\mathcal{L}_{\rm Im}$ in derivative expansion.

All the vertices of the meson physical processes of normal parity can be found from $\mathcal{L}_{\rm Re}$ [Eq. (13)] and all the vertices of abnormal parity can be derived from $\mathcal{L}_{\rm Im}$. $\mathcal{L}_{\rm Re}$ and $\mathcal{L}_{\rm Im}$ are derived from the same Lagrangian [Eq. (1)]. On the other hand, the meson fields are coupled to corresponding bilinear quark fields in the Lagrangian [Eq. (1)] and both the meson vertices of normal parity and abnormal parity can be found by bosonizing the bilinear quark fields of these couplings in terms of the quark propagator determined by Eq. (1). In this paper both methods obtaining the vertices will be employed and it will be shown that the vertices obtained by different methods are the same.

DEFINING PHYSICAL MESON FIELDS

In Eq. (13) there are divergences. The theory studied in this paper is an effective theory and it is not renormalizable. In order to build a physical effective meson theory, the introduction of a cutoff to the theory is necessary and the cutoff will be determined in this paper. We define

$$\frac{F^2}{16} = \frac{N_c}{(4\pi)^2}m^2\frac{D}{4}\Gamma\left(2-\frac{D}{2}\right), \tag{14}$$

$$g^2 = \frac{8}{3}\frac{N_c}{(4\pi)^2}\frac{D}{4}\Gamma\left(2-\frac{D}{2}\right) = \frac{1}{6}\frac{F^2}{m^2}. \tag{15}$$

The relationship between the cutoff and F^2, g will be explored. From the kinetic terms of the meson fields in Eq. (13) we can see that they are not physical. The physical meson fields can be defined in the following way that makes the corresponding kinetic terms in the standard form:

$$\pi \to \frac{2}{f_\pi}\pi, \quad \eta \to \frac{2}{f_\eta}\eta,$$
$$\rho \to \frac{1}{g}\rho, \quad \omega \to \frac{1}{g}\omega, \tag{16}$$

where f_π and f_η are pion and η decay constants, and in the chiral limit we take $f_\pi = f_\eta$. Use of these substitutions leads to the physical masses of ρ and ω mesons

$$m_\rho^2 = m_\omega^2 = \frac{1}{g^2}m_0^2. \tag{17}$$

We can also make a transformation to a_1 and f_1 fields:

$$a_\mu^i \to \frac{1}{g}a_\mu^i, \quad f_\mu \to \frac{1}{g}f_\mu. \tag{18}$$

However, there are other factors for the normalizations of axial-vector fields. In Eq. (13) there is mixing between a_μ^i and $\partial_\mu \pi_i$, f_μ and $\partial_\mu \eta$. In the chiral limit the mixing

$$\frac{F^2}{2g}\partial_\mu \pi_i a_i^\mu$$

comes from the first term of Eq. (13). The transformation

$$a_\mu^i \to a_\mu^i - c\partial_\mu \pi^i \tag{19}$$

is used to erase the mixing. In the chiral limit, c is determined by canceling the mixing term

$$c = \frac{\frac{F^2}{2g}}{m_\rho^2 + \frac{F^2}{g^2}}. \tag{20}$$

There is a similar mixing term between f_μ and $\partial_\mu \eta$ and the transformation

$$f_\mu \to f_\mu - c\partial_\mu \eta$$

is used to cancel the mixing term. In the chiral limit, c of this formula is the same as in Eq. (20). From the term

$$\frac{N_c}{6(4\pi)^2}\mathrm{Tr}D_\nu D_\mu U D^\nu D^\mu U^\dagger$$

of the Lagrangian [Eq. (13)], in the chiral limit another term related to the normalization of a_μ^i field is found, which can be written as

$$\frac{1}{8\pi^2 g^2}(\partial_\mu a_\nu^i - \partial_\nu a_\mu^i)(\partial^\mu a^{i\nu} - \partial^\nu a^{i\mu}). \tag{21}$$

By combining this term with the kinetic term of a_μ^i in Eq. (13), the physical a_μ^i field is defined as

$$a_\mu^i \to \frac{1}{g}\left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}} a_\mu^i - \frac{c}{g}\partial_\mu \pi^i. \tag{22}$$

In the same way, we obtain the physical f_μ field

$$f_\mu \to \frac{1}{g}\left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}} f_\mu - \frac{c}{g}\partial_\mu \eta. \tag{23}$$

After the transformations [Eq. (19)], in order to make the pion kinetic term in the standard form, in the chiral limit the following equation must be satisfied:

$$\frac{F^2}{8}\left(1 - \frac{2c}{g}\right)^2 + \frac{1}{2}m_\rho^2 c^2 = \frac{f_\pi^2}{8}. \tag{24}$$

Equation (24) makes the kinetic term of the η meson field in standard form too. Equations (20), (24) can be simplified as

$$c = \frac{f_\pi^2}{2gm_\rho^2}, \tag{25}$$

$$\frac{F^2}{f_\pi^2}\left(1 - \frac{2c}{g}\right) = 1. \tag{26}$$

NEW MASS FORMULAS OF VECTOR MESONS AND ITS CHIRAL PARTNERS

In the Lagrangian [Eq. (1)] the vector mesons and axial-vector mesons are chiral partners. However, it can be seen from Eq. (22) that vector and axial-vector meson fields behave differently. Because of Eqs. (22), (23), in the couplings of axial-vector fields to others there is an additional factor $(1 - 1/2\pi^2 g^2)^{-1/2}$.

The physical masses of vector mesons are defined by Eq. (17). For the masses of axial-vector mesons there are three contributors: the mass term in the Lagrangian [Eq. (1)]; the contribution of the first term of the Lagrangian [Eq. (13)], which is $\frac{F^2}{g^2}$; and the normalization factor $(1 - \frac{1}{2\pi^2 g^2})^{-1}$. By putting all these three factors together we find the a_1 mass to be

$$\left(1 - \frac{1}{2\pi^2 g^2}\right) m_a^2 = \frac{F^2}{g^2} + m_\rho^2. \tag{27}$$

In the same way, we obtain the mass formula of the f_1 meson:

$$\left(1 - \frac{1}{2\pi^2 g^2}\right) m_f^2 = \frac{F^2}{g^2} + m_\omega^2. \tag{28}$$

If we ignore the mass difference of ρ and ω mesons from these two mass formulas we obtain

$$m_f = m_a. \tag{29}$$

The deviation of this relation from physical values ($m_a =$ 1.26 GeV and $m_f = 1.285$ GeV) is about 2%.

In the chiral limit, there are three parameters in this theory, which can be chosen as g, f_π, and m_ρ. We could take f_π and m_ρ as input and choose

$$g = 0.35 \tag{30}$$

to get a better fit. The couplings in all the physical processes described by $\mathcal{L}_{\mathrm{Re}}$ and $\mathcal{L}_{\mathrm{Im}}$ are fixed by g and c. This is the universality of coupling in this theory.

VECTOR MESON DOMINANCE (VMD)

The formalism of VMD can be derived from the Lagrangian [Eq. (1)]. Before this paper the Nambu–Jona-Lasinio Lagrangian [20] was employed to simulate many properties of VMD [21] and in Ref. [22] a linear σ model was used to unify many properties of the VMD by computing various constituent quark loops and working in the chiral limit. In these studies [21,22] the vector mesons and pions are coupled to the quarks and these couplings play the key role in obtaining VMD. The Lagrangian [Eq. (1)] of this paper is different from Refs. [21,22]. In this theory the mesons are coupled to the quarks too and the VMD is expected in this theory by the same reason as in Refs. [21,22]. Comparing with Refs. [21,22], the couplings between the mesons and quarks are determined in a different way.

From Eq. (1) it can be seen that except for the kinetic term of the photon, photon and vector mesons always appear in the combinations

$$\frac{1}{g}\rho^0_\mu + \frac{1}{2}eA_\mu, \quad \frac{1}{g}\omega_\mu + \frac{1}{6}eA_\mu. \tag{31}$$

Therefore, the interaction of the photon with other fields can be found from the interactions of ρ^0 or ω with other fields by using the substitutions

$$\rho^0_\mu \to \frac{1}{2}egA_\mu,$$
$$\omega_\mu \to \frac{1}{6}egA_\mu. \tag{32}$$

Incorporating the photon field into the Lagrangian [Eq. (13)], from the kinetic terms of ρ^0 and ω mesons in the Lagrangian we obtain

$$-\frac{1}{4}\left\{\partial_\mu\left(\rho^0_\nu + \frac{1}{2}e_0gA_\nu\right) - \partial_\nu\left(\rho^0_\mu + \frac{1}{2}e_0gA_\mu\right)\right\}^2,$$
$$-\frac{1}{4}\left\{\partial_\mu\left(\omega_\nu + \frac{1}{6}e_0gA_\nu\right) - \partial_\nu\left(\omega_\mu + \frac{1}{6}e_0gA_\mu\right)\right\}^2. \tag{33}$$

In order to make the kinetic term of the photon field in standard form, it is necessary to redefine the photon field and the charge to be

$$A_\mu \to \left(1 + \frac{5e_0^2g^2}{18}\right)^{-\frac{1}{2}} A_\mu, \quad e_0 \to e\left(1 + \frac{5e_0^2g^2}{18}\right)^{\frac{1}{2}},$$

$$e_0A_\mu \to eA_\mu. \tag{34}$$

From Eq. (33) the couplings between the photon and vector mesons are derived:

$$-\frac{1}{2}\frac{e}{f_\rho}F_{\mu\nu}(\partial_\mu\rho_\nu - \partial_\nu\rho_\mu),$$
$$-\frac{1}{2}\frac{e}{f_\omega}F_{\mu\nu}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu), \tag{35}$$

where

$$\frac{1}{f_\rho} = \frac{1}{2}g, \quad \frac{1}{f_\omega} = \frac{1}{6}g. \tag{36}$$

The ratio of $\frac{1}{f_\rho}$ to $\frac{1}{f_\omega}$ is $1:\frac{1}{3}$, the same as the quark model. The comparison between theoretical and experimental values of f_ρ and f_ω can be found in Table I. The photon-vector meson couplings shown by Eqs. (35) are just the ones proposed in Ref. [23]. On the other hand, there are interactions between ρ and ω mesons with other mesons; in general, these interactions can be written as

$$\rho^{i\mu}j^i_\mu + \omega^\mu j^\omega_\mu.$$

Therefore, in addition to the direct coupling of the photon and ρ meson [Eq. (35)] another type of interaction between photons and other mesons can be found by the substitution [Eq. (32)]

$$\frac{e}{f_\rho}A^\mu j^0_\mu + \frac{e}{f_\omega}A_\mu j^\omega_\mu.$$

The complete expression of the interaction between the isovector photon and mesons is

$$\frac{e}{f_\rho}\left\{-\frac{1}{2}F^{\mu\nu}(\partial_\mu\rho^0_\nu - \partial_\nu\rho^0_\mu) + A^\mu j^0_\mu\right\}. \tag{37}$$

This is the exact expression of VMD proposed by Sakurai [3]. In the same way the isoscalar VMD is obtained:

$$\frac{e}{f_\omega}\left\{-\frac{1}{2}F^{\mu\nu}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu) + A^\mu j^\omega_\mu\right\}. \tag{38}$$

The reason leading to the explicit expression of VMD in this theory can be manifested in another way. In the Lagrangian [Eq. (1)] the vector mesons are coupled to the quark vector currents. As mentioned above, the meson vertex obtained from $\mathcal{L}_{\rm Re}$ can be obtained from the bosonization of the quark vector currents. It is well known that in QCD the electric current takes the form of

$$\bar\psi Q\gamma_\mu\psi = \frac{1}{2}\bar\psi\tau_3\gamma_\mu\psi + \frac{1}{6}\bar\psi\gamma_\mu\psi$$

which is in the Lagrangian [Eq. (1)]. The electric current $\bar\psi Q\gamma_\mu\psi$ can be bosonized in this theory. In Ref. [24] we have developed a method to find the effective currents in the case that only pseudoscalar fields are taken as background fields. In this paper this method is generalized to include vector and axial-vector mesons. From the Lagrangian [Eq. (1)] the equation satisfied by the quark propagator is obtained:

$$\left\{i\gamma\cdot\partial + \frac{1}{g}\gamma\cdot v(x) + \gamma\cdot a(x)\gamma_5 - mu(x)\right\}s_F(x,y) = \delta^4(x-y), \tag{39}$$

where a_μ is defined by Eqs. (22), (23). In the momentum picture we have

$$s_F(x,y) = \frac{1}{(2\pi)^4}\int d^4y e^{-ip(x-y)}s_F(x,p). \tag{40}$$

Equation (39) becomes

$$\left\{i\gamma\cdot\partial+\gamma\cdot p+\frac{1}{g}\gamma\cdot v(x)+\gamma\cdot a(x)\gamma_5-mu(x)\right\}\times s_F(x,p)=1. \quad (41)$$

Equation (41) is solved

$$s_F(x,p)=s_F^0\sum_{n=0}^{\infty}(-)^n\left(\left\{i\gamma\cdot\partial+\frac{1}{g}\gamma\cdot v(x)+\gamma\cdot a(x)\gamma_5\right\}s_F^0\right)^n, \quad (42)$$

where

$$s_F^0=-\frac{\gamma\cdot p-m\hat{u}}{p^2-m^2}. \quad (43)$$

In terms of Eqs. (40), (42) the bosonization of quark electric current is done in the following way:

$$\langle\bar\psi(x)Q\gamma_\mu\psi(x)\rangle=\frac{1}{2}\langle\bar\psi(x)\tau_3\gamma_\mu\psi(x)\rangle+\frac{1}{6}\langle\bar\psi(x)\gamma_\mu\psi(x)\rangle,$$

$$\langle\bar\psi(x)\tau_3\gamma_\mu\psi(x)\rangle=-i\,\mathrm{Tr}\tau_3\gamma_\mu s_F(x,x),$$

$$\langle\bar\psi(x)\gamma_\mu\psi(x)\rangle=-i\,\mathrm{Tr}\gamma_\mu s_F(x,x). \quad (44)$$

The leading terms of Eq. (44) are obtained at $n=3$ and the effective currents take the forms

$$\langle\bar\psi\tau_3\gamma_\mu\psi\rangle=g\partial^2\rho_\mu^0+gj_\mu^0,\quad \langle\bar\psi\gamma_\mu\psi\rangle=g\partial^2\omega_\mu+gj_\mu^\omega. \quad (45)$$

In obtaining the first term of Eqs. (45), Eq. (15) has been used. In Eq. (45) j_μ^0 and j_μ^ω have been defined as the rest parts of the currents and j_μ^ω will be evaluated explicitly below. From Eq. (1) it can be seen that j_μ^0 couples to ρ_μ^0 and j_μ^ω couples to ω. Therefore, these two currents are the currents mentioned in Eqs. (37), (38). After getting rid of total derivative terms we have

$$eA^\mu\langle\bar\psi Q\gamma_\mu\psi\rangle=\frac{1}{f_\rho}\left\{-\frac{1}{2}F^{\mu\nu}(\partial_\mu\rho_\nu-\partial_\nu\rho_\mu)+A^\mu j_\mu^0\right\}+\frac{1}{f_\omega}\left\{-\frac{1}{2}F^{\mu\nu}(\partial_\mu\omega_\nu-\partial_\nu\omega_\mu)+A^\mu j_\mu^\omega\right\}. \quad (46)$$

This is the same as Eqs. (37), (38).

THE DECAYS OF $\rho\to\pi\pi$, $\omega\to\pi\pi$, AND KSFR SUM RULE

In this theory the pion is associated with γ_5 [see Eq. (1)] and only even numbers of γ_5 involved in $\rho\pi\pi$ vertex

TABLE I. Summary of the results.

	Experimental	Theoretical
f_π	186 MeV	Input
m_ρ	769.9±0.8 MeV	Input
m_π	138 MeV	input
m_η	547.45±0.19 MeV	Input
g		0.35 Input
m_ω	781.94±0.12 MeV	770 MeV
m_a	1230±40 MeV	1389 MeV
m_{f_1}	1282 ± 5 MeV	1389 MeV
π form factor	Consistent with ρ pole	ρ pole
Radius of π	0.663±0.023 fm	0.63 fm
$g_{\rho\gamma}$	0.116(1 ± 0.05) GeV^2	0.104 GeV^2
$g_{\omega\gamma}$	0.0359(1 ± 0.03) GeV^2	0.0357 GeV^2
$\Gamma(\rho\to\pi\pi)$	151.2 ± 1.2 MeV	135 MeV
$\Gamma(\omega\to\pi\pi)$	0.186(1 ± 0.15) MeV	0.136 MeV
$\Gamma(a_1\to\rho\pi)$	~400 MeV	325 MeV
$\Gamma(a_1\to\gamma\pi)$	(640±246) keV	252 keV
$\frac{d}{s}(a_1\to\rho\pi)$	−0.11 ± 0.02	−0.097
$\Gamma(\tau\to a_1\nu)$	$(2.42\pm0.76)\times10^{-13}$ GeV	1.56×10^{-13} GeV
$\Gamma(\tau\to\rho\nu)$	$(0.495\pm0.023)\times10^{-12}$ GeV	4.84×10^{-13} GeV
$\Gamma(\pi^0\to\gamma\gamma)$	7.74(1 ± 0.072)eV	7.64 eV
a (form factor of $\pi^0\to\gamma\gamma$)	0.032±0.004	0.03
$\Gamma(\omega\to\pi\gamma)$	717(1±0.07) keV	724 keV
$\Gamma(\rho\to\pi\gamma)$	68.2(1±0.12) keV	76.2 keV
$\Gamma(\omega\to\pi\pi\pi)$	7.43(1± 0.02) MeV	5 MeV
$\Gamma(f_1\to\rho\pi\pi)$	6.96(1±0.33) MeV	6.01 MeV
$B(f_1\to\eta\pi\pi)$	(10^{+7}_{-6})%	1.15×10^{-3}
$\Gamma(f_1\to\gamma\pi\pi)$		18.5 keV
$B(\rho\to\gamma\eta)$	$(3.8\pm0.7)\times10^{-4}$	3.04×10^{-4}
$B(\omega\to\gamma\eta)$	$(8.3\pm2.1)\times10^{-4}$	6.96×10^{-4}

occur. Therefore, the vertex of $\rho\pi\pi$ can be found from Eq. (13), which is

$$\mathcal{L}_{\rho\pi\pi} = \frac{2}{g}\epsilon_{ijk}\rho_i^\mu \pi_j \partial_\mu \pi_k + \frac{2}{\pi^2 g f_\pi^2}\left[4\pi^2 c^2 - \left(1-\frac{2c}{g}\right)^2\right] \times \epsilon_{ijk}\rho_i^\mu \partial_\nu \pi_j \partial_{\mu\nu}\pi_k. \quad (47)$$

In deriving Eq. (47), Eq. (26) has been used. For the decay of $\rho \to \pi\pi$ the mesons are on mass shell and in the chiral limit Eq. (47) becomes

$$\mathcal{L}_{\rho\pi\pi} = f_{\rho\pi\pi}\epsilon_{ijk}\rho_i^\mu \pi_j \partial_\mu \pi_k,$$

$$f_{\rho\pi\pi} = \frac{2}{g}\left\{1 + \frac{m_\rho^2}{2\pi^2 f_\pi^2}\left[\left(1-\frac{2c}{g}\right)^2 - 4\pi^2 c^2\right]\right\}. \quad (48)$$

The choice of $g = 0.35$ makes

$$f_{\rho\pi\pi} = \frac{2}{g}. \quad (49)$$

Using Eq. (48) we obtain

$$\Gamma(\rho \to \pi\pi) = \frac{f_{\rho\pi\pi}^2}{48\pi} m_\rho \left(1 - \frac{4m_\pi^2}{m_\rho^2}\right)^{\frac{3}{2}} = 135 \text{ MeV}. \quad (50)$$

The experimental value is 151 MeV and the deviation is about 10%. The KSFR sum rule [6]

$$g_{\rho\gamma} = \frac{1}{2} f_{\rho\pi\pi} f_\pi^2 \quad (51)$$

is the result of current algebra and PCAC (partial conservation of axial vector current). From Eq. (35) we have

$$g_{\rho\gamma} = \frac{1}{2} g m_\rho^2. \quad (52)$$

Substituting Eqs. (49), (52) into the KSFR sum rule we obtain

$$g^2 = 2\frac{f_\pi^2}{m_\rho^2}, \quad g = 0.342. \quad (53)$$

Comparing the value of g with Eq. (30) it can be seen that the KSFR sum rule is satisfied reasonably well.

In the chiral limit, the mixing between ω and ρ is caused by the electromagnetic interaction. The Lagrangian of this mixing is

$$\mathcal{L}_i = -\frac{1}{4} e g F^{\mu\nu}(\partial_\mu \rho_\nu - \partial_\nu \rho_\mu) - \frac{1}{12} e g F^{\mu\nu}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu). \quad (54)$$

The mixing angle is found from Eq. (54):

$$\sin 2\theta = \frac{\frac{2\pi}{3}\alpha g^2 \bar{m}_\rho^2}{m_\omega^2 - m_\rho^2}, \quad \theta = 1.74°, \quad (55)$$

where $\bar{m}_\rho^2 = \frac{1}{2}(m_\rho^2 + m_\omega^2)$. The decay width is

$$\Gamma(\omega \to \pi\pi) = \sin^2\theta \Gamma(\rho \to \pi\pi) \frac{p^{*3}}{p^3} \frac{m_\rho^2}{m_\omega^2} = 0.136 \text{ MeV}, \quad (56)$$

where p^* is the momentum of the pion when the mass of ρ is m_ω and p is the momentum of the pion when the mass of ρ is really m_ρ. The experimental value is $0.186(1 \pm 0.15)$ MeV.

PION FORM FACTOR

According to VMD [Eq. (37)], the vertex of $\pi\pi\gamma$ consists of two parts: direct coupling and indirect coupling through a ρ meson. The Lagrangian of direct coupling can be found either from Eq. (13) or by substituting $\rho^0 \to \frac{e}{f_\rho} A$ in Eq. (48):

$$\mathcal{L}_{\pi\pi\gamma} = e\frac{f_{\rho\pi\pi}}{f_\rho}\epsilon_{3jk} A^\mu \pi_j \partial_\mu \pi_k. \quad (57)$$

Because of the coupling $-\frac{1}{2}\frac{e}{f_\rho}F_{\mu\nu}(\partial^\mu \rho^\nu - \partial^\nu \rho^\mu)$ the indirect coupling of $\pi\pi\gamma$ is proportional to q^2 (q is the photon momentum); therefore the charge normalization of π^+ is satisfied by $\frac{f_{\rho\pi\pi}}{f_\rho} = 1$. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\frac{e}{f_\rho}\{F^{\mu\nu}(\partial_\mu \rho_\nu - \partial_\nu \rho_\mu)\} + \mathcal{L}_{\pi\pi\gamma} + \mathcal{L}_{\rho\pi\pi}. \quad (58)$$

The pion form factor is found to be

$$F_\pi(q^2) = \frac{1}{1 - \frac{q^2}{m_\rho^2}}. \quad (59)$$

VMD results in the ρ pole in the pion form factor [25]. The radius of the pion is

$$\sqrt{\langle r^2 \rangle_\pi} = 0.63 \text{ fm}. \quad (60)$$

The experimental value is 0.663 ± 0.023 fm [26].

DECAYS OF $A_1 \to \rho\pi$ AND $\pi\gamma$

The decay $a_1 \to \rho\pi$ is a process with normal parity. In the chiral limit, this vertex is derived from Eq. (13):

$$\mathcal{L}_{a_1 \to \rho\pi} = \epsilon_{ijk}\{A a_i^\mu \rho_{j\mu} \pi_k + B a_i^\mu \rho_j^\nu \partial_{\mu\nu} \pi_k\},$$

$$A=\frac{2}{f_\pi}\left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}\left\{\frac{F^2}{g^2}+\frac{m_a^2}{2\pi^2g^2}-\frac{2c}{g}(p_\pi\cdot p_\rho+p_\pi\cdot p_a)-\frac{3}{2\pi^2g^2}\left(1-\frac{2c}{g}\right)p_\pi\cdot p_\rho\right\}$$
$$=\frac{2}{f_\pi}\left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}\left\{\frac{F^2}{g^2}+\frac{m_a^2}{2\pi^2g^2}-\left[\frac{2c}{g}+\frac{3}{4\pi^2g^2}\left(1-\frac{2c}{g}\right)\right](m_a^2-m_\rho^2)\right\}$$
$$=\frac{2}{f_\pi}\left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}(m_a^2-m_\rho^2)\left(1-\frac{2c}{g}\right)\left(1-\frac{3}{4\pi^2g^2}\right),$$

$$B=-\frac{2}{f_\pi}\left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}\frac{1}{2\pi^2g^2}\left(1-\frac{2c}{g}\right). \tag{61}$$

The three expressions of A in Eq. (61) have different uses in this paper and in obtaining the last two expressions of A, Eq. (27) has been used. The width of the decay is calculated to be 326 MeV which is comparable with data [27]. From Eq. (61) it can be seen that there are s wave and d wave in this decay. The ratio [28] of these two waves obtained in this theory is

$$\frac{s}{d}=-\frac{1}{3}p_\pi^2\frac{\frac{A}{m_\rho(m_\rho+E_\rho)}-B\frac{m_a}{m_\rho}}{A(1+\frac{1}{3}\frac{p_\pi^2}{m_\rho(m_\rho+E_\rho)})-\frac{B}{3}\frac{m_a}{m_\rho}p_\pi^2}=-0.097. \tag{62}$$

The quark model [29] predicts that $\frac{d}{s}=-0.15$, while the experimental value is -0.11 ± 0.02 [30].

The vertex of $a_1\to\pi\gamma$ is obtained by substituting Eq. (32) in Eq. (61):

$$\mathcal{L}_{a_1\to\pi\gamma}=-\epsilon_{3jk}\{Aa_j^\mu A_\mu\pi_k+Ba_j^\mu A^\nu\partial_{\mu\nu}\pi_k\},$$
$$A=\frac{2}{f_\pi}\left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}\left\{\frac{F^2}{g^2}+\frac{m_a^2}{2\pi^2g^2}-\left[\frac{2c}{g}+\frac{3}{4\pi^2g^2}\left(1-\frac{2c}{g}\right)\right](m_a^2-q^2)\right\}, \tag{63}$$

where q is the photon momentum and A is obtained from Eq. (61). Before presenting the numerical result, we will prove current conservation in the case of the real photon. In order to have current conservation the following equation should be satisfied:

$$A(q^2=0)=\frac{1}{2}m_a^2B. \tag{64}$$

The left-hand side of this equation can be written as

$$\frac{2}{f_\pi}\left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}\left\{\left(1-\frac{2c}{g}\right)\left(1-\frac{1}{2\pi^2g^2}\right)m_a^2-m_\rho^2-\frac{1}{4\pi^2g^2}\left(1-\frac{2c}{g}\right)m_a^2\right\}. \tag{65}$$

By using the mass formula [Eq. (27)] and the expression of c [Eq. (25)], it can be found that the left hand of the equation [Eq. (65)] is just $\frac{1}{2}m_a^2B$. Therefore, current conservation in the process of $a_1\to\pi\gamma$ is satisfied. The decay width of $a_1\to\pi\gamma$ is computed to be 252 keV. The experimental value is 640 ± 246 keV [31].

Using $\mathcal{L}_{\rm Re}$ the decay width of $a_1\to3\pi$ can be calculated. It is found that the branching ratio is 5×10^{-4} which is consistent with data 0.003 ± 0.003 [32].

REEXAMINATION OF WEINBERG'S SUM RULES

From chiral symmetry, current algebra, and VMD Weinberg has found the first sum rule [5]:

$$\frac{g_\rho^2}{m_\rho^2}-\frac{g_a^2}{m_a^2}=\frac{1}{4}f_\pi^2, \tag{66}$$

where

$$\left\langle 0\left|\bar\psi\frac{\tau_i}{2}\gamma_\mu\psi\right|\rho_\lambda^j\right\rangle=g_\rho\epsilon_\mu^\lambda\delta_{ij},$$
$$\left\langle 0\left|\bar\psi\frac{\tau_i}{2}\gamma_\mu\gamma_5\psi\right|a_\lambda^j\right\rangle=g_a\epsilon_\mu^\lambda\delta_{ij}. \tag{67}$$

Assuming an additional condition [5], Weinberg's second sum rule has been derived:

$$g_a=g_\rho. \tag{68}$$

Equations (66), (68) and the KSFR sum rule together lead to $m_a^2=2m_\rho^2$ which is not in good agreement with the present value of m_a. The theory presented in this paper can be considered as a realization of chiral symmetry, current algebra, and VMD. In this theory, the isovector vector and isovector axial-vector currents in Eq. (1) are the same as the ones in Eqs. (67). Therefore, g_ρ and g_a can be evaluated explicitly. Using Eqs. (40), (42) the following expressions are found:

$$\left\langle\bar\psi\frac{\tau_i}{2}\gamma_\mu\gamma_5\psi\right\rangle=-\frac{1}{2}gm_\rho^2\left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}a_\mu^i+\cdots,$$
$$\left\langle\bar\psi\frac{\tau_i}{2}\gamma_\mu\psi\right\rangle=-\frac{1}{2}gm_\rho^2\rho_\mu^i+\cdots. \tag{69}$$

We obtain

$$g_\rho = -\frac{1}{2} g m_\rho^2,$$
$$g_a = -\frac{1}{2} g m_\rho^2 \left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}}. \tag{70}$$

The relation (68) is not satisfied and $m_a^2 = 2m_\rho^2$ is not confirmed by this theory. This is the reason why the mass relation [Eq. (27)] obtained in this paper is not the same as the one of Ref. [5]. However, Weinberg's first sum rule [Eq. (66)] only depends on chiral symmetry, VMD, and current algebra. Therefore, it should be achieved in this theory. Substituting g_ρ and g_a [Eq. (70)] into the left-hand side of Eq. (66), we obtain

$$\frac{g^2}{4} m_\rho^2 \left\{1 - \frac{m_\rho^2}{m_a^2}\left(1 - \frac{1}{2\pi^2 g^2}\right)^{-1}\right\}. \tag{71}$$

Substituting the mass formula [Eq. (27)] and Eq. (26) into Eq. (71), indeed Weinberg's first sum rule is satisfied. The factor $(1 - \frac{1}{2\pi^2 g^2})^{-\frac{1}{2}}$ plays an important role in this theory.

In Ref. [33] two new formulas for the amplitude of $a_1 \to \rho\pi$ in the limit of $p_\pi \to 0$ have been obtained from the Ward identity found by Weinberg [5]:

$$g_a f_\pi A(m_\rho^2) = 2 g_\rho (m_a^2 - m_\rho^2),$$
$$g_\rho f_\pi A(m_a^2) = 2 g_a (m_a^2 - m_\rho^2), \tag{72}$$

where A is the amplitude of $a_1 \to \rho\pi$ in the limit of $p_\pi \to 0$. It needs to be checked if these two relations are satisfied in this theory. In the limit of $p_\pi = 0$ the amplitude A [Eq. (61)] of $a_1 \to \rho\pi$ can be written as

$$A(k^2) = \frac{2}{f_\pi}\left\{\frac{F^2}{g^2} + \frac{k^2}{2\pi^2 g^2}\right\}\left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}}. \tag{73}$$

Using Eq. (73) and the mass formula [Eq. (27)], the two relations [Eq. (72)] are indeed satisfied.

THE DECAYS OF $\tau \to \rho\nu$ AND $a_1\nu$

The decay widths of $\tau \to \rho\nu$ and $a_1\nu$ can be calculated in terms of the two matrix elements (67), (70)

$$\Gamma(\tau \to \rho\nu) = \frac{G^2}{8\pi}\cos^2\theta g_\rho^2 \frac{m_\tau^3}{m_\rho^2}\left(1 - \frac{m_\rho^2}{m_\tau^2}\right)^2\left(1 + 2\frac{m_\rho^2}{m_\tau^2}\right)$$
$$= 4.84 \times 10^{-13}\ \text{GeV},$$

$$\Gamma(\tau \to a_1\nu) = \frac{G^2}{8\pi}\cos^2\theta g_a^2 \frac{m_\tau^3}{m_a^2}\left(1 - \frac{m_a^2}{m_\tau^2}\right)^2\left(1 + 2\frac{m_a^2}{m_\tau^2}\right)$$
$$= 1.56 \times 10^{-13}\ \text{GeV}. \tag{74}$$

The experimental values are $(0.495 \pm 0.023)10^{-12}$ GeV and $(2.42 \pm 0.76)10^{-13}$ GeV for $\tau \to \rho\nu$ and $\tau \to a_1\nu$, respectively.

These calculations can also be done in terms of the effective Lagrangian [Eq. (13)] of mesons, in which the couplings between the mesons and W bosons are determined. In the chiral limit, the $\pi - W$ coupling is found from the first term of the Lagrangian [Eq. (13)]:

$$-\frac{F^2}{4f_\pi}\left(1 - \frac{2c}{g}\right) g_w \partial_\mu \pi_i W_i^\mu = -\frac{f_\pi}{4} g_w \partial_\mu \pi_i W_i^\mu. \tag{75}$$

From π_{l2} decay, f_π is determined to be 186 MeV.

Like the photon, from the Lagrangian [Eq. (1)] it can be seen that W bosons always appear in the combinations either $\rho_\mu + \frac{g_w}{4} g(\tau_1 W_\mu^1 + \tau_2 W_\mu^2)$ or $a_\mu - \frac{g_w}{4} g_a'(\tau_1 W_\mu^1 + \tau_2 W_\mu^2)$. The W boson fields of Eq. (1) need to be normalized:

$$W \to \left(1 + 2g^2 \frac{g_w^2}{4}\right)^{-\frac{1}{2}} W,$$

$$g_w \to \left(1 + 2g^2 \frac{g_w^2}{4}\right)^{\frac{1}{2}} g_w, \quad g_w W_\mu \to g_w W_\mu.$$

Like VMD, the coupling of $\rho - W$ is obtained:

$$-\frac{g_w}{4}\frac{1}{2} g(\partial_\mu \rho_\nu^i - \partial_\nu \rho_\mu^i)(\partial^\mu W^{i\nu} - \partial^\nu W^{i\mu}). \tag{76}$$

When the ρ is on mass shell the coupling becomes g_ρ. Of course, like VMD, there is direct coupling between the W boson and other mesons:

$$\frac{g_w}{4}\frac{1}{2} g W_\mu^i j^{i\mu}. \tag{77}$$

The axial-vector part of the interactions of the W boson with mesons is derived to be

$$-\frac{g_w}{4} g \left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}}\left(\frac{F^2}{g^2} + \frac{1}{2\pi^2 g^2} m_a^2\right) a_\mu^i W^{i\mu} - \frac{g_w}{4} f_\pi W_{i\mu} \partial_\mu \pi_i$$
$$+ \frac{g_w}{4}\frac{1}{2} g \left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}} (\partial_\mu a_\nu^i - \partial_\nu a_\mu^i)(\partial^\mu W^{i\nu} - \partial^\nu W^{i\mu}) - \frac{g_w}{4}\frac{1}{2} g W_\mu^i j_a^{i\mu}, \tag{78}$$

where $j_a^{i\mu}$ is defined as isovector axial-vector current, and a_1 fields couple to this current. Using the mass formula [Eq. (27)], it can be seen from Eq. (78) that the coupling $a_1 - W$ is just g_a [Eq. (70)].

$\pi\pi$ SCATTERING AND DETERMINATION OF PARAMETERS OF CPT

The pion is nearly the Goldstone boson associated with the dynamically broken $SU(2)_L\times SU(2)_R$ chiral symmetry which is a symmetry of QCD in the limit $m_{u,d}\to 0$. $\pi\pi$ scattering has been studied by Weinberg, using a nonlinear chiral Lagrangian [34]. The modern study of $\pi\pi$ scattering utilizes chiral perturbation theory [2,35]. In the present theory the $\pi\pi$ scattering is related to an even number of factors of γ_5; hence, the Lagrangian of this process can be found from Eq. (13). As discussed in Ref. [36] the pion mass term can be introduced to the theory by adding the quark mass term $-\bar{\psi}M\psi$ (M is the quark mass matrix) to the Lagrangian [Eq. (1)]. According to Ref. [36] the pion mass term obtained from the quark mass term is

$$\frac{1}{8}f_\pi^2 m_\pi^2 \mathrm{Tr}(U-1). \tag{79}$$

The $\pi_{j_1}(k_1)+\pi_{j_2}(k_2)\to\pi_{i_1}(p_1)+\pi_{i_2}(p_2)$ scattering amplitudes are written as

$$T_{j_1j_2,i_1i_2} = A(s,t,u)\delta_{i_1i_2}\delta_{j_1j_2} + A(t,s,u)\delta_{i_1j_1}\delta_{i_2j_2} + A(u,t,s)\delta_{i_1j_2}\delta_{i_2j_1}, \tag{80}$$

where $s=(k_1+k_2)^2$, $t=(k_1-p_1)^2$, and $u=(k_1-p_2)^2$. In the center-of-mass frame $s=4m_\pi^2+4k^2$, $t=-2k^2(1-\cos\theta)$, $u=-2k^2(1+\cos\theta)$, where k is the pion momentum and θ is the scattering angle. The partial wave amplitudes are defined as

$$\begin{aligned}
T_l^I(s) &= \frac{1}{64\pi}\int_{-1}^{1} d\cos\theta P_l(\cos\theta)T^I(s,t,u),\\
T^0 &= 3A(s,t,u)+A(t,s,u)+A(u,t,s),\\
T^1 &= A(t,s,u)-A(u,t,s),\\
T^2 &= A(t,s,u)+A(u,t,s).
\end{aligned} \tag{81}$$

At low energies, the partial wave amplitudes can be expanded in terms of the scattering length a_l^I and slope b_l^I:

$$\mathrm{Re}T_l^I(s) = \left(\frac{k^2}{m_\pi^2}\right)^l\left(a_l^I+\frac{k^2}{m_\pi^2}b_l^I\right). \tag{82}$$

The Lagrangian of $\pi\pi$ scattering derived from Eq. (13) contains two parts: direct coupling and ρ meson exchange. To the leading order of chiral perturbation, the amplitudes obtained from direct coupling (with an index D) are

$$\begin{aligned}
A(s,t,u)_D &= \frac{16}{f_\pi^4}\left\{\frac{1}{3}f_\pi^2\left(1-\frac{6c}{g}\right)(2m_\pi^2+3k^2)+\frac{c^2}{g^2}\left[\frac{3}{\pi^2}\left(1-\frac{2c}{g}\right)^2-4c^2\right](6k^4-2k^4\cos^2\theta)\right.\\
&\quad\left.-\frac{4}{(4\pi)^2}\left(1-\frac{2c}{g}\right)^4(10k^4-2k^4\cos^2\theta)+\frac{8}{(4\pi)^2}\left(1-\frac{2c}{g}\right)^2\left[-\frac{16c}{g}k^4+4k^4+\frac{4c^2}{g^2}k^4(1+\cos^2\theta)\right]\right\},\\
A(t,s,u)_D &= \frac{16}{f_\pi^4}\left\{\frac{1}{6}f_\pi^2\left(1-\frac{6c}{g}\right)(-2m_\pi^2-3k^2+3k^2\cos\theta)\right.\\
&\quad+\frac{c^2}{g^2}\left[\frac{3}{\pi^2}\left(1-\frac{2c}{g}\right)^2-4c^2\right][-3k^4+k^4\cos^2\theta-6k^4\cos\theta]-\frac{4}{(4\pi)^2}\left(1-\frac{2c}{g}\right)^4[-2k^4+2k^4\cos^2\theta-8k^4\cos\theta]\\
&\quad+\frac{8}{(4\pi)^2}\left(1-\frac{2c}{g}\right)^2\left[\frac{2c}{g}(-2k^4-2k^4\cos^2\theta+4k^4\cos\theta)+(-k^2+k^2\cos\theta)^2\right.\\
&\quad\left.\left.+\frac{2c^2}{g^2}(5k^4+2k^4\cos\theta+k^4\cos^2\theta)\right]\right\}.
\end{aligned} \tag{83}$$

Equation (26) has been used in deriving Eqs. (83). The amplitude $A(u,t,s)$ can be obtained by using the substitution of $\cos\theta\to-\cos\theta$ in $A(t,s,u)$. The amplitudes from ρ exchange are obtained by using Eqs. (48), (49),

$$\begin{aligned}
A(s,t,u)_\rho &= \frac{8}{g^2}\frac{2m_\pi^2+3k^2+k^2\cos\theta}{m_\rho^2+2k^2-2k^2\cos\theta}+\frac{8}{g^2}\frac{2m_\pi^2+3k^2-k^2\cos\theta}{m_\rho^2+2k^2+2k^2\cos\theta},\\
A(t,s,u)_\rho &= \frac{16}{g^2}\frac{k^2\cos\theta}{m_\rho^2-s+im_\rho\Gamma(k)}-\frac{8}{g^2}\frac{2m_\pi^2+3k^2-k^2\cos\theta}{m_\rho^2+2k^2+2k^2\cos\theta},\\
A(u,t,s)_\rho &= \frac{-16}{g^2}\frac{k^2\cos\theta}{m_\rho^2-s+im_\rho\Gamma(k)}-\frac{8}{g^2}\frac{2m_\pi^2+3k^2+k^2\cos\theta}{m_\rho^2+2k^2-2k^2\cos\theta},
\end{aligned} \tag{84}$$

where $\Gamma(k)$ is the decay width of the ρ meson:

$$\Gamma(k) = \frac{2}{3\pi g^2}\frac{k^3}{m_\rho^2}. \tag{85}$$

For kinematic reasons the decay width of the ρ meson appears only when the virtual momentum squared of ρ is equal to s.

The scattering lengths and slopes are found from direct coupling [Eqs. (83)],

$$a_0^0=\frac{5m_\pi^2}{24\pi f_\pi^2}+\frac{2m_\pi^2}{3\pi f_\pi^2}\left(1-\frac{6c}{g}\right),$$
$$b_0^0=\frac{m_\pi^2}{\pi f_\pi^2}\left(1-\frac{6c}{g}\right),$$
$$a_0^2=\frac{m_\pi^2}{12\pi f_\pi^2}-\frac{m_\pi^2}{3\pi f_\pi^2}\left(1-\frac{6c}{g}\right),$$
$$b_0^2=-\frac{m_\pi^2}{2\pi f_\pi^2}\left(1-\frac{6c}{g}\right),$$
$$a_1^1=\frac{m_\pi^2}{6\pi f_\pi^2}\left(1-\frac{6c}{g}\right), \tag{86}$$

and from ρ exchange [Eq. (84)] we obtain

$$a_0^0=\frac{2m_\pi^2}{\pi g^2m_\rho^2},\ b_0^0=\frac{3m_\pi^2}{\pi g^2m_\rho^2},\ a_0^2=-\frac{m_\pi^2}{\pi g^2m_\rho^2},$$
$$b_0^2=-\frac{3m_\pi^2}{2\pi g^2m_\rho^2},\ a_1^1=\frac{m_\pi^2}{2\pi g^2m_\rho^2}. \tag{87}$$

Numerical calculation shows that in these quantities the contribution of ρ exchange is dominant; for instance, the contribution of ρ exchange to a_0^0 is ten times more than the one from direct coupling. Adding Eqs. (86), (87) together and using Eq. (25) we obtain

$$a_0^0=\frac{7m_\pi^2}{8\pi f_\pi^2},\ b_0^0=\frac{m_\pi^2}{\pi f_\pi^2},\ a_0^2=-\frac{m_\pi^2}{4\pi f_\pi^2},\ b_0^2=-\frac{m_\pi^2}{2\pi f_\pi^2},$$
$$a_1^1=\frac{m_\pi^2}{6\pi f_\pi^2}. \tag{88}$$

These are just the scattering lengths and slopes obtained by Weinberg [34]. In Eqs. (86), there are terms with the factor of $\frac{c}{g}$ obtained from the shift $a_\mu\to a_\mu-c\partial_\mu\pi$. Because of Eq. (25) these terms are canceled by the corresponding terms obtained from ρ exchange. These cancellations lead to Weinberg's results in this theory. As a matter of fact, the cancellation is the result of chiral symmetry. In the Lagrangian [Eq. (1)], there is a term $\frac{1}{2}m_0^2(a_\mu a^\mu+v_\mu v^\mu)$ introduced by chiral symmetry and due to this term, c [see Eq. (25)] has m_ρ^2 in the denominator of the expression [Eq. (25)] which leads to the cancellation. All these quantities are only related to the zeroth and the second orders of derivatives. The scattering lengths and slope a_2^0, a_0^2, and b_1^1 are obtained from the terms with the derivatives at fourth order:

$$b_1^1=\frac{m_\pi^4}{6\pi^3f_\pi^4}(-0.036)+\frac{2m_\pi^4}{\pi g^2m_\rho^4},$$
$$a_2^0=\frac{m_\pi^4}{10\pi^3f_\pi^4}(0.0337)+\frac{4m_\pi^4}{15\pi g^2m_\rho^4},$$
$$a_2^2=\frac{m_\pi^4}{10\pi^3f_\pi^4}(0.0207)-\frac{2m_\pi^4}{15\pi g^2m_\rho^4}. \tag{89}$$

In these quantities [Eq. (89)], the contributions of ρ exchange are ten times more than the ones from direct coupling. Therefore, ρ meson exchange is dominant, ar d

TABLE II. The pion scattering lengths and slopes.

	Experimental	Theoretical
a_0^0	0.26 ± 0.05	0.16
b_0^0	0.25 ± 0.03	0.18
a_0^2	-0.028 ± 0.012	-0.045
b_0^2	-0.082 ± 0.008	-0.089
a_1^1	0.038 ± 0.002	0.030
b_1^1		5.56×10^{-3}
a_2^0	$(17\pm3)\times10^{-4}$	7.84×10^{-4}
a_2^2	$(1.3\pm3)\times10^{-4}$	-3.53×10^{-4}

numerical results are listed in Table II.

It is well known that chiral perturbation theory [2,35] describes $\pi\pi$ scattering reasonably well. In particular, the parameters of chiral perturbation theory can be calculated by the present theory. In addition to f_π and m_π there are two other parameters appearing in $\pi\pi$ scattering [35]:

$$\mathcal{L}_4=\frac{\alpha_1}{4}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)^2+\frac{\alpha_2}{4}\mathrm{Tr}(\partial_\mu U\partial_\nu U^\dagger)\mathrm{Tr}(\partial^\mu U\partial^\nu U^\dagger). \tag{90}$$

The amplitudes T_2^0 and T_2^2 are determined by $\mathcal{L}_4$ [35]:

$$T_2^0=\frac{2\alpha_2+\alpha_1}{15\pi f_\pi^4}(s-4m_\pi^2)^2,\quad T_2^2=\frac{2\alpha_1+\alpha_2}{30\pi f_\pi^4}(s-4m_\pi^2)^2. \tag{91}$$

From the amplitudes [Eqs. (83),(84)] [to $O(k^4)$], we obtain

$$T_2^0=\frac{1}{15\pi f_\pi^4}(s-4m_\pi^2)^2(0.00698),$$
$$T_2^2=-\frac{1}{30\pi f_\pi^4}(s-4m_\pi^2)^2(0.00656). \tag{92}$$

The two parameters in Eq. (91) are determined to be

$$\alpha_1=-0.0068,\quad \alpha_2=0.0070. \tag{93}$$

They are compatible with the values of CPT [35],

$$\alpha_1=-0.0092,\quad \alpha_2=0.0080. \tag{94}$$

The same values of α_1 and α_2 can also be found by comparing T_0^0, T_0^2, and T_1^1 of Ref. [35] with the corresponding combinations of Eqs. (83), (84). The numerical calculation shows that the contribution of ρ exchange to α_1 and α_2 is higher than the one from direct coupling by two orders of magnitude. The contribution of ρ exchange to T_2^0 and T_2^2 can be obtained from Eqs. (84):

$$T_2^0=\frac{4}{15\pi g^2m_\rho^4}k^4,\quad T_2^2=-\frac{2}{15\pi g^2m_\rho^4}k^4.$$

Then from ρ exchange only we have

$$-\alpha_1=\alpha_2=\frac{1}{4}\frac{1}{g^2}\frac{f_\pi^4}{m_\rho^4}=0.007.$$

Using the decay width of the ρ meson [Eq. (85)], we obtain

$$\alpha_1 = -\alpha_2 = -3\pi \frac{f_\pi^4}{m_\rho^5} \frac{\Gamma_\rho}{(1 - \frac{4m_\pi^2}{m_\rho^2})^{\frac{2}{3}}}.$$

This is just the expression presented in Ref. [9]. In the present theory the reason for ρ dominance is the consequence of cancellation between the original pion and the pion obtained from the shift of $a_\mu \to a_\mu - c\partial_\mu \pi$. On the other hand, the energy dependence of the amplitudes [Eqs. (83),(84)] can be predicted by the present theory. The amplitudes $A(t,s,u)_\rho$ and $A(u,t,s)_\rho$ [Eq. (84)] predict the resonance structure of $\rho(770)$ in the channel of $I = 1$ and $l = 1$. The experimental data [37] clearly shows the $\rho(770)$ resonance in the amplitude T_1^1. The comparison between theoretical predictions and experimental data are given in Fig. 1, which shows that $|T_1^1|$ is in good agreement with data and T_2^0, $\mathrm{Re}T_2^2$, and $\mathrm{Re}T_0^2$ agree with data well. However, this theory does not provide an imaginary part for T_0^0 and the experimental data shows that there is an imaginary part in T_0^0. On the other hand, the data shows (Table II) that the theoretical predictions of a_0^0 and b_0^0 are lower than the experimental data. Therefore, in the channel of $I = 0$ and $l = 0$ something is missing in this theory. In Ref. [38] the 0^{++} $f_0(1300)$ meson has been introduced to the effective meson theory to improve the theoretical value of a_0^0. The $f_0(1300)$ meson can be introduced to the Lagrangian [Eq. (1)]; however, this is beyond the scope of this paper.

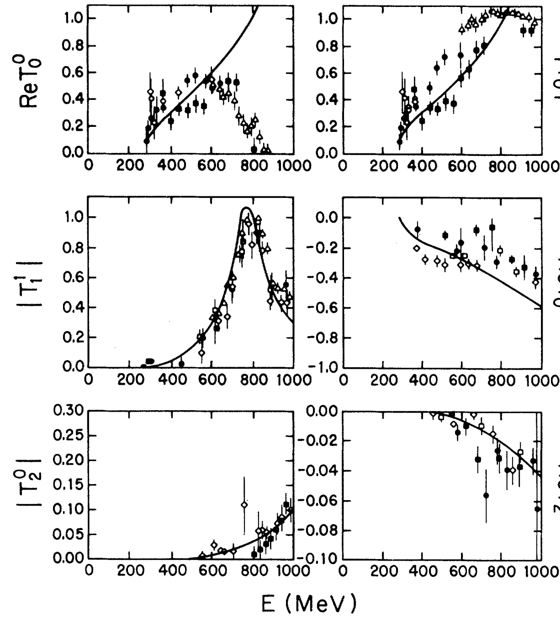

FIG. 1. Energy dependence of $\pi\pi$ scattering amplitudes.

$\omega \to \rho\pi$ AND OTHER RELATED PROCESSES

In the 1960s, Gell-Mann, Sharp, and Wagner [39] used the coupling of $\omega \to \rho\pi$ to compute the decay rates of $\omega \to \pi\gamma$ and $\pi^0 \to \gamma\gamma$ in terms of VMD. The Syracuse group [14] has used the generalized Wess-Zumino action to study these processes.

In the process $\omega \to \rho\pi$, only the pion is associated with γ_5 in this theory. Therefore, the vertex of the decay $\omega \to \rho\pi$ should be found from $\mathcal{L}_{\mathrm{Im}}$. On the other hand, in the Lagrangian [Eq. (1)] the interactions between the ω meson and others are given by the term $\frac{1}{g}\omega^\mu \bar{\psi}\gamma_\mu\psi$. The quark vector current can be bosonized by the quark propagator [Eq. (42)] and the vertices related to the ω field can be derived. In above sections we have used this method to derive the formula of the VMD [Eq. (46)] and the expressions of g_ρ and g_a. The results are the same as the ones obtained from $\mathcal{L}_{\mathrm{Re}}$. Now we use this method to find the interaction vertices between ω and other mesons. In Ref. [24] we have used the quark propagator without vector and axial-vector mesons to calculate the effective baryon current $\frac{1}{3}\langle\bar{\psi}\gamma_\mu\psi\rangle$ and it has been found that the current found in Ref. [24] is just the topological current induced from the Wess-Zumino term. As in Ref. [24], the current $\langle\bar{\psi}\gamma_\mu\psi\rangle$ can be bosonized by equation

$$\langle\bar{\psi}\gamma_\mu\psi\rangle = \frac{-i}{(2\pi)^D}\int d^D p\, \mathrm{Tr}\gamma_\mu s_F(x,p). \tag{95}$$

The leading terms come from $n = 3$. After a lengthy and very careful derivation it is found that except for the term $g\partial^2\omega_\mu$ in Eq. (45) all other terms with even numbers of γ_5 in Eq. (95) cancel each other. This is consistent with the fact that in the two flavor case, the ω field is a flavor singlet. Except for the kinetic term, ω field does not appear in $\mathcal{L}_{\mathrm{Re}}$ [Eq. (13)] which describes the processes with normal parity. Using Eq. (42) and taking $n = 3$, we have

$$\langle\bar{\psi}\gamma_\mu\psi\rangle = \frac{i}{(2\pi)^D}\int \frac{d^D p}{(p^2-m^2)^4}\mathrm{Tr}\gamma_\mu(\gamma\cdot p - m\hat{u})\gamma\cdot D(\gamma\cdot p - m\hat{u})\gamma\cdot D(\gamma\cdot p - m\hat{u})\gamma\cdot D(\gamma\cdot p - m\hat{u}), \tag{96}$$

where $\gamma\cdot D = i\gamma\cdot\partial + \frac{1}{g}\gamma\cdot v + \gamma\cdot a\gamma_5$ [a_μ is defined by Eqs. (22), (23)]. In the calculation of Eq. (96) we use dimensional regularization. However, there is one term in Eq. (96),

$$\frac{i}{(2\pi)^D}\int \frac{d^D p}{(2\pi)^D}\mathrm{Tr}\gamma_\mu\gamma\cdot p\gamma\cdot D\gamma\cdot p\gamma\cdot D\gamma\cdot p\gamma\cdot D\gamma\cdot p, \tag{97}$$

which has divergence and contains γ_5. Therefore this part of the integral needs special treatment in dimensional regularization. We use the 't Hooft–Veltman prescription [40] used to treat the triangle anomaly to calculate this integral. We define

$$p = \underline{p} + q,$$

where $\underline{p}$ has only four components in the four-dimensional space and q is defined in $D-4$ dimensional space. According to Ref. [40] $\underline{p}$ and q observe equations

$$\gamma \cdot \underline{p}\gamma_5 = -\gamma_5\gamma \cdot \underline{p}, \quad \gamma \cdot q\gamma_5 = \gamma_5\gamma \cdot q.$$

Following these treatments the integral is computed. The final expression for $\langle \bar{\psi}\gamma_\mu\psi\rangle$ is

$$\begin{aligned}\frac{1}{g}\omega^\mu\langle\bar{\psi}\gamma_\mu\psi\rangle &= \omega^\mu\partial^2\omega_\mu + \frac{N_c}{(4\pi)^2 g}\varepsilon^{\mu\nu\alpha\beta}\omega_\mu \mathrm{Tr}\left\{\frac{4}{3g^2}\left[-v_{\nu\alpha}a_\beta + \partial_\nu(a_\alpha v_\beta) - ia_\nu a_\alpha a_\beta - \frac{i}{2}a_\alpha[\rho_\beta,\rho_\nu]\right]\right.\\ &\quad \left. -\frac{2}{3}(V_{\nu\alpha}a_\beta - a_{\nu\alpha}v_\beta) - \frac{1}{3}(V_{\nu\alpha}a_\beta + a_{\nu\alpha}v_\beta)\right\}\\ &\quad + \frac{N_c}{(4\pi)^2 g}\frac{i}{3}\varepsilon^{\mu\nu\alpha\beta}\omega_\mu \mathrm{Tr}\left\{2(D_\nu U)U^\dagger(D_\alpha U)U^\dagger(D_\beta U)U^\dagger - \frac{3i}{g^2}V_{\nu\alpha}a_\beta - \frac{3i}{g^2}a_{\nu\alpha}\rho_\beta\right.\\ &\quad \left. + \frac{3i}{g}V_{\nu\alpha}[U(D_\beta U^\dagger) - U^\dagger(D_\beta U)] - \frac{3i}{g}a_{\nu\alpha}[U(D_\beta U^\dagger) + U^\dagger(D_\beta U)]\right\},\end{aligned} \tag{98}$$

where $v_{\nu\alpha} = \partial_\nu\rho_\alpha - \partial_\alpha\rho_\nu - \frac{i}{g}[\rho_\nu,\rho_\alpha]$, $V_{\nu\alpha} = \partial_\nu\rho_\alpha - \partial_\alpha\rho_\nu - \frac{i}{g}[\rho_\nu,\rho_\alpha] - \frac{i}{g}[a_\nu,a_\alpha]$, and $a_\nu \to (1-\frac{1}{2\pi^2 g^2})^{-\frac{1}{2}}a_\nu - c\partial_\nu\pi$. This expression can be simplified as

$$\begin{aligned}\frac{1}{g}\omega^\mu\langle\bar{\psi}\gamma_\mu\psi\rangle &= \omega^\mu\partial^2\omega_\mu + \frac{N_c}{(4\pi)^2 g}\frac{2}{3}\varepsilon^{\mu\nu\alpha\beta}\omega_\mu \mathrm{Tr}\partial_\nu UU^\dagger\partial_\alpha UU^\dagger\partial_\beta UU^\dagger\\ &\quad + \frac{N_c}{(4\pi)^2}\frac{2}{g}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\omega_\nu \mathrm{Tr}\left\{\frac{i}{g}[\partial_\beta UU^\dagger(\rho_\alpha + a_\alpha) - \partial_\beta U^\dagger U(\rho_\alpha - a_\alpha)]\right.\\ &\quad \left. -\frac{2}{g^2}(\rho_\alpha + a_\alpha)U(\rho_\beta - a_\beta)U^\dagger - \frac{2}{g^2}\rho_\alpha a_\beta\right\}.\end{aligned} \tag{99}$$

This formula is exactly the same as the one obtained by Kaymakcalan, Rajeev, and Schechter [14] and by Witten [13]. In Eq. (99) all the couplings are fixed by g and c. This is the universality of coupling in this theory.

The interaction Lagrangian of $\omega\rho\pi$ is derived from Eq. (99):

$$\mathcal{L}_{\omega\rho\pi} = -\frac{N_c}{\pi^2 g^2 f_\pi}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\omega_\nu\rho^i_\alpha\partial_\beta\pi^i, \tag{100}$$

which can be used to study related processes. Using the vertex of $\omega \to \pi\gamma$ obtained by the substitution $\rho^0_\mu \to \frac{e}{f_\rho}A_\mu$ in Eq. (100) the decay width is calculated to be

$$\Gamma(\omega \to \gamma\pi) = 724 \text{ keV}.$$

The experimental value is $717(1 \pm 0.07)$ keV. In the same way, we obtain the vertex of $\rho \to \pi\gamma$ by the substitution $\omega_\mu \to \frac{e}{f_\omega}A_\mu$ in Eq. (100). The decay width calculated is

$$\Gamma(\rho^0 \to \pi^0\gamma) = 76.2 \text{ keV}.$$

The experimental data are $68.2(1 \pm 0.12)$ keV.

$\pi^0 \to \gamma\gamma$ is an evidence of the Adler-Bell-Jackiw anomaly [41]. This process is a crucial test of the present theory. According to VMD, the $\pi^0\gamma\gamma$ vertex should be obtained by using the substitutions [Eq. (32)] in Eq. (100):

$$\mathcal{L}_{\pi^0\to\gamma\gamma} = -\frac{\alpha}{\pi f_\pi}\varepsilon^{\mu\nu\alpha\beta}\pi^0\partial_\mu A_\nu\partial_\alpha A_\beta, \tag{101}$$

which is the expression given by the Adler-Bell-Jackiw anomaly [41]. The decay width obtained from Eq. (101) is

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{\alpha^2}{16\pi^3}\frac{m_\pi^3}{f_\pi^2}, \tag{102}$$

which is the result of the triangle anomaly. The numerical result is 7.64 eV and the data is $7.74(1 \pm 0.072)$ eV. If one of the two photons $\pi^0 \to \gamma\gamma$ is virtual, according to VMD, there are vector meson poles in the decay amplitude. The amplitude is written as

$$\mathcal{M} = \frac{2\alpha}{\pi f_\pi}\left\{1 + \frac{1}{2}\frac{q^2}{m_\rho^2 - q^2} + \frac{1}{2}\frac{q^2}{m_\omega^2 - q^2}\right\}, \tag{103}$$

where q is the momentum of the virtual photon. Since q^2 is much less than the mass of the vector meson, the form factor takes approximate form

$$F(q^2) = 1 + \frac{1}{2}\frac{q^2}{m_\rho^2 - q^2} + \frac{1}{2}\frac{q^2}{m_\omega^2 - q^2} = 1 + a\frac{q^2}{m_{\pi^0}^2},$$

$$a = \frac{m_{\pi^0}^2}{2}\left(\frac{1}{m_\omega^2} + \frac{1}{m_\rho^2}\right) = 0.03. \tag{104}$$

The data are $a = 0.032 \pm 0.004$.

In $\omega \to \pi\pi\pi$, aside from the process $\omega\rho\pi$ and $\rho \to \pi\pi$ there is direct coupling $\omega\pi\pi\pi$ which is derived from Eq. (99):

$$\mathcal{L}_{\omega\pi\pi\pi} = \frac{2}{g\pi^2 f_\pi^3}\left(1 + \frac{6c^2}{g^2} - \frac{6c}{g}\right) \times \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{ijk}\omega_\mu \partial_\nu \pi_i \partial_\alpha \pi_j \partial_\beta \pi_k. \tag{105}$$

In both the $\mathcal{L}_{\omega\pi\pi\pi}$ and $\mathcal{L}_{\omega\to\rho\pi}$ there is a factor of $\frac{1}{\pi^2}$. Therefore, qualitatively speaking, this theory predicts a narrower width for ω decay. Using the formula of c [Eq. (25)] and the value of g it is found that

$$1 + \frac{6c^2}{g^2} - \frac{6c}{g} = -0.083.$$

The last two terms come from the shift $a_\mu \to a_\mu - c\partial_\mu \pi$ and there is very strong cancellation. Using Eqs. (49),(100),(105) the decay width is obtained:

$$\Gamma(\omega \to \pi\pi\pi) = \frac{1}{24 m_\omega (2\pi)^3}\int dq_1^2 dq_2^2 \{|\vec{p}_1|^2|\vec{p}_2|^2 - (\vec{p}_1 \cdot \vec{p}_2)^2\} \times \left\{3f_{\omega 3\pi} + f_{\omega\rho\pi} f_{\rho\pi\pi}\left(\frac{1}{q_1^2 - m_\rho^2} + \frac{1}{q_2^2 - m_\rho^2} + \frac{1}{q_3^2 - m_\rho^2}\right)\right\}^2, \tag{106}$$

where

$$f_{\omega 3\pi} = \frac{2}{g\pi^2 f_\pi^3}\left(1 + \frac{6c^2}{g^2} - \frac{6c}{g}\right), \quad f_{\omega\rho\pi} = \frac{N_c}{\pi^2 g^2 f_\pi}, \tag{107}$$

and $q_i^2 = (p - p_i)^2$, p is ω momentum and p_i is pion momentum. We obtain

$$\Gamma(\omega \to 3\pi) = 5 \text{ MeV}.$$

If only $\omega \to \rho\pi$ and $\rho \to \pi\pi$ are taken into account the width of $\omega \to 3\pi$ is 5.4 MeV. The experimental value is $7.43(1 \pm 0.02)$ MeV. From this study we can see that the process of $\omega \to \rho\pi$ is dominant in the decay of $\omega \to 3\pi$, as proposed by the authors [39]. The direct coupling of $\omega \to 3\pi$ is responsible for about 20% of the decay rate. The agreement between theoretical and experimental decay rates of $\pi^0 \to \gamma\gamma$, $\omega \to \pi\gamma$, and $\rho \to \pi\gamma$ shows that $\omega \to \rho\pi$ obtained in this theory is more reliable. However, at the tree level due to the cancellation, $f_{\omega 3\pi}$ is too small and has the wrong sign. Therefore, corrections from loop diagrams and terms with higher order derivatives to $f_{\omega 3\pi}$ are needed.

DECAYS OF THE $f_1(1285)$ MESON

In this theory the $f_1(1285)$ meson is the chiral partner of the ω meson [see Eq. (1)]. The decay of $f_1 \to 4\pi$ consists of two processes: direct coupling $f_1 \to \rho\pi\pi$ and $f_1 \to a_1\pi \to \rho\pi\pi$. The f_1 meson is associated with γ_5 [see Eq. (1)]; hence odd numbers of γ_5 are involved in these two processes. Therefore, the vertex of the decay $f_1 \to \rho\pi\pi$ cannot be found in Eq. (13) and should be found in the Wess-Zumino Lagrangian with vector and axial-vector mesons. In this paper the method used to find the effective Lagrangian of these processes is the same as the one used to study the decays of the ω meson. From the Lagrangian [Eq. (1)] and normalization of the f_1 meson the vertices of the f_1 decays can be found from the formula

$$\mathcal{L}_i = \frac{1}{g}\left(1 - \frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}} f_\mu \langle \bar{\psi}\gamma_\mu\gamma_5\psi\rangle. \tag{108}$$

The bosonization of $\langle \bar{\psi}\gamma_\mu\gamma_5\psi\rangle$ can be carried out by using the equation

$$\langle \bar{\psi}\gamma_\mu\gamma_5\psi\rangle = \frac{-i}{(2\pi)^D}\int d^D p\, \mathrm{Tr}\gamma_\mu\gamma_5 s_F(x,p). \tag{109}$$

Substituting Eq. (42) into Eq. (109) the flavor singlet axial-vector current of mesons can be achieved. Because of the fact that only odd numbers of γ_5 are involved in $f_1 \to \rho\pi\pi$, we are only interested in the terms with the antisymmetric tensor. The leading terms with antisymmetric tensor appear at $n = 3$. As in the case of the ω meson the divergent terms can be treated by the prescription provided in Ref. [40]. Finally, the terms with $\varepsilon^{\mu\nu\alpha\beta}$ in the effective Lagrangian [Eq. (108)] take the form

$$\mathcal{L}_i = -\frac{N_c}{3g^3(4\pi)^2}\left(1-\frac{1}{2\pi^2 g^3}\right)^{-\frac{1}{2}}\varepsilon^{\mu\nu\alpha\beta}f_\mu \mathrm{Tr}(3V_{\nu\alpha}v_\beta - a_{\nu\alpha}a_\beta)$$
$$+\frac{iN_c}{3g^2(4\pi)^2}\left(1-\frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}}\varepsilon^{\mu\nu\alpha\beta}f_\mu \mathrm{Tr}\{-\partial_\alpha(\partial_\nu U U^\dagger d^+_\beta + \partial_\nu U^\dagger U d^-_\beta)$$
$$-2(\partial_\nu U U^\dagger \partial_\alpha U U^\dagger d^+_\beta + \partial_\nu U^\dagger U \partial_\alpha U^\dagger U d^-_\beta)$$
$$-\frac{i}{g}(d^+_\nu \partial_\alpha d^+_\beta + d^-_\nu \partial_\alpha d^-_\beta + 2U^\dagger d^+_\nu U \partial_\alpha d^-_\beta + 2U d^-_\nu U^\dagger \partial_\alpha d^+_\beta - 2\partial_\nu U U^+ d^+_\alpha d^+_\beta - 2\partial_\nu U^\dagger U d^-_\alpha d^-_\beta)$$
$$-\frac{1}{g^2}(2Ud^-_\nu U^\dagger d^+_\alpha d^+_\beta + 2U^\dagger d^+_\nu U d^-_\alpha d^-_\beta + d^-_\nu d^-_\alpha d^-_\beta + d^+_\nu d^+_\alpha d^+_\beta))\}. \quad (110)$$

For the definitions of $d^{\pm}_\mu$, see Eq. (117). From this Lagrangian, the same conclusion as Ref. [16] is reached, that the decays of $f_1 \to \rho\rho$ and $\omega\omega$ are forbidden. Therefore, the f_1 meson cannot decay to two real photons. This is Yang's theorem [42].

The Lagrangians of the decay of $f \to a_1\pi$ and $f \to \rho\pi\pi$ are derived from Eq. (110):

$$\mathcal{L}_{fa_1\pi} = -\frac{4N_c}{3(4\pi)^2 f_\pi}\frac{1}{g^2}\left(1-\frac{1}{2\pi^2 g^2}\right)^{-1}\varepsilon^{\mu\nu\alpha\beta}f_\mu \partial_\nu \pi^i \partial_\alpha a^i_\beta,$$

$$\mathcal{L}_{f\rho\pi\pi} = \frac{4N_c}{3(4\pi)^2 g^2 f^2_\pi}\left(1-\frac{4c}{g}\right)\left(1-\frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}}\varepsilon^{\mu\nu\alpha\beta}\epsilon_{ijk}f_\mu\{\partial_\alpha(\partial_\nu\pi_i\pi_j\rho^k_\beta) - 2\partial_\nu\pi_i\partial_\alpha\pi_j\rho^k_\beta\}. \quad (111)$$

In both $\mathcal{L}_{f\rho\pi\pi}$ and $\mathcal{L}_{fa_1\pi}$ there is a factor of $\frac{1}{\pi^2}$. Therefore, this theory predicts a narrower width for $f \to \rho\pi\pi$. On the other hand, the factor of $1-\frac{4c}{g}$ in $\mathcal{L}_{f\rho\pi\pi}$ is very small; hence the process $f \to a_1\pi \to \rho\pi\pi$ is dominant over the decay of $f \to \rho\pi\pi$. Using Eqs. (61), (111) the decay width is calculated to be

$$\Gamma(f_1 \to \rho\pi\pi) = 6.01 \text{ MeV}. \quad (112)$$

The experimental value of the decay width is $6.96(1 \pm 0.33)$ MeV. The prediction of $\Gamma(f_1 \to \rho\pi\pi)$ is in agreement with data.

In another decay of the $f_1(1285)$ meson, $f \to \eta\pi\pi$ [excluding $a_0(980)\pi$], even numbers of γ_5 are involved. Therefore, the effective Lagrangian of this decay should be found from the Lagrangian $\mathcal{L}_{\mathrm{Re}}$ [Eq. (13)]. Of course, the vertex of $f\eta\pi\pi$ can also be derived from Eq. (108) and it should be the same as the one obtained from Eq. (13). The calculation shows that in the Lagrangian $\mathcal{L}_{\mathrm{Re}}$ the term at the second order in derivatives,

$$\frac{F^2}{16}\mathrm{Tr}D_\mu U D^\mu U^\dagger,$$

does not contribute to this decay. The effective Lagrangian of this decay comes from the terms at the fourth order in derivatives which have a factor of $\frac{1}{(4\pi)^2}$. Therefore, this theory predicts a narrow width for the process $f_1 \to \eta\pi\pi$. In the chiral limit, the effective Lagrangian is obtained from Eq. (13):

$$\mathcal{L}_{f\eta\pi\pi} = \frac{4N_c}{3(4\pi)^2 f^3_\pi}\frac{1}{g}\left(1-\frac{1}{2\pi^2 g^2}\right)^{-\frac{1}{2}}\left(1-\frac{2c}{g}\right)^3 0.7104 f_\mu\{\partial^\mu\eta\partial_\nu\pi_i\partial^\nu\pi_i + 2\partial_\nu\eta\partial^\nu\pi_i\partial^\mu\pi_i\}, \quad (113)$$

where the factor 0.7104 is from the mixing between η and η', $0.7104 = \frac{1}{\sqrt{3}}(\cos\theta - \sqrt{2}\sin\theta)$, and $\theta = -10°$. The numerical result of the decay width is

$$\Gamma(f \to \eta\pi\pi) = 27.5 \text{ keV}. \quad (114)$$

Theoretical prediction of the branching ratio is $1.15 \times 10^{-3}(1 \pm 0.13)$ and the data of the branching ratio is $(10^{+7}_{-6})\%$. In principle, the meson $a_0(980)[1^-(0^{++})]$ can be incorporated into the Lagrangian, and then the decay of $f \to a_0\pi$ can be studied. However, this is beyond the scope of the present paper.

Using VMD and $\mathcal{L}_{f\rho\pi\pi}$, the decay width of $f \to \gamma\pi\pi$ is computed:

$$\Gamma(f \to \gamma\pi\pi) = 18.5 \text{ keV}. \quad (115)$$

In this theory the decay of $f_1 \to$ virtual photon $+\rho$ involves loop diagrams whose calculation is beyond the scope of this paper.

THE DECAYS OF $\rho \to \eta\gamma$ AND $\omega \to \eta\gamma$

According to VMD, the decays of $\rho \to \eta\gamma$ and $\omega \to \eta\gamma$ are related to the vertices $\eta\rho\rho$ and $\eta\omega\omega$ in which odd numbers of γ_5 are involved and these vertices cannot be found from $\mathcal{L}_{\mathrm{Re}}$ [Eq. (13)]. From the Lagrangian [Eq. (1)] it can be seen that the interaction between the η and other mesons can be found from

$$\mathcal{L}_\eta = -\frac{2i}{f_\pi}0.7104m\eta\langle\bar{\psi}\gamma_5\psi\rangle,$$
$$\langle\bar{\psi}\gamma_5\psi\rangle = \frac{-i}{(2\pi)^D}\int d^Dp\,\mathrm{Tr}\gamma_5 s_F(x,p). \tag{116}$$

Substituting the solution [Eq. (42)] into Eq. (116) $\mathcal{L}_\eta$ is obtained. The leading terms come from $n=4$. We are only interested in the terms containing the vertices ηvv, which are found to be

$$\langle\bar{\psi}\gamma_5\psi\rangle = \frac{N_c}{(4\pi)^2}\frac{i}{6mg}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(F^+_{\mu\nu}D^-_\alpha D^+_\beta + F^-_{\mu\nu}D^+_\alpha D^-_\beta) + \frac{N_c}{(4\pi)^2}\frac{i}{6mg}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(D^+_\mu D^-_\nu F^+_{\alpha\beta} + D^-_\mu D^+_\nu F^-_{\alpha\beta})$$
$$+\frac{N_c}{(4\pi)^2}\frac{i}{3m}\varepsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(D^+_\mu D^-_\nu D^+_\alpha D^-_\beta + D^-_\mu D^+_\nu D^-_\alpha D^+_\beta),$$

$$D^\pm_\mu = \partial_\mu - \frac{i}{g}d^\pm_\mu,\quad d^\pm_\mu = v_\mu \mp a_\mu,\quad a_\mu \to \left(1-\frac{1}{2\pi^2g^2}\right)^{-\frac{1}{2}}a_\mu - c\partial_\mu\pi,$$

$$F^\pm_{\mu\nu} = \partial_\mu d^\pm_\nu - \partial_\nu d^\pm_\mu - \frac{i}{g}[d^\pm_\mu, d^\pm_\nu]. \tag{117}$$

The vertices of $\eta\rho\rho$ and $\eta\omega\omega$ are obtained from Eqs. (117):

$$\mathcal{L}_{\eta vv} = -\frac{8}{g^2f_\pi}\frac{N_c}{(4\pi)^2}\varepsilon^{\mu\nu\alpha\beta}0.7104\eta(\partial_\mu\rho^i_\nu\partial_\alpha\rho^i_\beta + \partial_\mu\omega_\nu\partial_\alpha\omega_\beta). \tag{118}$$

Using VMD we obtain

$$\mathcal{L}_{\rho\eta\gamma} = -\frac{8e}{gf_\pi}\frac{N_c}{(4\pi)^2}\varepsilon^{\mu\nu\alpha\beta}0.7104\eta\partial_\mu\rho^0_\nu\partial_\alpha A_\beta,$$
$$\mathcal{L}_{\omega\eta\gamma} = -\frac{8e}{3gf_\pi}\frac{N_c}{(4\pi)^2}\varepsilon^{\mu\nu\alpha\beta}0.7104\eta\partial_\mu\omega_\nu\partial_\alpha A_\beta. \tag{119}$$

The decay widths of $\rho\to\eta\gamma$ and $\omega\to\eta\gamma$ are calculated:

$$\Gamma(\rho\to\eta\gamma) = 46.1\ \mathrm{keV},\quad B(\rho\to\eta\gamma) = 3.04\times10^{-4},$$
$$\Gamma(\omega\to\eta\gamma) = 5.87\ \mathrm{keV},\quad B(\omega\to\eta\gamma) = 6.96\times10^{-4}. \tag{120}$$

The experimental data are

$$B(\rho\to\eta\gamma) = (3.8\pm0.7)\times10^{-4},$$
$$B(\omega\to\eta\gamma) = (8.3\pm2.1)\times10^{-4}. \tag{121}$$

Theoretical predictions are in good agreement with the data. For $\eta\to\gamma\gamma$, in addition to $\eta\to\rho\rho$ and $\eta\to\omega\omega$, the process $\eta\to\phi\phi$ also contributes. We will study $\eta\to\gamma\gamma$ in another paper in which the strange flavor is included.

After the study of these physical processes, three problems of this theory should be discussed. They are loop diagrams, dynamical chiral symmetry breaking, and momentum expansion.

LARGE N_C EXPANSION

According to 't Hooft [7], in the large N_c limit QCD is equivalent to a meson theory at low energies. Therefore, large N_c expansion plays a crucial role in the connection between QCD and effective meson theory, even though we do not know how to derive the Lagrangian of effective meson theory from QCD directly. In the present theory the large N_c expansion plays an important role too. The quark fields in the Lagrangian [Eq. (1)] carry colors. In order to obtain the effective Lagrangian of mesons from the Lagrangian [Eq. (1)], the quark fields have been integrated out by path integral methods. After this integration the trace in the color space generates the number of color N_c. The parameter m of the Lagrangian [Eq. (1)] is $O(1)$ in the large N_c expansion. Equation (14) determines that F^2 is of order N_c, hence f_π is of order $O(\sqrt{N_c})$. The coupling constant g, defined by Eq. (15), is $O(\sqrt{N_c})$. After normalization, the physical meson fields, pion, η, ρ, ω, a_1, and f_1 are all of order $O(\sqrt{N_c})$. It should be pointed out that the original form of the factor $(1-\frac{1}{2\pi^2g^2})^{-\frac{1}{2}}$ in the normalization of the axial-vector meson is $(1-\frac{N_c}{6\pi^2g^2})^{-\frac{1}{2}}$. Therefore, this factor is of order $O(1)$. The masses of mesons are of order $O(1)$. Using all these results, it is not difficult to find out that all the vertices of this paper are of order N_c and it is obvious that the meson propagator is order of $O(1)$. Therefore, the order of magnitude of a Feynman diagram of mesons in large N_c expansion is given by

$$N_c^{N_v-N_p}, \tag{122}$$

where N_v is the number of vertices and N_p is the number of internal lines. Equation (122) tells that all tree diagrams are of order $O(N_c)$; hence, they are leading contributions. A diagram with loops is at higher order in large N_c expansion. For instance, a diagram of one loop with two internal lines is of order $O(1)$. Therefore, the large N_c expansion is the loop expansion in this theory. In this paper all calculations have been done at the tree level. Most of the theoretical predictions are in agreement with data. This success can be viewed as a support of the large N_c expansion. In other words, the large N_c

expansion provides a good argument for the success of the theory at the tree level. However, it would be better to raise all the calculations done in this paper to one loop level to test the theory and the large N_c expansion. In this paper we only present the calculations done at the tree level.

DYNAMICAL CHIRAL SYMMETRY BREAKING

The parameter m in Eq. (1) is associated with the quark condensate which is defined as

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle = -\frac{i}{(2\pi)^D}\int d^Dp\,\mathrm{Tr}\langle 0|s_F(x,p)|0\rangle. \quad (123)$$

At the tree level, using Eq. (42) we obtain the relation between m and the quark condensate:

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle = 3m^3g^2\left(1+\frac{1}{2\pi^2g^2}\right). \quad (124)$$

This is u and d quark condensate. Nonzero quark condensate means dynamical chiral symmetry breaking. Therefore, there is dynamical chiral symmetry breaking in this theory. On the other hand, the quark mass term $-\bar{\psi}M\psi$ can be introduced to the Lagrangian [Eq. (1)], where M is the mass matrix of u and d quark. Using Eqs. (40), (42) and removing a constant, the leading term in quark mass expansion is obtained:

$$\begin{aligned} -\langle\bar{\psi}M\psi\rangle &= \frac{i}{(2\pi)^D}\int d^Dp\,\mathrm{Tr}Ms_F(x,p) \\ &= -\frac{1}{f_\pi^2}(m_u+m_d)\langle 0|\bar{\psi}\psi|0\rangle. \end{aligned} \quad (125)$$

The pion mass [Eq. (79)] obtained from this equation is

$$m_\pi^2 = -\frac{2}{f_\pi^2}(m_u+m_d)\langle 0|\bar{\psi}\psi|0\rangle. \quad (126)$$

Detailed discussion of masses of pseudoscalar mesons can be found in Ref. [36]. Equation (126) is a well known formula obtained by the theory of chiral symmetry breaking proposed by Gell-Mann, Oakes, and Renner [43], and by Glashow and Weinberg [44]. Equation (126) shows that the quark condensate is negative; hence the parameter m is negative too. The parameter m is determined from Eqs. (15), (25), (26),

$$m = -300 \text{ MeV}, \quad (127)$$

and from Eq. (124), we obtain

$$\langle 0|\bar{\psi}\psi|0\rangle = -(241 \text{ MeV})^3. \quad (128)$$

The quark mass is determined to be

$$m_u + m_d = 23.5 \text{ MeV},$$

whose current value is 15.5 ± 2.2 MeV [27]. As pointed out in Ref. [45], the determination of the absolute value of quark mass is model dependent.

DERIVATIVE EXPANSION

This theory is an effective meson theory at low energies. Like the chiral perturbation theory [2], the derivative expansion (to be accurate, covariant derivative expansion) has been applied. It seems that the derivative expansion works in the studies presented in this paper. The calculations of decay widths are good examples. If the terms at the second order in derivatives in the Lagrangian [Eq. (13)] contribute to the decay of a meson, the decay width of this meson is broader. ρ decay and a_1 decay are two examples. If only the terms at the fourth order in derivatives contribute to the decay, the decay width is narrower. The reason is that in the decay amplitude there is a factor of $\frac{1}{\pi^2}$. The predictions of narrower widths for the decays, $\omega \to 3\pi$, $f \to \rho\pi\pi$, and $f \to \eta\pi\pi$, support this argument. From Table II it can be seen that the scattering lengths and slopes (a_0^0, b_0^0, a_1^1, a_0^2, b_0^2), which are obtained from the terms at the zeroth order or the second order in derivatives, are much greater than a_2^0, a_2^2, and b_1^1 which are from the terms at the fourth order in derivatives.

This theory is an effective theory and it is not renormalizable, as mentioned above. Therefore, a cutoff of momentum has to be introduced into this theory. Equation (15) can be used to determine the cutoff. Using a cutoff instead of dimensional regularization, Eq. (15) is rewritten as

$$\frac{N_c}{(4\pi)^2}\left\{\ln\left(1+\frac{\Lambda^2}{m^2}\right)+\frac{1}{1+\frac{\Lambda^2}{m^2}}-1\right\} = \frac{1}{16}\frac{F^2}{m^2} = \frac{3}{8}g^2. \quad (129)$$

Using the values of m and g, we obtain

$$\Lambda = 1.6 \text{ GeV}. \quad (130)$$

Derivative expansion is a momentum expansion and Λ is the maximum momentum. The momentum expansion requires that the momentum must be less than Λ. The masses of ρ, ω, a_1, and f_1 are less than Λ. However, there is a physical case in which momentum expansion is not suitable. In $\pi\pi$ scattering there is a ρ resonance in the scattering amplitudes [Eq. (84)]. At very low energy the momentum expansion has been applied to $\pi\pi$ scattering amplitudes; however, in the region of ρ resonance the momentum expansion is not working because it destroys the resonance. Therefore, we do not apply the momentum expansion to resonance amplitudes in this paper.

SUMMARY OF THE RESULTS

To summarize, the results achieved by this theory are as follows. VMD is revealed from this theory. Weinberg's first sum rule and new relations about a_1 meson decay are satisfied. The KSFR sum rule is satisfied reasonably well. New mass relations between vector and axial-vector meson are found. Weinberg's $\pi\pi$ scattering lengths and slopes are obtained. This theory provides a mechanism of the ρ meson dominance in $\pi\pi$ scattering. The amplitude of $\pi^0 \rightarrow \gamma\gamma$ obtained by this theory is exactly the same as the prediction of the triangle anomaly. This theory unifies the description of the physical processes with normal and abnormal parity and the universality of coupling is realized. The effective Lagrangians used to treat the processes of abnormal parity are exactly the same as with the ones obtained from the gauging Wess-Zumino Lagrangian with vector and axial-vector mesons. The results of $\pi\pi$ scattering have been shown in Table II and Fig. 1. Other results are listed in Table I.

As mentioned in the paper, many of the studies have been done separately before. The theory studied in this paper provides a unified description of meson physics at low energies. In this unified description universal coupling in all the physical processes has been found and the inputs are the cutoff Λ, m (related to quark condensate), and ρ meson mass in the chiral limit and the quark mass in the phase space. Some of the results are new, for instance, the mass relations [Eqs. (27),(28)], the expression of g_a [Eq. (7)] and $g_\rho \neq g_a$, etc. This theory has dynamical chiral symmetry breaking which is an important feature of nonperturbative QCD.

ACKNOWLEDGMENTS

The author wishes to thank G.P. Lepage for helpful discussion and to thank T. Barnes, E. Swanson, and A. MacKellar for help. This research was partially supported by DOE Grant No. DE-91ER75661.

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