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Angular distribution in the decays of the triplet D_2 state of charmonium directly produced in unpolarized $\bar{p}p$ collisions

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We calculate the combined angular distribution of the final electron and of the two γ photons in the cascade process $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1$ ($J=0,1,2$) $\to (\psi\gamma_2)+\gamma_1 \to (e^+e^-)+\gamma_2+\gamma_1$, where $\bar{p}$ and p are unpolarized. Our final result is valid in the $\bar{p}p$ c.m. frame and it is expressed in terms of the Wigner D^J functions and the spherical harmonics whose arguments are the angles representing the various directions involved. The coefficients of the terms involving the spherical harmonics and the Wigner D^J functions are functions of the angular momentum helicity amplitudes or equivalently of the multipole amplitudes of the individual processes. Once the combined angular distribution is measured, our expressions will enable one to calculate the relative magnitudes as well as the relative phases of all the helicity amplitudes in the processes $1\,{}^3D_2 \to 1\,{}^3P_J + \gamma_1$ and $1\,{}^3P_J \to \psi + \gamma_2$ for the $J=2$ case. For the $J=1$ case, we can determine the relative magnitudes of all the helicity amplitudes as well as the cosines of all their relative phases. The sines are not completely determined. If the sine of the relative phase between any two amplitudes is known, then the sines of the relative phases among other amplitudes can be determined. For the $J=0$ case, there is only one helicity amplitude in each decay and that is fixed by our normalization. We also present the partially integrated angular distributions in six different cases, which can all be expressed in terms of the spherical harmonics. We also calculate the angular distribution of the γ photon in the process $\bar{p}p \to {}^3D_2 \to {}^1S_0 + \gamma$ where again $\bar{p}$ and p are unpolarized. In this case, the angular distribution has a very simple form: namely $W(\theta)=(1/\sqrt{4\pi})[Y_{00}+(\sqrt{5}/14)Y_{20}(\theta)+\frac{8}{21}Y_{40}(\theta)]$, where θ is the angle γ makes with the $\bar{p}$ direction. So the observation of a γ photon with an energy of about 840 MeV and with the above angular distribution can be used as a signal for the formation of the ${}^3D_2$ state in $\bar{p}p$ collisions.

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I. INTRODUCTION

Recently we calculated [1,2] the angular distributions of the decay products in the various decay schemes of the singlet D state of charmonium directly produced in $\bar{p}p$ collisions. The main interest in these calculations is that the singlet D state of charmonium, even though it is above the charm threshold, is expected to have a narrow width since its decay into $D+\bar{D}$ is forbidden by parity conservation and its decay into $D+\bar{D}^*$ or $D^*+\bar{D}$ is forbidden by conservation of energy. But the singlet D state has one drawback. By C and P invariances, p and $\bar{p}$ should have the same helicities for the resonance formation of the ${}^1D_2$ state in $\bar{p}p$ collisions. Now if we assume that the constituent u and d quarks of the proton are massless and they have the same helicities as the proton then the formation of the ${}^1D_2$ state in $\bar{p}p$ collisions is forbidden in perturbative QCD since $\bar{U}_{\lambda_1}\gamma^\mu V_{\lambda_2}$ vanishes for massless spinors when the helicity indices λ_1 and λ_2 are equal. It is quite possible this helicity selection rule is strongly violated in nature as is seen from the processes $\bar{p}p \to \eta_c$, ${}^1P_1$, etc. It is interesting to note that the formation of the ${}^3D_2$ state in $\bar{p}p$ collisions is not suppressed due to this helicity selection rule since C and P invariances now dictate that p and $\bar{p}$ should have opposite helicities for the formation of the triplet D states of quarkonium. Furthermore the triplet D state is also expected to have a narrow width since its decay into $D+\bar{D}$ is also forbidden by parity conservation and the predicted mass of $1\,{}^3D_2$ in the potential models is such that its decay into $D+\bar{D}^*$ or $D^*+\bar{D}$ is also forbidden by the conservation of energy. In fact, the predicted mass of ${}^3D_2$ in most potential models lies very close to the singlet D state mass. In this paper we study the angular distribution of the decay products of the ${}^3D_2$ state formed in unpolarized $\bar{p}p$ collisions in the following decay schemes:

$$(1)\quad \bar{p}p \to 1\,{}^3D_2 \to 1\,{}^3P_J(J=0,1,2)+\gamma_1 \to (1\,{}^3S_1+\gamma_2) +\gamma_1 \to (e^+e^-)+\gamma_2+\gamma_1$$

and

$$(2)\quad \bar{p}p \to 1\,{}^3D_2 \to 1\,{}^1S_0+\gamma\ .$$

Our final expression for the angular distribution in each case is valid in the $\bar{p}p$ c.m. frame or the ${}^3D_2$ rest frame and it is given in terms of the angles measured in that frame. In process (1) the expression for the combined angular distribution of the electron and of the two photons is given in terms of the angular-momentum helicity amplitudes in the individual processes ${}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$. Once the angular distribution is experimentally measured our expressions will enable one to calculate the relative magnitudes as well as the relative phases of all the angular-momentum helicity amplitudes or equivalently of the radiative multipole amplitudes in the transitions ${}^3D_2 \to \chi_2 + \gamma_1$ and $\chi_2 \to \psi + \gamma_2$. For the $J=1$ case, we can only determine the relative magnitudes of all the helicity amplitudes as well as the cosines of their relative phases. The sines of the relative phases are not completely determined. If the sine of one of the relative phases is known, the sines of the others can be determined. For the $J=0$ case, there is only one helicity or multipole amplitude and that is fixed by our normalization. In the second cascade process $\bar{p}p \to 1\,{}^3D_2 \to 1\,{}^1S_0 + \gamma$, the normalized angular distribution of the photon has a very simple form without any unknown amplitudes: namely,

$$W(\theta) = \frac{1}{\sqrt{4\pi}} \left[Y_{00} + \frac{\sqrt{5}}{14} Y_{20}(\theta) + \frac{8}{21} Y_{40}(\theta) \right] , \quad (1)$$

where θ is the angle γ makes with the $\bar{p}$ direction.

The format of the rest of the paper is as follows. In Sec. II, we derive the combined angular distribution of γ_1, γ_2, and e^- in the cascade process,

$$\bar{p}p \to 1\,{}^3D_2 \to \chi_J(J=0,1,2) + \gamma_1 \to \psi + \gamma_2 + \gamma_1 \to (e^- + e^+) + \gamma_2 + \gamma_1 .$$

We express our result in terms of the orthonormal Wigner D^J functions and show how the measurement of this angular distribution enables us to obtain complete information about the helicity amplitudes for the $J=2$ case and almost complete information for the $J=1$ case. In Sec. III, we present the results for the partially integrated angular distributions in six different cases. They can all be expressed in terms of the orthonormal spherical harmonic functions. We also show how the measurement of these partially integrated angular distributions alone will give a wealth of information on the helicity amplitudes. In Sec. IV, we give the relations between the angular momentum helicity amplitudes and the radiative multipole amplitudes in the decays $1\,{}^3D_2 \to \chi_J + \gamma_1$ and $\chi_J \to \psi + \gamma_2$ $(J=0,1,2)$. Finally, in Sec. V we derive the strikingly simple angular distribution of the γ photon in the cascade process $\bar{p}p \to 1\,{}^3D_2 \to 1\,{}^1S_0 + \gamma$.

II. THE COMBINED ANGULAR DISTRIBUTION OF γ_1, γ_2 AND e^- IN $\bar{p}p \to {}^3D_2 \to \chi_J + \gamma_1$ $(J=0,1,2)$ $\to \psi + \gamma_2 + \gamma_1 \to e^- + e^+ + \gamma_2 + \gamma_1$

In this section we will consider the cascade process

$$\begin{aligned} &\bar{p}(\lambda_1)p(\lambda_2) \to {}^3D_2(\nu) \to \chi_J(\sigma) + \gamma_1(\mu) \ (J=0,1,2) \\ &\to [\psi(\rho) + \gamma_2(\kappa)] + \gamma_1(\mu) \\ &\to (e^-(\alpha_1) + e^+(\alpha_2)) + \gamma_2(\kappa) + \gamma_1(\mu) \end{aligned}$$

in the $\bar{p}p$ c.m. frame, where the Greek symbols after the particle symbols represent their helicities except for the stationary ${}^3D_2$ resonance in which case the symbol ν represents the Z component of the angular momentum. We choose the Z axis along the direction of motion of χ_J. The X and Y axes of the right-handed coordinate system are otherwise arbitrary. The probability amplitude for the above cascade process can be written (within constant) as a product of the matrix elements for the individual processes. Only the helicities of the initial and the final particles are observed. So we write the probability amplitude as

$$\begin{aligned} T^{\alpha_1\alpha_2,\kappa\mu}_{\lambda_1\lambda_2} = \sum_{\nu}^{-2\to+2} \sum_{\sigma}^{-J\to+J} \sum_{\rho}^{-1\to+1} {}_D\langle {}^3D_2(\nu)|B|\bar{p}(\lambda_1)p(\lambda_2)\rangle_{D\,D}\langle \chi_J(\sigma)\gamma_1(\mu)|A|{}^3D_2(\nu)\rangle_D \\ \times {}_D\langle \psi(\rho)\gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_{D\,D}\langle e^-(\alpha_1)e^+(\alpha_2)|C|\psi(\rho)\rangle_D . \end{aligned} \quad (2)$$

We sum over the helicities of the unobserved particles in Eq. (2). The symbols B, A, E, and C represent the appropriate transition operators. The subscript D attached to the bra or the ket vector indicates that each individual matrix element is evaluated in the ${}^3D_2$ rest frame. In the first two matrix elements the ${}^3D_2$ rest frame is the same as the c.m. frame of the two particles. In the last two matrix elements $\langle \psi\gamma_2|E|\chi_J\rangle$ and $\langle e^-e^+|C|\psi\rangle$ this is not the case. To avoid confusion, we should clarify what we mean by the two-particle helicity states when they are not in their c.m. frame. For example, the two-particle state $|\psi(\rho)\gamma_2(\kappa)\rangle_D$ defined in the ${}^3D_2$ rest frame, which is not the c.m. frame of ψ and γ_2, has the following meaning. First construct the two-particle helicity state $|\psi(\rho)\gamma_2(\kappa)\rangle_{\chi_J}$ in the χ_J rest frame (which is the same as the c.m. frame of ψ and γ_2) according to the usual conventions [3] with ψ and γ_2 having exactly equal and opposite momenta and helicities ρ and κ, respectively. Then,

$$|\psi(\rho)\gamma_2(\kappa)\rangle_D = U_\Lambda({}^3D_2, \chi_J)|\psi(\rho)\gamma_2(\kappa)\rangle_{\chi_J} , \quad (3)$$

where $U_\Lambda(A,B)$ is the unitary operator corresponding to the Lorentz transformation $\Lambda(A,B)$ which takes the system from the Lorentz frame where B is at rest to the Lorentz frame where A is at rest. It is important to c trify this point since in general ψ and γ_2 do not have definite helicities ρ and κ in the $^3D_2$ rest frame. A similar meaning also holds for the two-particle state $|e^-(\alpha_1)e^+(\alpha_2)\rangle_D$.

Let us now consider the matrix elements in Eq. (2), one by one. First,

$$_D\langle {}^3D_2(\nu)|B|\bar{p}(\lambda_1)p(\lambda_2)\rangle_D=\langle 2\nu|B|p(\theta,\phi);\lambda_1\lambda_2\rangle \ , \quad (4)$$

where $p(\theta,\phi)$ is the magnitude of the c.m. momentum of $\bar{p}$ which is taken to be in the direction (θ,ϕ) in the coordinate system we have chosen. Using the usual expansion [3,4] of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we find

$$_D\langle {}^3D_2(\nu)|B|\bar{p}(\lambda_1)p(\lambda_2)\rangle_D=\sqrt{5/4\pi}B_{\lambda_1\lambda_2}D^2_{\nu\lambda}(\phi,\theta,-\phi) \ , \quad (5)$$

where

$$\lambda=\lambda_1-\lambda_2 \ . \quad (6)$$

Because of C and P invariances [3], the angular-momentum helicity amplitudes $B_{\lambda_1\lambda_2}$ are not all independent. We have

$$B_{\lambda_1\lambda_2}\overset{C}{=}-B_{\lambda_2\lambda_1}\overset{P}{=}-B_{-\lambda_1-\lambda_2} \ . \quad (7)$$

Because of Eq. (7)

$$B_{++}=-B_{++}=0$$

and

$$B_{--}=-B_{--}=0 \ . \quad (8)$$

Also we call

$$B_1=\sqrt{2}B_{+-}=-\sqrt{2}B_{-+} \ . \quad (9)$$

Next we consider the matrix element

$$\begin{aligned} _D\langle \chi_J(\sigma)\gamma_1(\mu)|A|^3D_2(\nu)\rangle_D &=\langle p_1(0,0,0);\sigma\mu|A|2\nu\rangle \\ &=\sqrt{5/4\pi}A^J_{\sigma\mu}D^{2*}_{\nu,\sigma-\mu}(0,0,0) \\ &=\sqrt{5/4\pi}A^J_{\sigma\mu}\delta_{\nu,\sigma-\mu} \ . \end{aligned} \quad (10)$$

The C invariance is trivially satisfied in the process $^3D_2\to\chi_J+\gamma_1$. The P invariance of the transition operation leads to

$$A^J_{\sigma\mu}=(-1)^{J+1}A^J_{-\sigma-\mu} \ . \quad (11)$$

We label the $(2J+1)$ linearly independent A amplitudes as

$$A^J_\sigma=A^J_{\sigma 1}=(-1)^{J+1}A^J_{-\sigma-1}, \quad \sigma=-J,-J+1,\ldots,+J \quad (12)$$

with the restriction $-2\le\nu=\sigma-\mu\le+2$. This would mean that for $J=2$, σ cannot take the value -2.

The matrix element for the process $\chi_J\to\psi+\gamma_2$, in the $^3D_2$ and the χ_J rest frames are equal. That is,

$$\begin{aligned} &_D\langle \psi(\rho)\gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_D \\ &\quad={}_{\chi_J}\langle \psi(\rho)\gamma_2(\kappa)|U^\dagger_\Lambda(^3D_2,\chi_J)EU_\Lambda(^3D_2,\chi_J)|\chi_J(\sigma)\rangle_{\chi_J} \\ &\quad={}_{\chi_J}\langle \psi(\rho)\gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_{\chi_J} \ . \end{aligned} \quad (13)$$

In Eq. (13) we used the fact that the transition operator E is invariant under Lorentz transformations: namely,

$$U^\dagger_\Lambda EU_\Lambda=E \ . \quad (14)$$

Using Eq. (13) we can now write

$$_D\langle \psi(\rho)\gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_D={}_{\chi_J}\langle p'(\theta',\phi');\rho\kappa|E|J\sigma\rangle_{\chi_J} \ , \quad (15)$$

where p' is the magnitude of the ψ three-momentum in the χ_J rest frame or the $\psi\gamma_2$ c.m. frame. Again using the expansion [4] of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we obtain

$$\begin{aligned} &_D\langle \psi(\rho)\gamma_2(\kappa)|E|\chi_J(\sigma)\rangle_D \\ &\qquad=\sqrt{2J+1/4\pi}E^J_{\rho\kappa}D^{J*}_{\sigma,\rho-\kappa}(\phi',\theta',-\phi') \ . \end{aligned} \quad (16)$$

The P invariance of the transition operator E leads to

$$E^J_{\rho\kappa}=(-1)^JE^J_{-\rho-\kappa} \ . \quad (17)$$

We label the three independent E amplitudes as

$$E^J_\rho=E^J_{\rho-1,-1}=(-1)^JE^J_{-\rho+1,1}, \quad \rho=0,1,\ldots,J \ . \quad (18)$$

For the matrix element of the final process $\psi(\rho)\to e^-(\alpha_1)+e^+(\alpha_2)$ the situation is more involved. We have

$$\begin{aligned} _D\langle e^-(\alpha_1)e^+(\alpha_2)|C|\psi(\rho)\rangle_D &={}_\psi\langle e^-(\alpha_1)e^+(\alpha_2)|U^\dagger_\Lambda(^3D_2,\psi)CU_\Lambda(^3D_2,\chi_J)U_\Lambda(\chi_J,\psi)|\psi(\rho)\rangle_\psi \\ &={}_\psi\langle e^-(\alpha_1)e^+(\alpha_2)|U^\dagger_\Lambda(^3D_2,\psi)CU_\Lambda(^3D_2,\psi)U^\dagger_\Lambda(^3D_2,\psi)U_\Lambda(^3D_2,\chi_J)U_\Lambda(\chi_J,\psi)|\psi(\rho)\rangle_\psi \\ &={}_\psi\langle e^-(\alpha_1)e^+(\alpha_2)|CU^\dagger_\Lambda(^3D_2,\psi)U_\Lambda(^3D_2,\chi_J)U_\Lambda(\chi_J,\psi)|\psi(\rho)\rangle_\psi \ . \end{aligned} \quad (19)$$

In the first equality of Eqs. (19) we made use of the fact that the single-particle state $|\psi(\rho)\rangle_D$ was also part of the two-particle state of Eq. (16). It was obtained by successively performing two unitary operations corresponding to the two Lorentz transformations, the first taking the ψ state from its rest to the χ_J rest frame and the second taking it from the χ_J rest frame to the $^3D_2$ rest frame. In the last equality of Eqs. (19) we now make use of the fact that

$$U_\Lambda({}^3D_2,\chi_J)U_\Lambda(\chi_J,\psi)=U_\Lambda({}^3D_2,\psi)U_{R_W}\ , \tag{20}$$

where U_{R_W} is the unitary operator corresponding to a pure rotation, usually called "Wigner rotation." Using Eq. (20) and the unitarity of U_Λ Eq. (19) now leads to

$$\begin{aligned}
&{}_D\langle e^-(\alpha_1)e^+(\alpha_2)|C|\psi(\rho)\rangle_D\\
&\quad={}_\psi\langle e^-(\alpha_1)e^+(\alpha_2)|CU_{R_W}|\psi(\rho)\rangle_\psi\\
&\quad={}_\psi\langle e^-(\alpha_1)e^+(\alpha_2)|U_{R_W}U^\dagger_{R_W}CU_{R_W}|\psi(\rho)\rangle_\psi\\
&\quad={}_\psi\langle e^-(\alpha_1)e^+(\alpha_2)|U_{R_W}C|\psi(\rho)\rangle_\psi
\end{aligned} \tag{21}$$

since

$$U^\dagger_{R_W}CU_{R_W}=C\ . \tag{22}$$

Using the expansion [4] of the two-particle helicity state in terms of the angular-momentum states, we can write the right-hand side of Eq. (21) as

$$\begin{aligned}
&{}_\psi\langle e^-(\alpha_1)e^+(\alpha_2)|U_{R_W}C|\psi(\rho)\rangle_\psi\\
&\quad=\sqrt{3/4\pi}D^{1*}_{\rho\alpha}(R_W^{-1}\hat{e}_\psi)C_{\alpha_1\alpha_2}\\
&\quad=\sqrt{3/4\pi}C_{\alpha_1\alpha_2}D^{1*}_{\rho\alpha}(\phi'',\theta'',-\phi'')\ ,
\end{aligned} \tag{23}$$

where

$$\alpha=\alpha_1-\alpha_2 \tag{24}$$

and $\hat{e}_\psi$ is a unit vector in the direction of e^- three-momentum in the ψ rest frame and R_W is the (3×3) rotation matrix and $C_{\alpha_1\alpha_2}$ are the angular-momentum helicity amplitudes.

The Wigner-rotated unit vector $R_W^{-1}\hat{e}_\psi$ can be obtained in the following way. Let R represent the (4×4) matrix whose spatial part gives the (3×3) matrix R_W mentioned above. Then from the definition of U_{R_W} in Eq. (20),

$$R=\Lambda^{-1}({}^3D_2,\psi)\Lambda({}^3D_2,\chi_J)\Lambda(\chi_J,\psi)\ , \tag{25}$$

where the Λ's are the (4×4) Lorentz transformation matrices. Now we note that the electron is highly relativistic in the ψ rest frame and its four-momentum vector P_{e_ψ} can be represented to a very good approximation as

$$P_{e_\psi}=\frac{M_\psi}{2}(1,\hat{e}_\psi)\ , \tag{26}$$

$$\begin{aligned}
R^{-1}P_{e_\psi}&=\Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}({}^3D_2,\chi_J)\Lambda({}^3D_2,\psi)P_{e_\psi}\\
&=\Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}({}^3D_2,\chi_J)\Lambda({}^3D_2,\psi)\\
&\quad\times\Lambda^{-1}({}^3D_2,\psi)P_{e_D}\\
&=\Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}({}^3D_2,\chi_J)P_{e_D}\ ,
\end{aligned} \tag{27}$$

where P_{e_D} is the four-momentum of e^- in the ${}^3D_2$ rest frame. From Eq. (26) we also have

$$R^{-1}P_{e_\psi}=\frac{M_\psi}{2}(1,R_W^{-1}\hat{e}_\psi)\ . \tag{28}$$

Combining Eqs. (27) and (28) we get

$$\frac{M_\psi}{2}(1,R_W^{-1}\hat{e}_\psi)=\Lambda^{-1}(\chi_J,\psi)\Lambda^{-1}({}^3D_2,\chi_J)E_{e_D}(1,\hat{e}_D)\ . \tag{29}$$

The spatial part of Eq. (29) gives, within a normalization factor, the Wigner-rotated unit vector $\hat{\tilde{e}}=(R_W^{-1}\hat{e}_\psi)$ in terms of the angles $(\tilde{\theta}'',\tilde{\phi}'')$ which give the direction of e^- in the ${}^3D_2$ rest frame.

The helicity amplitudes $C_{\alpha_1\alpha_2}$ of Eq. (23) are not all independent. The C and P invariances give

$$C_{\alpha_1\alpha_2}\overset{C}{=}C_{\alpha_2\alpha_1}\overset{P}{=}C_{-\alpha_1-\alpha_2}\ . \tag{30}$$

We call the independent C amplitudes C_0 and C_1, where

$$\begin{aligned}
C_0&=C_{++}=C_{--}\ ,\\
C_1&=C_{+-}=C_{-+}\ .
\end{aligned} \tag{31}$$

If e^+e^- is created through a virtual photon, C_0 can be neglected compared to C_1 if the energy of the electron is much larger than its restmass energy, which is true in the present case.

Using Eqs. (2), (5), (10), (16), and (23) we can write the amplitude for the cascade process as

$$\begin{aligned}
T^{\alpha_1\alpha_2,\kappa\mu}_{\lambda_1\lambda_2}&=\left[\frac{5}{4\pi}\right]\sqrt{2J+1/4\pi}\sqrt{3/4\pi}C_{\alpha_1\alpha_2}B_{\lambda_1\lambda_2}\\
&\quad\times\sum_\rho^{-1,0,+1}E^J_{\rho\kappa}D^{1*}_{\rho\alpha}(\phi'',\theta'',-\phi'')\sum_\sigma^{-J\to+J}A^J_{\sigma\mu}D^{J*}_{\sigma,\rho-\kappa}(\phi',\theta',-\phi')\sum_\nu^{-2\to+2}\delta_{\nu,\sigma-\mu}D^2_{\nu\lambda}(\phi,\theta,-\phi)\\
&=\left[\frac{5}{4\pi}\right]\sqrt{2J+1/4\pi}\sqrt{3/4\pi}C_{\alpha_1\alpha_2}B_{\lambda_1\lambda_2}\sum_\rho^{-1,0,+1}E^J_{\rho\kappa}D^{1*}_{\rho\alpha}(\phi'',\theta'',-\phi'')\\
&\qquad\times\sum_\sigma^{-J\to+J}A^J_{\sigma\mu}D^{J*}_{\sigma,\rho-\kappa}(\phi',\theta',-\phi')D^2_{\sigma-\mu,\lambda}(\phi,\theta,-\phi)\ .
\end{aligned} \tag{32}$$

The angles (θ',ϕ') giving the direction of ψ and the angles (θ'',ϕ'') giving the direction of $R_W^{-1}\hat{e}_\psi$ are measured in the χ_J and the ψ rest frames, respectively. Later we will relate them to the corresponding angles $(\tilde{\theta},\tilde{\phi}')$ and $(\tilde{\theta}'',\tilde{\phi}'')$ measured in the ${}^3D_2$ rest frame.

When $\bar{p}$ and p are unpolarized, the normalized function describing the angular distribution of γ_1, γ_2, and e^- is given by

$$W(\theta,\phi;\theta',\phi';\theta'',\phi'')=N\frac{1}{4}\sum_{\lambda_1\lambda_2}^{+1/2,-1/2}\sum_{\alpha_1\alpha_2}^{+1/2,-1/2}\sum_{\mu,\kappa}^{+1,-1}T^{\alpha_1\alpha_2,\kappa\mu}_{\lambda_1\lambda_2}T^{\alpha_1\alpha_2,\kappa\mu *}_{\lambda_1\lambda_2}\ , \tag{33}$$

where N is a normalization constant so chosen that W integrated over all the angles will give the value one. After we substitute Eq. (32) into Eq. (33) we have to perform the various sums. Before we do the sums we make use of the Clebsch-Gordan series relation for the D^J functions, namely,

$$D^{j_1}_{m_1m_2}D^{j_2}_{m_1'm_2'}=\sum_{J=|j_1-j_2|}^{j_1+j_2}\langle j_1j_2m_1m_1'|J,m_1+m_1'\rangle\langle j_1j_2m_2m_2'|J,m_2+m_2'\rangle D^J_{m_1+m_1',m_2+m_2'}\ , \tag{34}$$

and the relation

$$D^{j*}_{m_1m_2}=(-1)^{m_1-m_2}D^j_{-m_1,-m_2}\ . \tag{35}$$

Then we see that the various sums in Eq. (33) factor out, or in other words, the angular distribution function W becomes a product of four sums, one involving λ_1 and λ_2, a second involving α_1 and α_2, a third involving ρ, ρ', and κ, and a fourth involving σ, σ', and μ. The sums over λ_1, λ_2 and α_1, α_2 are trivial. The sums over ρ, ρ', κ and σ, σ', μ are performed after we make the following change of variables:

$$\begin{aligned} d&=\rho-\rho'\ ,\\ s&=\rho+\rho'\ ,\\ d'&=\sigma-\sigma'\ ,\\ s'&=\sigma+\sigma'\ . \end{aligned} \tag{36}$$

We also notice that the terms for negative d, d' are the complex conjugates of those with positive d, d'. So in the sums we can restrict ourselves to positive d, d'. The κ and μ sums lead to an expression for the right-hand side of Eq. (33) which is given in terms of the linearly independent angular-momentum helicity amplitudes defined through Eqs. (9), (12), (18), and (31). After a rather long algebra we finally obtain the following expression for the normalized angular distribution:

$$W(\theta,\phi;\theta',\phi';\theta'',\phi'')=\frac{1}{4(4\pi)^3}\sum_{L_1}^{0,2,4}\beta_{L_1}\sum_{L_3}^{0,2}\gamma_{L_3}\sum_{L_2}^{0,1,\to 2J}\sum_{d'}^{0\to \mathrm{Min}(L_3,L_2,J)}\varepsilon^{L_3L_2}_{d'}\sum_{d}^{0\to \mathrm{Min}(L_1,L_2,3)}\alpha^{L_1L_2}_{d}\mathcal{Y}^{L_1L_2L_3}_{dd'}(\theta,\phi;\theta',\phi';\theta'',\phi'')\ , \tag{37}$$

where

$$\beta_{L_1}=-\sqrt{5}\langle 22;1-1|L_10\rangle|B_1|^2\ , \tag{38}$$

$$\gamma_{L_3}=-\sqrt{3}\sum_{\alpha}^{0,1}(-1)^\alpha\langle 11;\alpha-\alpha|L_30\rangle|C_\alpha|^2\ , \tag{39}$$

$$\begin{aligned}\varepsilon^{L_3L_2}_{d'}(J)={}&(-1)^J\sqrt{3(2J+1)}\left[1-\frac{\delta_{d'0}}{2}\right]\sum_{s'(d')}\left[\frac{E^J_{s'+d'}}{2}\frac{E^{J*}_{s'-d'}}{2}+(-1)^{L_2}\frac{E^{J*}_{s'+d'}}{2}\frac{E^J_{s'-d'}}{2}\right]\left\langle JJ;\frac{s'+d'}{2},-\frac{s'-d'}{2}\middle|L_2d'\right\rangle\\ &\times\left\langle 11;\frac{s'+d'-2}{2},-\frac{s'-d'-2}{2}\middle|L_3d'\right\rangle\ ,\end{aligned} \tag{40}$$

$$s'(d')=d',J'+2,\ldots,2J-d'\ ,$$

$$\begin{aligned}\alpha^{L_1L_2}_{d}(J)={}&(-1)^{J+1}\sqrt{5(2J+1)}\left[1-\frac{\delta_{d0}}{2}\right]\sum_{s(d)}\left[\frac{A^J_{s+d}}{2}\frac{A^{J*}_{s-d}}{2}+(-1)^{L_2}\frac{A^{J*}_{s+d}}{2}\frac{A^J_{s-d}}{2}\right]\\ &\times\left\langle JJ;\frac{s+d}{2},-\frac{s-d}{2}\middle|L_2d\right\rangle\left\langle 22;\frac{s+d-2}{2},-\frac{s-d-2}{2}\middle|L_1d\right\rangle\ ,\end{aligned} \tag{41}$$

$$s(d)=-(2J-d),-(2J-d)+2,\ldots,+(2J-d)\ .$$

In the above equations the angular-momentum helicity amplitudes B_1, C_α, E_ρ^J, and A_σ^J are given by Eqs. (9), (31), (18), and (12), respectively. We also use the normalizations

$$|B_1|^2=|C_0|^2+|C_1|^2=\sum_{\rho}^{0\to J}|E_\rho^J|^2=\sum_{\sigma}^{-1\to J}|A_\sigma^J|^2=1 \ . \quad (42)$$

The explicit expressions for the nonzero coefficients β_{L_1}, γ_{L_3}, $\varepsilon_{d'}^{L_3L_2}(J)$, and $\alpha_d^{L_1L_2}(J)$ for $J=0,1,2$ will be given later. Finally the angular function $\mathcal{Y}_{dd'}^{L_1L_2L_3}$ is defined as

$$\begin{aligned}\mathcal{Y}_{dd'}^{L_1L_2L_3}=&(D_{d0}^{L_1}D_{dd'}^{L_2^*}D_{d'0}^{L_3^*}+D_{d0}^{L_1^*}D_{dd'}^{L_2}D_{d'0}^{L_3})\\&+(-1)^{L_2}(D_{d0}^{L_1}D_{d,-d'}^{L_2^*}D_{-d'0}^{L_3^*}\\&+D_{d0}^{L_1^*}D_{d,-d'}^{L_2}D_{-d'0}^{L_3}) \ . \quad (43)\end{aligned}$$

The arguments of the Wigner functions D^{L_1}, D^{L_2}, and D^{L_3} are $(\phi,\theta,-\phi)$ the direction of $\bar{p}$ with respect to χ_J in the $^3D_2$ rest frame, $(\phi',\theta',-\phi')$ the direction of ψ in the χ_J rest frame, and $(\phi'',\theta'',-\phi'')$ the direction of the Wigner-rotated e^- momentum in the ψ rest frame, namely, $R_W^{-1}\hat{e}_\psi$, respectively.

If $(\tilde{\theta},\tilde{\phi}')$ and $(\tilde{\theta}'',\tilde{\phi}'')$ represent the directions of ψ and of the electron in the $^3D_2$ rest frame, they are related to the angles (θ',ϕ') and (θ'',ϕ'') by the relations

$$\phi'=\tilde{\phi}' \ , \quad (44)$$

$$\cos\theta'=\frac{1}{1-\beta_2^2\cos^2\tilde{\theta}'}\left\{(\cos^2\tilde{\theta}'-1)\frac{\beta_2}{\beta_1}+\cos\tilde{\theta}'\sqrt{1-\beta_2^2}\sqrt{1-(\beta_2/\beta_1)^2+\cos^2\tilde{\theta}'[(\beta_2/\beta_1)^2-\beta_2^2]}\right\} . \quad (45)$$

Since $0\le\theta'\le\pi$, $\sin\theta'$ has to be positive and so it will be given by the positive square root

$$\sin\theta'=\sqrt{1-\cos^2\theta'} \quad (46)$$

where $\cos\theta'$ is given by Eq. (45). In Eq. (45), β_1 is the parameter v/c of ψ in the χ_J rest frame and β_2 is the v/c of χ_J in the $^3D_2$ rest frame:

$$\beta_1=\frac{M_{\chi_J}^2-M_\psi^2}{M_{\chi_J}^2+M_\psi^2} \ , \quad (47)$$

$$\beta_2=\frac{M_D^2-M_{\chi_J}^2}{M_D^2+M_{\chi_J}^2} \ , \quad (48)$$

where M_D is the mass of the $^3D_2$ state. The angles (θ'',ϕ'') are related to the angles $(\tilde{\theta}'',\tilde{\phi}'')$ measured in the $^3D_2$ frame by the relations

$$\begin{aligned}\cos\phi''=\frac{1}{\eta'}[&\gamma_2\beta_2\sin\theta'+\cos\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}''\\&+\cos\theta'\sin\phi\sin\tilde{\theta}''\sin\tilde{\phi}''\\&-\sin\theta'\cos\tilde{\theta}''\gamma_2] \ , \quad (49)\end{aligned}$$

$$\sin\phi''=\frac{1}{\eta'}[\cos\phi'\sin\tilde{\theta}''\sin\tilde{\phi}''-\sin\phi'\sin\tilde{\theta}''\cos\tilde{\phi}''] \ , \quad (50)$$

$$\begin{aligned}\cos\theta''=\frac{1}{\eta}[&-\gamma_1\gamma_2(\beta_1+\beta_2\cos\theta')\\&+\gamma_1(\sin\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}''\\&+\sin\theta'\sin\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'')\\&+\gamma_1\gamma_2(\beta_1\beta_2+\cos\theta')\cos\tilde{\theta}''] \ , \quad (51)\end{aligned}$$

$$\sin\theta''=+\sqrt{1-\cos^2\theta''}=\frac{\eta'}{\eta} \ , \quad (52)$$

where

$$\begin{aligned}\eta'=[&(\gamma_2\beta_2\sin\theta'+\cos\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}''\\&+\cos\theta'\sin\phi'\sin\tilde{\theta}''\sin\tilde{\phi}''-\sin\theta'\cos\tilde{\theta}''\gamma_2)^2\\&+(\cos\tilde{\phi}\sin\tilde{\theta}''\sin\tilde{\phi}''-\sin\phi'\sin\tilde{\theta}''\cos\tilde{\phi}'')^2]^{1/2} \ , \quad (53)\end{aligned}$$

$$\begin{aligned}\eta=[&\gamma_1\gamma_2(1+\beta_1\beta_2\cos\theta')\\&-\gamma_1\beta_1(\sin\theta'\cos\phi'\sin\tilde{\theta}''\cos\tilde{\phi}''+\sin\theta'\sin\phi'\sin\tilde{\theta}''\sin\tilde{\phi}'')\\&-\gamma_1\gamma_2(\beta_2+\beta_1\cos\theta')\cos\tilde{\theta}''] \ . \quad (54)\end{aligned}$$

The constants γ_i $(i=1,2)$ are related to β_i $(i=1,2)$ by the relations,

$$\gamma_1=\frac{1}{\sqrt{1-\beta_i^2}} \ . \quad (55)$$

From Eqs. (47) and (48),

$$\begin{aligned}\gamma_1&=\frac{M_{\chi_J}^2+M_\psi^2}{2M_{\chi_J}M_\psi} \ ,\\ \gamma_2&=\frac{M_D^2+M_{\chi_J^2}}{2M_DM_{\chi_J}} \ .\end{aligned} \quad (56)$$

It is also useful to notice that, in Eq. (43),

$$D_{MO}^L=\sqrt{4\pi/2L+1}\,Y_{LM}^* \ . \quad (57)$$

Finally the explicit expressions for the nonzero coefficients in Eq. (37) are

$$\beta_0=|B_1|^2=1\ ,$$
$$\beta_2=-\sqrt{\tfrac{5}{14}}\ ,\qquad (58)$$
$$\beta_4=-\sqrt{\tfrac{8}{7}}\ ,$$
$$\gamma_0=1\ ,$$
$$\gamma_2=\frac{1}{\sqrt{2}}(|C_1|^2-2|C_0|^2)\cong\frac{1}{\sqrt{2}}\ .\qquad (59)$$

$J=0$:

$$\varepsilon_0^{00}=1\ ,$$
$$\varepsilon_0^{20}=\frac{1}{\sqrt{2}}\ ,$$
$$\alpha_0^{00}=1\ ,\qquad (60)$$
$$\alpha_0^{20}=-\sqrt{\tfrac{5}{14}}\ ,$$
$$\alpha_0^{40}=-\sqrt{\tfrac{8}{7}}\ .$$

$J=1$:

$$\varepsilon_0^{00}=1\ ,$$
$$\varepsilon_0^{02}=-\sqrt{2}\left[|E_0|^2-\frac{1}{2}|E_1|^2\right]\ ,$$
$$\varepsilon_0^{20}=\frac{1}{\sqrt{2}}[|E_0|^2-2|E_1|^2]\ ,$$
$$\varepsilon_1^{21}=-3i\ \mathrm{Im}(E_1E_0^*)\ ,\qquad (61)$$
$$\varepsilon_0^{22}=-1\ ,$$
$$\varepsilon_1^{22}=-3\,\mathrm{Re}(E_1E_0^*)\ ,$$
$$\alpha_0^{00}=1\ ,$$
$$\alpha_0^{02}=\frac{1}{\sqrt{2}}[|A_{-1}|^2-2|A_0|^2+|A_1|^2]\ ,$$
$$\alpha_0^{20}=\sqrt{10/7}\left[|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2\right]\ ,$$
$$\alpha_1^{21}=-i\sqrt{15/7}\sqrt{6}\left[\mathrm{Im}(A_0A_{-1}^*)+\frac{1}{\sqrt{6}}\mathrm{Im}(A_1A_0^*)\right]\ ,$$
$$\alpha_0^{22}=\sqrt{5/7}[|A_{-1}|^2+|A_0|^2-|A_1|^2]\ ,$$
$$\alpha_1^{22}=\sqrt{15/7}\sqrt{6}\left[\mathrm{Re}(A_0A_{-1}^*)-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A_0^*)\right]\ ,\qquad (62)$$
$$\alpha_2^{22}=2\sqrt{30/7}\mathrm{Re}(A_1A_{-1}^*)\ ,$$
$$\alpha_0^{40}=\frac{1}{\sqrt{14}}[|A_{-1}|^2-4|A_0|^2+6|A_1|^2]\ ,$$
$$\alpha_1^{41}=-i\sqrt{15/7}[\mathrm{Im}(A_0A_{-1}^*)-\sqrt{6}\mathrm{Im}(A_1A_0^*)]\ ,$$
$$\alpha_0^{42}=\frac{1}{2\sqrt{7}}[|A_{-1}|^2+8|A_0|^2+6|A_1|^2]\ ,$$
$$\alpha_1^{42}=\sqrt{15/7}[\mathrm{Re}(A_0A_{-1}^*)+\sqrt{6}\mathrm{Re}(A_1A_0^*)]\ ,$$
$$\alpha_2^{42}=\sqrt{90/7}\mathrm{Re}(A_1A_{-1}^*)\ .$$

$J=2$:

$$\varepsilon_0^{00}=1\ ,$$
$$\varepsilon_0^{02}=-\sqrt{10/7}\left[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right]\ ,$$
$$\varepsilon_0^{20}=\frac{1}{\sqrt{2}}[|E_0|^2-2|E_1|^2+|E_2|^2]\ ,$$
$$\varepsilon_1^{21}=-3i[\mathrm{Im}(E_1E_0^*)-\sqrt{2/3}\mathrm{Im}(E_2E_1^*)]\ ,$$
$$\varepsilon_0^{22}=-\sqrt{5/7}[|E_0|^2-|E_1|^2-|E_2|^2]\ ,$$
$$\varepsilon_1^{22}=-\sqrt{15/7}[\mathrm{Re}(E_1E_0^*)-\sqrt{6}\mathrm{Re}(E_2E_1^*)]\ ,\qquad (63)$$
$$\varepsilon_2^{22}=2\sqrt{30/7}\mathrm{Re}(E_2E_0^*)\ ,$$
$$\varepsilon_1^{23}=i\sqrt{6}[\mathrm{Im}(E_1E_0^*)+\sqrt{3/2}\mathrm{Im}(E_2E_1^*)]\ ,$$
$$\varepsilon_2^{23}=i\sqrt{30}\mathrm{Im}(E_2E_0^*)\ ,$$
$$\varepsilon_0^{24}=\frac{3}{\sqrt{7}}\left[|E_0|^2+\frac{4}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right]\ ,$$
$$\varepsilon_1^{24}=\sqrt{15/7}[\sqrt{6}\mathrm{Re}(E_1E_0^*)+\mathrm{Re}(E_2E_1^*)]\ ,$$
$$\varepsilon_2^{24}=\sqrt{90/7}\mathrm{Re}(E_2E_0^*)\ ,$$
$$\varepsilon_0^{04}=\frac{1}{\sqrt{14}}[6|E_0|^2-4|E_1|^2+|E_2|^2]\ ,$$
$$\alpha_0^{00}=1\ ,$$
$$\alpha_0^{02}=-\sqrt{5/14}[|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2]\ ,$$
$$\alpha_0^{04}=-2\sqrt{2/7}\left[|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right]\ ,$$
$$\alpha_0^{20}=\sqrt{10/7}\left[|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2-\frac{1}{2}|A_2|^2\right]\ ,$$
$$\alpha_1^{21}=-3i\sqrt{10/7}\left[\mathrm{Im}(A_0A_{-1}^*)+\frac{1}{\sqrt{6}}\mathrm{Im}(A_1A_0^*)\right.$$
$$\left.-\frac{1}{3}\mathrm{Im}(A_2A_1^*)\right]\ ,$$
$$\alpha_0^{22}=-\frac{5}{7}[|A_{-1}|^2-|A_0|^2-|A_1|^2+|A_2|^2]\ ,$$
$$\alpha_1^{22}=\frac{5}{7}\sqrt{6}\left[\mathrm{Re}(A_0A_{-1}^*)-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A_0^*)\right.$$
$$\left.+\mathrm{Re}(A_2A_1^*)\right]\ ,$$
$$\alpha_2^{22}=\frac{10}{7}\sqrt{6}[\mathrm{Re}(A_1A_{-1}^*)+\mathrm{Re}(A_2A_0^*)]\ ,$$
$$\alpha_1^{23}=2i\sqrt{15/7}\left[\mathrm{Im}(A_0A_{-1}^*)+\frac{1}{\sqrt{6}}\mathrm{Im}(A_1A_0^*)\right.$$
$$\left.+\frac{1}{2}\mathrm{Im}(A_2A_1^*)\right]\ ,$$

$$\alpha_2^{23}=5i\sqrt{6/7}\,\mathrm{Im}(A_2A_0^*)\ ,$$

$$\alpha_0^{24}=-\frac{4\sqrt{5}}{7}\left[|A_{-1}|^2+\frac{3}{4}|A_0|^2-|A_1|^2+\frac{1}{8}|A_2|^2\right]\ ,$$

$$\alpha_1^{24}=-\frac{30}{7}\left[\mathrm{Re}(A_0A_{-1}^*)-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A_0^*)\right.\qquad(64)$$

$$\left.-\frac{1}{6}\mathrm{Re}(A_2A_1^*)\right]\ ,$$

$$\alpha_2^{24}=-\frac{20\sqrt{2}}{7}\left[\mathrm{Re}(A_1A_{-1}^*)-\frac{3}{4}\mathrm{Re}(A_2A_0^*)\right]\ ,$$

$$\alpha_0^{40}=\frac{1}{\sqrt{14}}[|A_{-1}|^2-4|A_0|^2+6|A_1|^2-4|A_2|^2]\ ,$$

$$\alpha_1^{41}=-i\sqrt{15/7}[\mathrm{Im}(A_0A_{-1}^*)-\sqrt{6}\,\mathrm{Im}(A_1A_0^*)$$

$$+2\,\mathrm{Im}(A_2A_1^*)]\ ,$$

$$\alpha_0^{42}=-\frac{\sqrt{5}}{14}[|A_{-1}|^2-8|A_0|^2+6|A_1|^2+8|A_2|^2]\ ,$$

$$\alpha_1^{42}=\frac{5}{7}[\mathrm{Re}(A_0A_{-1}^*)+\sqrt{6}\,\mathrm{Re}(A_1A_0^*)-6\,\mathrm{Re}(A_2A_1^*)]\ ,$$

$$\alpha_2^{42}=\frac{15\sqrt{2}}{7}\left[\mathrm{Re}(A_1A_{-1}^*)-\frac{4}{3}\mathrm{Re}(A_2A_0^*)\right]\ ,$$

$$\alpha_1^{43}=i\sqrt{10/7}[\mathrm{Im}(A_0A_{-1}^*)-\sqrt{6}\,\mathrm{Im}(A_1A_0^*)$$

$$-3\,\mathrm{Im}(A_2A_1^*)]\ ,$$

$$\alpha_2^{43}=-10i\sqrt{2/7}\,\mathrm{Im}(A_2A_0^*)\ ,$$

$$\alpha_3^{43}=-5i\,\mathrm{Im}(A_2A_{-1}^*)\ ,$$

$$\alpha_0^{44}=-\frac{2}{7}[|A_{-1}|^2+6|A_0|^2+6|A_1|^2+|A_2|^2]\ ,$$

$$\alpha_1^{44}=-\frac{5\sqrt{6}}{7}[\mathrm{Re}(A_0A_{-1}^*)+\sqrt{6}\,\mathrm{Re}(A_1A_0^*)$$

$$+\mathrm{Re}(A_2A_1^*)]\ ,$$

$$\alpha_2^{44}=-\frac{10\sqrt{6}}{7}[\mathrm{Re}(A_1A_{-1}^*)+\mathrm{Re}(A_2A_0^*)]\ ,$$

$$\alpha_3^{44}=-5\,\mathrm{Re}(A_2A_{-1}^*)\ .$$

Equation (37) together with the expressions for the nonvanishing coefficients [Eqs. (59)–(64)] give the angular distribution of the two γ phonons γ_1 and γ_2 and of the electron as a function of the angles (θ,ϕ), (θ',ϕ'), and (θ'',ϕ''). Equation (37) looks complicated only because it gives the combined angular distribution of three particles. Nevertheless, it is useful. Since the result is expressed as a sum of products of the orthonormal Wigner D^J functions, we can obtain the coefficient of the $\mathcal{Y}_{dd'}^{L_1L_2L_3}$ angular function as

$$\beta_{L_1}\gamma_{L_3}\varepsilon_{d'}^{L_3L_2}\alpha_d^{L_1L_2}[1+(-1)^{L_2}\delta_{d0}][1+(-1)^{L_2}\delta_{d'0}]$$
$$=(2L_1+1)(2L_2+1)(2L_3+1)\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\mathcal{Y}_{dd'}^{L_1L_2L_3^*}\,d\Omega\,d\Omega'\,d\Omega''\ ,\qquad(65)$$

where $\mathcal{Y}_{dd'}^{L_1L_2L_3}$ is defined by Eq. (43). If the angular distribution W is determined at sufficiently large number of points, the integral on the right-hand side can be performed numerically for all possible allowed values of L_1, L_2, L_3, d, and d'. A close examination of the expressions for β_{L_1}, γ_{L_3}, $\varepsilon_{d'}^{L_3L_2}$, and $\alpha_d^{L_1L_2}$ given by Eqs. (58)–(64) shows that this will enable us to determine not only all the relative magnitudes of the helicity amplitudes but also the cosines of their relative phases. For the $J=2$ case we can also determine the sines of all the relative phases which will then uniquely determine all the relative phases. For the $J=1$ case, the sines of the relative phases are not completely determined. If the sine of the relative phase between any two amplitudes is known, it will then fix the relative phases of all the other helicity amplitudes. We can also determine the absolute magnitudes of the helicity amplitudes, once the branching ratios for the different decays are known.

The angles in the expression of Eq. (37) are not all measured in the $^3D_2$ rest frame or equivalently the lab frame. But they are related to the angles measured in the lab frame through Eqs. (44)–(56). Even though these equations may look formidable, once the angular distribution is known in terms of the laboratory angles, they can be easily expressed in terms of the angles (θ,ϕ), (θ',ϕ'), and (θ'',ϕ'') through a computer program generated with the help of Eqs. (44)–(56). This kind of transformation is routinely done by experimentalists.

III. PARTIALLY INTEGRATED ANGULAR DISTRIBUTIONS

In order to get further physical insight into the angular distributions, we consider below, the partially integrated angular distributions. They will look a lot simpler and they can all be expressed in terms of the spherical harmonics. From the partially integrated angular distributions alone we can get, for the $J=1$ case, all the information we obtained for the angular momentum helicity amplitudes by considering the combined angular distribution of γ_1, γ_2, and e^-. For the $J=2$ case, we can only get the relative magnitudes of all the helicity amplitudes and the cosines of their relative phases from the partially integrated angular distributions. To get all the relative phases uniquely we have to consider the combined angular distribution of γ_1, γ_2, and e^-. We consider six different cases of partially integrated results. In deriving these results, we make use of the identities

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} D_{mm'}^{j*}(\alpha,\beta,\gamma) D_{\mu\mu'}^{j'}(\alpha,\beta,\gamma)\sin\beta d\beta = \frac{8\pi^2}{(2j+1)}\delta_{m\mu}\delta_{m'\mu'}\delta_{jj'} , \quad (66)$$

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} D_{mm'}^{j}(\alpha,\beta,\gamma)\sin\beta d\beta = \frac{8\pi^2}{2j+1}\delta_{m0}\delta_{m'0}\delta_{j0} , \quad (67)$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} D_{MM'}^{L*}(\phi,\theta,-\phi)\sin\theta d\theta = \int_0^{2\pi} d\phi \int_0^{\pi} D_{MM'}^{L}(\theta)\sin\theta d\theta = 2\pi\delta_{M-M',0}\int_{-1}^{+1} d_{MM'}^{L}(\theta)\sin\theta d\theta . \quad (68)$$

Case 1: We will integrate over (θ',ϕ') and (θ'',ϕ''). Only the angular distribution of the γ photon γ_1 is measured. We obtain

$$\tilde{W}(\theta,\phi) = \int W(\theta,\phi;\theta',\phi';\theta'',\phi'')d\Omega' d\Omega'' = \frac{1}{\sqrt{4\pi}}\left[Y_{00}(\theta) - \frac{1}{\sqrt{14}}\alpha_0^{20}Y_{20}(\theta) - \frac{1}{3}\sqrt{8/7}\alpha_0^{40}Y_{40}(\theta)\right] , \quad (69)$$

where θ is the angle between p and γ_1 in the ${}^3D_2$ rest frame. The coefficients α_0^{20} and α_0^{40} have the following expressions for the $J=1$ and $J=2$ cases:

$J=1$:

$$\alpha_0^{20} = \sqrt{10/7}\left[|A_{-1}|^2 - \frac{1}{2}|A_0|^2 - |A_1|^2\right] , \qquad \alpha_0^{40} = \frac{1}{\sqrt{14}}[|A_{-1}|^2 - 4|A_0|^2 + 6|A_1|^2] . \quad (70)$$

Normalization gives

$$|A_{-1}|^2 + |A_0|^2 + |A_1|^2 = 1 .$$

$J=2$:

$$\alpha_0^{20} = \sqrt{10/7}\left[|A_{-1}|^2 - \frac{1}{2}|A_0|^2 - |A_1|^2 - \frac{1}{2}|A_2|^2\right] , \qquad \alpha_0^{40} = \frac{1}{\sqrt{14}}[|A_{-1}|^2 - 4|A_0|^2 + 6|A_1|^2 - 4|A_2|^2] . \quad (71)$$

Normalization, $|A_{-1}|^2+|A_0|^2+|A_1|^2+|A_2|^2=1$. For the $J=1$ case, we can determine the magnitudes of all the helicity amplitudes from Eqs. (70) since there are three equations and three unknowns. For the $J=2$ case, this is not possible since there are four unknowns and only three equations.

Case 2: We will integrate over (θ,ϕ) and (θ'',ϕ''). Only the angular distribution of γ_2 is measured:

$$\tilde{W}(\theta',\phi') = \int W(\theta,\phi;\theta',\phi';\theta'',\phi'')d\Omega d\Omega'' . \quad (72)$$

We will write the results separately for the $J=1$ and the $J=2$ cases.

$J=1$:

$$\tilde{W}(\theta') = \frac{1}{\sqrt{4\pi}}\left[Y_{00}(\theta') - \frac{1}{\sqrt{5}}\left[|E_0|^2 - \frac{1}{2}[E_1|^2\right](|A_{-1}|^2 - 2|A_0|^2 + |A_1|^2)Y_{20}(\theta')\right] . \quad (73)$$

Since we already know $|A_{-1}|$, $|A_0|$, and $|A_1|$, we can now determine $(|E_0|^2 - \frac{1}{2}|E_1|^2)$. Since $|E_0|^2+|E_1|^2=1$, this determines $|E_0|$ and $|E_1|$.

$J=2$:

$$\tilde{W}(\theta') = \frac{1}{\sqrt{4\pi}}\left[Y_{00}(\theta') + \frac{\sqrt{5}}{7}\left[|E_0|^2 + \frac{1}{2}|E_1|^2 - |E_2|^2\right](|A_{-1}|^2 + 2|A_0|^2 + |A_1|^2 - 2|A_2|^2)Y_{20}(\theta') - \frac{2}{21}(6|E_0|^2 - 4|E_1|^2 + |E_2|^2)\left[|A_{-1}|^2 - \frac{3}{2}|A_0|^2 + |A_1|^2 - \frac{1}{4}|A_2|^2\right]Y_{40}(\theta')\right] . \quad (74)$$

Also, $|E_0|^2+|E_1|^2+|E_2|^2=1$. Here, $(\pi-\theta')$ is the angle between χ_J and γ_2 in the χ_J rest frame. It is related to $\tilde{\theta}'$, the angle measured in the ${}^3D_2$ rest frame or the lab frame, by a Lorentz transformation given by Eqs. (45) and (46).

Case 3: We will integrate over (θ,ϕ) and (θ',ϕ'). Only the angular distribution of e^- is measured:

$$\tilde{W}(\theta'',\phi'') = \int W(\theta,\phi;\theta',\phi';\theta''\phi'')d\Omega d\Omega'$$
$$= \frac{1}{\sqrt{4\pi}}\left[Y_{00}(\theta'') + \frac{1}{\sqrt{20}}(|E_0|^2 - 2|E_1|^2)Y_{20}(\theta'')\right] \quad \text{for } J=1 ,$$
$$= \frac{1}{\sqrt{4\pi}}\left[Y_{00}(\theta'') + \frac{1}{\sqrt{20}}(|E_0|^2 - 2|E_1|^2 + |E_2|^2)Y_{20}(\theta'')\right] \quad \text{for } J=2 . \quad (75)$$

By measuring the partially integrated angular distributions of cases (1)–(3) we can determine the magnitudes of all the helicity amplitudes for the $J=1$ and for the $J=2$ cases.

We should mention that θ'' is the "Wigner-rotated" angle of e^- with the χ_J momentum, in the ψ rest frame. It can be related to $\tilde{\theta}''$, the angle between e^- and χ_J in the $^3D_2$ rest frame, by Eqs. (49)–(56).

Case 4: We will integrate over (θ'',ϕ''), the direction of the final electron. The combined angular distribution of the two photons γ_1 and γ_2 are measured:

$$\tilde{\tilde{W}}(\theta,\phi;\theta',\phi')=\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')d\Omega''$$

$$=\frac{1}{8\pi}\sum_{L_1}^{0,2,4}\beta_{L_1}\sum_{L_2}^{0,2,\to 2J}\varepsilon_0^{0L_2}\sum_{d}^{0\to\mathrm{Min}(L_1,L_2,3)}\alpha_d^{L_1L_2}\frac{2}{\sqrt{(2L_1+1)(2L_2+1)}}\mathrm{Re}\{Y^*_{L_1d}(\theta,\phi)Y_{L_2d}(\theta',\phi')\}\ . \quad (76)$$

Using Eqs. (58)–(64), we now express the right-hand side of Eq. (76) in terms of the helicity amplitudes. We consider the $J=1$ and the $J=2$ cases separately.

(a) $J=1$:

$$\begin{aligned}\tilde{\tilde{W}}(\theta,\phi;\theta',\phi')=\frac{1}{16\pi^2}\Bigg[&1-\sqrt{4\pi/5}\left[|E_0|^2-\frac{1}{2}|E_1|^2\right](|A_{-1}|^2-2|A_0|^2+|A_1|^2)Y_{20}(\theta')\\&-\frac{5}{7}\sqrt{4\pi/5}(|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2)Y_{20}(\theta)\\&+\frac{4\pi}{7}(|A_{-1}|^2+|A_0|^2-|A_1|^2)Y_{20}(\theta)Y_{20}(\theta')\\&+\frac{12\sqrt{2}\pi}{7}\left[\mathrm{Re}(A_0A^*_{-1})-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A^*_0)\right]\mathrm{Re}(Y^*_{21}(\theta,\phi)Y_{21}(\theta',\phi'))\\&+\frac{8\sqrt{6}\pi}{7}\mathrm{Re}(A_1A^*_{-1})\mathrm{Re}(Y^*_{22}(\theta,\phi)Y_{22}(\theta',\phi'))-\frac{4\sqrt{\pi}}{21}(|A_{-1}|^2-4|A_0|^2+6|A_1|^2)Y_{40}(\theta)\\&+\frac{8\pi}{7}\frac{1}{45}\left[|E_0|^2-\frac{1}{2}|E_1|^2\right](|A_{-1}|^2+8|A_0|^2+6|A_1|^2)Y_{40}(\theta)Y_{20}(\theta')\\&+\frac{16\pi}{7\sqrt{3}}\left[|E_0|^2-\frac{1}{2}|E_1|^2\right](\mathrm{Re}(A_0A^*_{-1})+\sqrt{6}\mathrm{Re}(A_1A^*_0))\mathrm{Re}(Y^*_{41}(\theta,\phi)Y_{21}(\theta',\phi'))\\&+\frac{8\sqrt{2}\pi}{7}\left[|E_0|^2-\frac{1}{2}|E_1|^2\right]\mathrm{Re}(A_1A^*_{-1})\mathrm{Re}(Y^*_{42}(\theta,\phi)Y_{22}(\theta',\phi'))\Bigg]\ .\end{aligned} \quad (77)$$

(b) $J=2$:

$$\begin{aligned}\tilde{\tilde{W}}(\theta,\phi;\theta',\phi')=\frac{1}{16\pi^2}\Bigg[&1+\frac{5}{7}\sqrt{4\pi/5}(|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2)(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2)Y_{20}(\theta')\\&-\frac{4}{21}\sqrt{\pi}(6|E_0|^2-4|E_1|^2+|E_2|^2)\left[|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right]Y_{40}(\theta')\\&-\frac{2\sqrt{5}}{7}\sqrt{\pi}\left[|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2-\frac{1}{2}|A_2|^2\right]Y_{20}(\theta)-\frac{20}{49}\pi\left[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right]\\&\times(|A_{-1}|^2-|A_0|^2-|A_1|^2+|A_2|^2)Y_{20}(\theta)Y_{20}(\theta')\\&+\frac{20\sqrt{6}}{49}\pi\left[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right]\left[\mathrm{Re}(A_0A^*_{-1})-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A^*_0)+\mathrm{Re}(A_2A^*_1)\right]\\&\times\mathrm{Re}(Y^*_{21}(\theta,\phi)Y_{21}(\theta',\phi'))\\&+\frac{40\sqrt{6}\pi}{49}\Big[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2)(\mathrm{Re}(A_1A^*_{-1})+\mathrm{Re}(A_2A^*_0))\mathrm{Re}(Y^*_{22}(\theta,\phi)Y_{22}(\theta',\phi'))\\&+\frac{8\sqrt{5}\pi}{147}(6|E_0|^2-4|E_1|^2+|E_2|^2)\left[|A_{-1}|^2+\frac{3}{4}|A_0|^2-|A_1|^2+\frac{1}{8}|A_2|^2\right]Y_{20}(\theta)Y_{40}(\theta')\end{aligned}$$

$$+\frac{20}{49}\pi(6|E_0|^2-4|E_1|^2+|E_2|^2)\left[\mathrm{Re}(A_0A_{-1}^*)-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A_0^*)-\frac{1}{6}\mathrm{Re}(A_2A_1^*)\right]$$
$$\times\mathrm{Re}(Y_{21}^*(\theta,\phi)Y_{41}(\theta',\phi'))$$
$$+\frac{80\pi}{147\sqrt{2}}(6|E_0|^2-4|E_1|^2+|E_2|^2)\left[\mathrm{Re}(A_1A_{-1}^*)-\frac{3}{4}\mathrm{Re}(A_2A_0^*)\right]\mathrm{Re}(Y_{22}^*(\theta,\phi)Y_{42}(\theta',\phi'))$$
$$-\frac{2\sqrt{2}}{21}\sqrt{\pi}(|A_{-1}|^2-4|A_0|^2+6|A_1|^2-4|A_2|^2)Y_{40}(\theta)-\frac{8\sqrt{5}\pi}{147}\left[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right]$$
$$\times(|A_{-1}|^2-8|A_0|^2+6|A_1|^2+8|A_2|^2)Y_{40}(\theta)Y_{20}(\theta')+\frac{80\pi}{147}\left[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right]$$
$$\times(\mathrm{Re}(A_0A_{-1}^*)+\sqrt{6}\mathrm{Re}(A_1A_0^*)-6\,\mathrm{Re}(A_2A_1^*))\mathrm{Re}(Y_{41}^*(\theta,\phi)Y_{21}(\theta',\phi'))$$
$$+\frac{80\sqrt{2}\pi}{49}\left[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right]\left[\mathrm{Re}(A_1A_{-1}^*)-\frac{4}{3}\mathrm{Re}(A_2A_0^*)\right]\mathrm{Re}(Y_{42}^*(\theta,\phi)Y_{22}(\theta',\phi'))$$
$$+\frac{8\sqrt{2}\pi}{9\times 49}(6|E_0|^2-4|E_1|^2+|E_2|^2)(|A_{-1}|^2+6|A_0|^2+6|A_1|^2+|A_2|^2)Y_{40}(\theta)Y_{40}(\theta')$$
$$+\frac{40\sqrt{3}\pi}{9\times 49}(6|E_0|^2-4|E_1|^2+|E_2|^2)(\mathrm{Re}(A_0A_{-1}^*)+\sqrt{6}\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*))$$
$$\times\mathrm{Re}(Y_{41}^*(\theta,\phi)Y_{41}(\theta';\phi'))$$
$$+\frac{80\sqrt{3}\pi}{9\times 49}(6|E_0|^2-4|E_1|^2+|E_2|^2)(\mathrm{Re}(A_1A_{-1}^*)+\mathrm{Re}(A_2A_0^*))\mathrm{Re}(Y_{42}^*(\theta,\phi)Y_{42}(\theta',\phi'))$$
$$\left.+\frac{40\pi}{63}(6|E_0|^2-4|E_1|^2+|E_2|^2)\mathrm{Re}(A_2A_{-1}^*)\mathrm{Re}(Y_{43}^*(\theta,\phi)Y_{43}(\theta',\phi'))\right] . \tag{78}$$

In Eqs. (76)–(78), the angles (θ,ϕ) represent the direction of $\bar{p}$ with Z axis chosen along the χ_J momentum. We can also take this as the direction of γ_1 in the ${}^3D_2$ rest frame with the proton moving along the Z axis. The X and Y axes are arbitrary. The angles (θ',ϕ') represent the direction of ψ in the χ_J rest frame. They are related to the direction of ψ in the lab frame, namely $(\tilde{\theta}',\tilde{\phi}')$, through Eqs. (44)–(46). The angles $(\tilde{\theta}',\tilde{\phi}')$ in the lab frame can be determined by measuring the direction of γ_2 or by measuring the total momentum of e^- and e^+ in the lab frame. By measuring the partially integrated angular distributions in cases (1)–(4) we can determine the magnitudes of all of the E and the A helicity amplitudes as well as the cosines of the relative phases among the A amplitudes. For the $J=1$ case, we can determine the magnitudes of all the A and the E helicity amplitudes by just measuring the partially integrated angular distributions in cases (1) and (2). For the $J=2$ case, however, we need the partially integrated angular distributions in cases (1)–(4).

Case 5: Here we will integrate over (θ,ϕ) or the direction of γ_1. We will only measure the combined angular distribution of γ_2 and e^-. We obtain

$$\tilde{\tilde{W}}(\theta',\phi';\theta'',\phi'')=\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta d\theta d\phi$$
$$=\frac{1}{8\pi}\sum_{L_3}^{0,2}\sum_{L_2}^{0,2,\to 2J}\sum_{d'}^{0\to\mathrm{Min}(L_3,L_2,J)}\varepsilon_{d'}^{L_3L_2}\alpha_0^{0L_2}\frac{2}{\sqrt{(2L_2+1)(2L_3+1)}}\mathrm{Re}\{Y_{L_2d'}(\theta',\phi')Y_{L_3d'}(\theta'',\phi'')\} . \tag{79}$$

Using Eqs. (58)–(64) we now express the right-hand side of Eq. (79) in terms of the A and the E helicity amplitudes. We consider the $J=1$ and the $J=2$ cases separately.

(a) $J=1$:

$$\tilde{\tilde{W}}(\theta',\phi';\theta'',\phi'')=\frac{1}{16\pi^2}\left[1-\sqrt{4\pi/5}\left[|E_0|^2-\frac{1}{2}|E_1|^2\right](|A_{-1}|^2-2|A_0|^2+|A_1|^2)Y_{20}(\theta')\right.$$
$$+\sqrt{\pi/5}(|E_0|^2-2|E_1|^2)Y_{20}(\theta'')-\frac{\sqrt{8}\pi}{5}(|A_{-1}|^2-2|A_0|^2+|A_1|^2)Y_{20}(\theta')Y_{20}(\theta'')$$
$$\left.-\frac{12\pi}{5\sqrt{2}}\mathrm{Re}(E_1E_0^*)(|A_{-1}|^2-2|A_0|^2+|A_1|^2)\mathrm{Re}(Y_{21}(\theta',\phi')Y_{21}(\theta'',\phi''))\right] . \tag{80}$$

(b) $J=2$:

$$\tilde{\tilde{W}}(\theta',\phi';\theta'',\phi'')=\frac{1}{16\pi^2}\left[1+\frac{2}{7}\sqrt{5\pi}\left[|E_0|^2+\frac{1}{2}|E_1|^2-|E_2|^2\right](|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2)Y_{20}(\theta')\right.$$
$$-\frac{4}{21}\sqrt{\pi}(6|E_0|^2-4|E_1|^2+|E_2|^2)\left[|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right]Y_{40}(\theta')$$
$$+\sqrt{2\pi/5}(|E_0|^2-2|E_1|^2+|E_2|^2)Y_{20}(\theta'')+\frac{2\sqrt{2}\pi}{7}(|E_0|^2-|E_1|^2-|E_2|^2)$$
$$\times(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2)\mathrm{Re}(Y_{20}(\theta')Y_{20}(\theta''))$$
$$+\frac{2\sqrt{6}\pi}{7}(\mathrm{Re}(E_1E_0^*)-\sqrt{6}\mathrm{Re}(E_2E_1^*))(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2)$$
$$\times\mathrm{Re}(Y_{21}(\theta',\phi')Y_{21}(\theta'',\phi''))-\frac{8\sqrt{3}\pi}{7}\mathrm{Re}(E_2E_0^*)(|A_{-1}|^2+2|A_0|^2+|A_1|^2-2|A_2|^2)$$
$$\times\mathrm{Re}(Y_{22}(\theta',\phi')Y_{22}(\theta'',\phi''))-\frac{8}{7}\sqrt{2/5}\pi\left[|E_0|^2+\frac{4}{3}|E_1|^2+\frac{1}{6}|E_2|^2\right]$$
$$\times\left[|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right]\mathrm{Re}(Y_{40}(\theta',\phi')Y_{20}(\theta'',\phi''))$$
$$-\frac{8\sqrt{6}\pi}{21}(\sqrt{6}\mathrm{Re}(E_1E_0^*)+\mathrm{Re}(E_2E_1^*))\left[|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right]$$
$$\times\mathrm{Re}(Y_{41}(\theta',\phi')Y_{21}(\theta'',\phi''))-\frac{16\pi}{7}\mathrm{Re}(E_2E_0^*)\left[|A_{-1}|^2-\frac{3}{2}|A_0|^2+|A_1|^2-\frac{1}{4}|A_2|^2\right]$$
$$\left.\times\mathrm{Re}(Y_{42}(\theta',\phi')Y_{22}(\theta'',\phi''))\right]. \tag{81}$$

In Eqs. (80) and (81) the angles (θ',ϕ') give the direction of ψ in the χ_J rest frame, with χ_J momentum in the ${}^3D_2$ rest frame (lab frame) taken as the Z axis. They can be related to the angles $(\tilde{\theta}',\tilde{\phi}')$, the direction of ψ in the ${}^3D_2$ rest frame or the lab frame, through Eqs. (44)–(48). The direction of ψ in the lab frame can be measured by measuring the total momentum of e^- and e^+ in the lab frame or by measuring the direction of γ_2 in the lab frame. The angles (θ'',ϕ'') give the direction of Wigner rotated e^- in the ψ rest frame. They can be related to the angles $(\tilde{\theta}'',\tilde{\phi}'')$ which give the direction of e^- in the ${}^3D_2$ rest frame by Eqs. (49)–(56).

Case 6: Here we will integrate over the angles (θ',ϕ') to obtain the combined angular distribution of γ_1 and e^- alone. We have

$$\tilde{\tilde{W}}(\theta,\phi;\theta'',\phi'')=\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')d\Omega' . \tag{82}$$

We give the results for the $J=1$ and for the $J=2$ cases separately.

(a) $J=1$:

$$\tilde{\tilde{W}}(\theta,\phi;\theta'',\phi'')=\frac{1}{(4\pi)^2}\left[1-\frac{5}{7}\sqrt{4\pi/5}\left[|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2\right]Y_{20}(\theta)\right.$$
$$\left.-\frac{2}{7}\sqrt{4\pi/9}(|A_{-1}|^2-4|A_0|^2+6|A_1|^2)Y_{40}(\theta)\right]\left[1+\frac{1}{2}(|E_0|^2-2|E_1|^2)\sqrt{4\pi/5}Y_{20}(\theta'')\right]$$
$$+\frac{1}{(4\pi)^2}\left[\frac{9\pi}{7\sqrt{2}}\left\{\mathrm{Im}(E_1E_0^*)\left[\mathrm{Im}(A_0A_{-1}^*)+\frac{1}{\sqrt{6}}\mathrm{Im}(A_1A_0^*)\right]\right.\right.$$
$$\left.-\frac{1}{3}\mathrm{Re}(E_1E_0^*)\left[\mathrm{Re}(A_0A_{-1}^*)-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A_0^*)\right]\right\}$$
$$\times\mathrm{Re}(Y_{21}^*(\theta,\phi)Y_{21}(\theta'',\phi''))$$
$$+\frac{2}{7}\sqrt{3}\pi\left\{\mathrm{Im}(E_1E_0^*)(\mathrm{Im}(A_0A_{-1}^*)-\sqrt{6}\mathrm{Im}(A_1A_0^*))-\frac{1}{3}\mathrm{Re}(E_1E_0^*)(\mathrm{Re}(A_0A_{-1}^*)\right.$$
$$\left.\left.+\sqrt{6}\mathrm{Re}(A_1A_0^*))\right\}\mathrm{Re}(Y_{41}^*(\theta,\phi)Y_{21}(\theta'',\phi''))\right]. \tag{83}$$

(b) $J=2$:

$$
\begin{aligned}
\tilde{\tilde{W}}(\theta,\phi;\theta'',\phi'')=\frac{1}{(4\pi)^2}&\left[1-\frac{5}{7}\sqrt{4\pi/5}\left[|A_{-1}|^2-\frac{1}{2}|A_0|^2-|A_1|^2-\frac{1}{2}|A_2|^2\right]Y_{20}(\theta)\right.\\
&\left.-\frac{2}{7}\sqrt{4\pi/9}(|A_{-1}|^2-4|A_0|^2+6|A_1|^2-4|A_2|^2)Y_{40}(\theta)\right]\\
&\times\left[1+\frac{1}{2}\sqrt{4\pi/5}(|E_0|^2-2|E_1|^2+|E_2|^2)Y_{20}(\theta'')\right]\\
&+\frac{1}{(4\pi)^2}\left[\frac{\pi}{2\sqrt{35}}\left\{9\sqrt{10/7}(\mathrm{Im}(E_1E_0^*)-\sqrt{2/3}\,\mathrm{Im}(E_2E_1^*))\right.\right.\\
&\times\left[\mathrm{Im}(A_0A_{-1}^*)+\frac{1}{\sqrt{6}}\mathrm{Im}(A_1A_0^*)-\frac{1}{3}\mathrm{Im}(A_2A_1^*)\right]\\
&-\left[\frac{5}{7}\right]^{3/2}\sqrt{2}(\mathrm{Re}(E_1E_0^*)-\sqrt{6}\,\mathrm{Re}(E_2E_1^*))\\
&\times\left[\mathrm{Re}(A_0A_{-1}^*)-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*)\right]\\
&+\sqrt{10/7}(\mathrm{Im}(E_1E_0^*)+\sqrt{3/2}\,\mathrm{Im}(E_2E_1^*))\\
&\times\left[\mathrm{Im}(A_0A_{-1}^*)+\frac{1}{\sqrt{6}}\mathrm{Im}(A_1A_0^*)+\frac{1}{2}\mathrm{Im}(A_2A_1^*)\right]\\
&-\frac{3}{7}\sqrt{15/7}(\sqrt{6}\,\mathrm{Re}(E_1E_0^*)+\mathrm{Re}(E_2E_1^*))\\
&\left.\times\left[\mathrm{Re}(A_0A_{-1}^*)-\frac{1}{\sqrt{6}}\mathrm{Re}(A_1A_0^*)-\frac{1}{6}\mathrm{Re}(A_2A_1^*)\right]\right\}\\
&\times\mathrm{Re}(Y_{21}^*(\theta,\phi)Y_{21}(\theta'',\phi''))\\
&-\frac{\pi}{3\sqrt{35}}\left\{\frac{40}{7}\sqrt{45/7}\,\mathrm{Re}(E_2E_0^*)(\mathrm{Re}(A_1A_{-1}^*)+\mathrm{Re}(A_2A_0^*))\right.\\
&+15\sqrt{5/7}\,\mathrm{Im}(E_2E_0^*)\mathrm{Im}(A_2A_0^*)-\frac{36}{7}\sqrt{5/7}\,\mathrm{Re}(E_2E_0^*)\\
&\left.\times\left[\mathrm{Re}(A_1A_{-1}^*)-\frac{3}{4}\mathrm{Re}(A_2A_0^*)\right]\right\}\mathrm{Re}(Y_{22}^*(\theta,\phi)Y_{22}(\theta'',\phi''))\\
&+\frac{2\pi}{3\sqrt{35}}\left\{3\sqrt{15/7}(\mathrm{Im}(E_1E_0^*)-\sqrt{2/3}\,\mathrm{Im}(E_2E_1^*))\right.\\
&\times(\mathrm{Im}(A_0A_{-1}^*)-\sqrt{6}\,\mathrm{Im}(A_1A_0^*)+2\,\mathrm{Im}(A_2A_1^*))\\
&-\frac{5}{7}\sqrt{5/21}(\mathrm{Re}(E_1E_0^*)-\sqrt{6}\,\mathrm{Re}(E_2E_1^*))\\
&\times(\mathrm{Re}(A_0A_{-1}^*)+\sqrt{6}\,\mathrm{Re}(A_1A_0^*)-6\,\mathrm{Re}(A_2A_1^*))\\
&+\frac{1}{3}\sqrt{15/7}(\mathrm{Im}(E_1E_0^*)+\sqrt{3/2}\,\mathrm{Im}(E_2E_1^*))\\
&\times(\mathrm{Im}(A_0A_{-1}^*)-\sqrt{6}\,\mathrm{Im}(A_1A_0^*)-3\,\mathrm{Im}(A_2A_1^*))\\
&-\frac{3}{14}\sqrt{10/7}(\sqrt{6}\,\mathrm{Re}(E_1E_0^*)+\mathrm{Re}(E_2E_1^*))\\
&\left.\times(\mathrm{Re}(A_0A_{-1}^*)+\sqrt{6}\,\mathrm{Re}(A_1A_0^*)+\mathrm{Re}(A_2A_1^*))\right\}
\end{aligned}
$$

$$\times \mathrm{Re}(Y^*_{41}(\theta,\phi)Y_{21}(\theta'',\phi''))$$
$$-\frac{4\pi}{9\sqrt{35}}\left\{\frac{60}{6}\sqrt{15/7}\mathrm{Re}(E_2E_0^*)\left[\mathrm{Re}(A_1A^*_{-1})-\frac{4}{3}\mathrm{Re}(A_2A_0^*)\right]\right.$$
$$-10\sqrt{15/7}\mathrm{Im}(E_2E_0^*)\mathrm{Im}(A_2A_0^*)-\frac{18}{7}\sqrt{15/7}\mathrm{Re}(E_2E_0^*)$$
$$\left.\times(\mathrm{Re}(A_1A^*_{-1})+\mathrm{Re}(A_2A_0^*))\right\}\mathrm{Re}(Y^*_{42}(\theta,\phi)Y_{22}(\theta'',\phi''))\bigg] . \quad (84)$$

In Eqs. (83) and (84), the angles (θ,ϕ) represent the direction of $\bar{p}$ in the $^3D_2$ rest frame with Z axis chosen along the χ_J momentum. We can also take θ to be the angle between p and the γ_1 photon in the $^3D_2$ rest frame which we assume to be the same as the lab frame. The angles (θ'',ϕ'') represent the Wigner-rotated direction of e^- in the ψ rest frame. They are related to the angles $(\tilde{\theta}'',\tilde{\phi}'')$ which represent the direction of e^- in the lab frame through Eqs. (49)–(56).

A close examination of the partially integrated results in cases (1)–(5) shows that we can determine the magnitudes of all the helicity amplitudes as well as the cosines of their relative phases by just measuring these angular distributions alone. We can determine the relative magnitudes of all the A helicity amplitudes for the $J=1$ case by just measuring the angular distribution of γ_1 alone, namely case (1). By measuring the angular distribution of γ_2 also [case (2)] we can determine the relative magnitudes of the E helicity amplitudes also for $J=1$. For determining the relative magnitudes of all the A and the E helicity amplitudes for the $J=2$ case we need to measure the angular distributions in cases (1)–(3), namely the angular distributions of γ_1, γ_2, and e^-, separately. By measuring the combined angular distributions of γ_1 and γ_2 (case 4) and of γ_2 and e^- (case 5) we are able to determine all the $\mathrm{Re}(E_1E_j^*)$ and $\mathrm{Re}(A_iA_j^*)$ for both $J=1$ and $J=2$ cases. They, in turn, enable us to determine the cosines of all the relative phases of the A and of the E helicity amplitudes. By measuring the combined angular distribution of γ_1 and e^- we are also able to get some information on the products, $\mathrm{Im}(E_iE_j^*)\mathrm{Im}(A_kA_l^*)$, as is seen from Eqs. (83) and (84). But it is not enough to determine all the sines uniquely. In fact, we can show that if we know the sine of one of the relative phases, the sines of the other relative phases can be determined for both $J=1$ and $J=2$ cases. For the $J=2$ case, we can determine the sines of all the relative phases uniquely, by measuring the combined angular distributions of the three particles, γ_1, γ_2, and e^-. For the $J=1$ case, however, this is not possible. For the $J=0$ case, there is only one helicity amplitude each for both $^3D_2\to\chi_0+\gamma_1$ and $\chi_0\to\psi+\gamma_2$. They are fixed by our normalization. So there is nothing to be determined for the $J=0$ case.

It is of great advantage that we expressed all the angular distributions in terms of orthonormal functions such as Wigner D^J functions and the spherical harmonics. Because of this feature of our results, we can get the coefficients of these functions, which are bilinear functions of the angular-momentum helicity amplitudes, by just doing a numerical integration of the measured angular distributions.

IV. THE RELATIONSHIP BETWEEN THE ANGULAR-MOMENTUM HELICITY AMPLITUDES AND THE RADIATIVE MULTIPOLE AMPLITUDES

The A helicity amplitudes describe the $^3D_2\to\chi_J+\gamma_1$ $(J=0,1,2)$ transition and the E helicity amplitudes describe the transition $\chi_J\to\psi+\gamma_2$ $(J=0,1,2)$. For $J=0$, there is only one independent helicity amplitude in each transition. On the other hand, for $J=1$, there are three helicity amplitudes in $^3D_2\to\chi_J+\gamma_1$ and two in $\chi_J\to\psi+\gamma_2$. Finally for $J=2$, there are four independent angular-momentum helicity amplitudes in $^3D_2\to\chi_J+\gamma_1$ and three in $\chi_J\to\psi+\gamma_2$. The number of independent helicity amplitudes is also equal to the number of radiative multipole amplitudes present in these decays. The relationship between the helicity amplitudes and the multipole amplitudes are given by [5,6] the orthogonal transformations

$$A^J_\sigma=\sum_{\mathrm{Max}(k=|2-J|;1)} a^J_k\left[\frac{2k+1}{5}\right]^{1/2}\langle kJ;-1,\sigma|2,\sigma-1\rangle ,$$
$$E^J_\rho=\sum_{k=1}^{J+1} e^J_k\left[\frac{2k+1}{2J+1}\right]^{1/2}\langle k1;1,\rho-1|J\rho\rangle,\quad J=0,1,2 , \quad (85)$$

where a^J_k and e^J_k are the radiative multipole amplitudes in $^3D_2\to\chi_J+\gamma_1$ and $\chi_J\to\psi+\gamma_2$, respectively. Since the transformations of Eqs. (85) are orthogonal:

$$\sum_\sigma |A^J_\sigma|^2=\sum_k |a^J_k|^2=1 ,$$
$$\sum_\rho |E^J_\rho|^2=\sum_k |e^J_k|^2=1 . \quad (86)$$

The decay process $^3D_2\to\chi_2+\gamma_1$ is especially noteworthy. There are four multipole amplitudes E_1, M_2, E_3, and M_4 in this case as there are four independent angular-momentum helicity amplitudes. In principle, we can calculate all these multipole amplitudes from the experimentally measured combined angular distribution of γ_1, γ_2, and e^-. In any potential model of quarkonium if we work out the radiative transition operator to relative order v^2/c^2, there are no terms whose rank is above three. So the M_4 multipole amplitude is zero to order v^2/c^2 in any potential model. So by measuring the M_4 amplitude one can test the approximate nonrelativistic nature of quarkonium and the validity of the potential models.

V. THE ANGULAR DISTRIBUTION OF THE PHOTON IN THE PROCESS $\bar{p}p \to 1\,^3D_2 \to 1\,^1S_0 + \gamma$

Next we turn to the angular distribution of the photon in the cascade process

$$\bar{p}p \to 1\,^3D_2 \to 1\,^1S_0 + \gamma \ .$$

The predicted mass [7] of $1\,^3D_2$ state of charmonium in the nonsingular potential model of Gupta *et al.* [8] is around 3826 MeV. So the above γ photon will have an energy of about 840 MeV. The calculation of the angular distribution of the photon in the above process is very straightforward. The amplitude for the process

$$\bar{p}(\lambda_1)p(\lambda_2) \to {}^3D_2(\nu) \to {}^1S_0 + \gamma(\mu)$$

can be written as

$$T^{\mu}_{\lambda_1\lambda_2} = \sum_{\nu=-2}^{+2} \langle \gamma(\mu), {}^1S_0 | A | {}^3D_2(\nu) \rangle \times \langle {}^3D_2(\nu) | B | \bar{p}(\lambda_1)p(\lambda_2) \rangle \ . \tag{87}$$

Both of the above matrix elements are calculated in the $^3D_2$ rest frame or the $\bar{p}p$ c.m. frame. We will choose the Z axis along the direction of the γ momentum in the $^3D_2$ rest frame. Then $^1S_0(\eta_c)$ will be along the negative Z axis. The X and Y axes are otherwise arbitrary. The antiproton $\bar{p}$ will have its momentum in the direction (θ,ϕ). Then

$$\langle {}^3D_2(\nu) | B | \bar{p}(\lambda_1)p(\lambda_2) \rangle = \langle 2\nu | B | p(\theta,\phi); \lambda_1\lambda_2 \rangle = \sqrt{5/4\pi} B_{\lambda_1\lambda_2} D^2_{\nu\lambda}(\phi,\theta,-\phi) \ , \tag{88}$$

where

$$\lambda = \lambda_1 - \lambda_2 \ , \tag{89}$$

$$\langle \gamma(\mu), \eta_c | A | {}^3D_2(\nu) \rangle = \langle p'(0,0,0); \mu 0 | A | 2\nu \rangle = \sqrt{5/4\pi} A_\mu D^{2*}_{\nu\mu}(0,0,0) = \sqrt{5/4\pi} A_\mu \delta_{\nu\mu} \ . \tag{90}$$

As before, by C and P invariances of the transition operators and the transformation properties of the states under C and P we obtain

$$B_{\lambda_1\lambda_2} \overset{C}{=} -B_{\lambda_2\lambda_1} \ , \qquad B_{\lambda_1\lambda_2} \overset{P}{=} -B_{-\lambda_1-\lambda_2} \ . \tag{91}$$

We call

$$B_1 = \sqrt{2} B_{+-} = -\sqrt{2} B_{-+} \ ,$$

$$B_0 = \sqrt{2} B_{++} = -\sqrt{2} B_{++} = -\sqrt{2} B_{--} = 0 \ , \tag{92}$$

$$A_\mu \overset{P}{=} A_{-\mu} (\mu = \pm 1) \ . \tag{93}$$

We will call

$$A_1 = A_{-1} = A \ . \tag{94}$$

Substituting Eqs. (88) and (90) into Eq. (87) we obtain

$$T^{\mu}_{\lambda_1\lambda_2} = \frac{5}{4\pi} B_{\lambda_1\lambda_2} A_\mu D^2_{\mu\lambda}(\phi,\theta,-\phi) \ . \tag{95}$$

If N is the normalization constant the normalized angular distribution function of the photon will be given by

$$W(\theta,\phi) = N \frac{1}{4} \sum_{\lambda_1\lambda_2}^{\pm 1/2} \sum_{\mu}^{\pm 1} T^{\mu}_{\lambda_1\lambda_2} T^{\mu *}_{\lambda_1\lambda_2} = N \frac{1}{4} \frac{5}{4\pi} \sum_{\lambda_1\lambda_2}^{\pm 1/2} |B_{\lambda_1\lambda_2}|^2 \sum_{\mu}^{\pm 1} |A_\mu|^2 D^2_{\mu\lambda} D^{2*}_{\mu\lambda} \ . \tag{96}$$

Using Eqs. (34) and (35), Eq. (96) reduces to

$$W(\theta,\phi) = \frac{5N}{16\pi} \sum_{\lambda_1\lambda_2}^{\pm 1/2} |B_{\lambda_1\lambda_2}|^2 (-1)^\lambda \sum_{\mu}^{\pm 1} |A_\mu|^2 (-1)^\mu \sum_{L}^{\Delta(2,2,L)} \langle 22; \mu-\mu | L0 \rangle \langle 22; \lambda-\lambda | L0 \rangle D^{L*}_{00}(\phi,\theta-\phi)$$

$$= \frac{5N}{16\pi} \sum_{L}^{\Delta(2,2,L)} \beta_L \alpha_L \sqrt{4\pi/2L+1}\, Y_{L0}(\theta,\phi) \ . \tag{97}$$

In Eq. (97),

$$\begin{aligned}\beta_L &= \sum_{\lambda_1\lambda_2}^{\pm 1/2} B_{\lambda_1\lambda_2}|^2(-1)^{\lambda}\langle 22;\lambda-\lambda|L0\rangle \\ &= -|B_{++}|^2\langle 22;1-1|L0\rangle - |B_{-+}|^2\langle 22;-11|L0\rangle \\ &= -\frac{1}{2}|B_1|^2(1+(-1)^L)\langle 22;1-1|L0\rangle \\ &= -|B_1|^2\langle 22;1-1|L0\rangle \ , \qquad (98)\end{aligned}$$

where in the last line L is always an even integer. Also,

$$\begin{aligned}\alpha_L &= \sum_{\mu}^{\pm 1} |A_\mu|^2(-1)^{\mu}\langle 22;\mu-\mu|L0\rangle \\ &= -|A|^2\langle 22;1-1|L0\rangle - |A|^2\langle 22;-11|L0\rangle \\ &= -|A|^2(1+(-1)^L)\langle 22;1-1|L0\rangle \\ &= -2|A|^2\langle 22;1-1|L0\rangle \ , \qquad (99)\end{aligned}$$

where in the last line, L is again always even. Substituting for the Clebsch-Gordan coefficients and choosing the normalization constant N such that $W(\theta,\phi)$ integrated over θ and ϕ will give the value one, Eq. (97) will simplify to

$$W(\theta,\phi)=\frac{1}{\sqrt{4\pi}}\left[Y_{00}+\frac{\sqrt{5}}{14}Y_{20}(\theta)+\frac{8}{21}Y_{40}(\theta)\right] , \qquad (100)$$

where θ is the angle between γ and $\bar{p}$ in the $\bar{p}p$ c.m. frame. This strikingly simple angular distribution of the photon with no unknown coefficients can be used as a signal for the formation of the triplet D state in unpolarized $\bar{p}p$ collisions. In other words the confirmation of a γ photon of energy around 840 MeV and with the above angular distribution will confirm the discovery of the $^3D_2$ state of charmonium in $\bar{p}p$ collisions.

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[1] F. L. Ridener, Jr. and K. J. Sebastian, Phys. Rev. D **49**, 4617 (1994).
[2] F. L. Ridener, Jr. and K. J. Sebastian, Phys. Rev. D **49**, 5830 (1994).
[3] A. D. Martin and T. D. Spearman, *Elementary Particle Theory* (North-Holland, Amsterdam, 1970).
[4] F. L. Ridener, Jr., K. J. Sebastian, and H. Grotch, Phys. Rev. D **45**, 3173 (1992).
[5] G. Karl, S. Meshkov, and J. L. Rosner, Phys. Rev. D **13**, 1203 (1976).
[6] M. G. Olsson and C. J. Suchyta III, Phys. Rev. D **34**, 2043 (1986).
[7] X. Zhang (private communication).
[8] S. N. Gupta, W. W. Repko, and C. J. Suchyta III, Phys. Rev. D **39**, 974 (1989); S. N. Gupta, J. M. Johnson, W. W. Repko, and C. J. Suchyta III, *ibid.* **49**, 1551 (1994).