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Angular distribution in the decay of the singlet D state of charmonium directly produced in unpolarized $\bar{p}p$ collisions

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We calculate the combined angular distribution of the final electron, of the γ photon, and of the π^0 meson produced in the cascade process $\bar{p}p \to {}^1D_2 \to {}^1P_1 + \gamma \to (\psi\pi^0) + \gamma \to (e^+e^-) + \pi^0 + \gamma$, where $\bar{p}$ and p are unpolarized. Our final result is valid in the $\bar{p}p$ c.m. frame and it is expressed in terms of the Wigner D^J functions and the spherical harmonics whose arguments are the angles representing the various directions involved. The coefficients of the terms involving the spherical harmonics and the Wigner D^J functions are functions of the angular momentum helicity amplitudes or equivalently of the multipole amplitudes of the individual processes. Once the combined angular distribution is measured, our expressions will enable one to calculate the relative magnitude as well as the cosines of the relative phases of all the angular momentum helicity amplitudes or equivalently of all the multipole amplitudes in the decay processes ${}^1D_2 \to {}^1P_1 + \gamma$ and ${}^1P_1 \to \psi + \pi^0$. We also derive six different partially integrated angular distribution functions which give the angular distributions of one or two particles in the final state. They can all be expressed entirely in terms of the spherical harmonics. By measuring these simpler angular distributions in the six different cases we get as much information on the helicity amplitudes as we obtained by measuring the combined angular distribution of the three particles, namely, the electron, the γ photon, and the π^0 meson.

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The potential models [1] predict the mass of the singlet D state of charmonium to be around 3820 MeV. Even though this mass is above the charm threshold, the state is expected to have a narrow width since ${}^1D_2 \to D+\bar{D}$ is forbidden by parity conservation and ${}^1D_2 \to D + \bar{D}^*$ or $D^* + \bar{D}$ is forbidden by energy conservation. In fact, the ${}^1D_2$ state would have a narrow width so long as its mass is less than 3875 MeV. In previous works [2], we have calculated the angular distribution of the two γ's and of the γ and the electron in the cascade processes, (1) $\bar{p}p \to {}^1D_2 \to {}^1P_1+\gamma_1 \to (1\,{}^1S_0+\gamma_2)+\gamma_1$ and (2) $\bar{p}p \to {}^1D_2 \to 1\,{}^3S_1 + \gamma \to (e^+e^-) + \gamma$ when $\bar{p}$ and p are unpolarized. The angular distributions of γ_1 alone in process (1) and of γ alone in process (2), have strikingly simple forms [3] in potential models. These distributions can be used as a signal for the formation of the singlet D state or as a check on the validity of the potential models. It should be noted that the individual processes ${}^1D_2 \to {}^1P_1+\gamma$ and ${}^1P_1 \to {}^1S_0 + \gamma_2$ should have significant branching ratios, but the process ${}^1S_0(\eta_c) \to 2\gamma$ has an insignificantly small branching ratio.

As a result, the full decay chain of process (1) may be hard to identify. It may be easier to observe the full process (3) $\bar{p}p \to {}^1D_2 \to {}^1P_1 + \gamma \to (\psi\pi^0) + \gamma \to (e^+e^-)\pi^0 + \gamma$, for the following reasons. Even though the decay ${}^1P_1 \to \psi + \pi^0$ does not conserve isospin, it may have a decay rate of a few keV [4]. Moreover, the processes $\pi^0 \to (\gamma^1\gamma^2)$ and $\psi \to e^+e^-$ have significant branching ratios of about 99 and 6 %, respectively, which makes the above full cascade process more likely to be seen than either process (1) or (2). In this work we derive the combined angular distribution of the photon, the electron, and the π^0 in the cascade process (3) in terms of the angular-momentum helicity amplitudes or equivalently of the multipole amplitudes of the individual processes. We also derive the angular distribution in the $\bar{p}p$ c.m. frame or the ${}^1D_2$ rest frame and express it in terms of the angles measured in that frame. Finally, our expression is a sum of products of Wigner D^J functions whose coefficients are functions of the angular-momentum helicity amplitudes of the individual processes. It will be clear from our final expression that one can determine the relative magnitudes, as well as the cosines of the relative phases of all the angular momentum helicity amplitudes in the processes ${}^1D_2 \to {}^1P_1 + \gamma$ and ${}^1P_1 \to \psi + \pi^0$, once the angular distribution is measured. The angular distribution of π^0 can be obtained by measuring the angular distribution of the total momentum of the two γ photons into which it decays almost all the time with a lifetime of only about 10^{-16} s. Since the multipole amplitudes can be expressed as a linear combination of the helicity amplitudes, we can also determine the relative magnitudes, as well as the cosines of the relative phases of all the multipole amplitudes in these processes. The sines of the relative phases are not determined uniquely. If the sine of one of the phases is known, the sines of the other phases can be determined.

We briefly sketch the derivation of our results now. We consider the process

$$\bar{p}(\lambda_1)p(\lambda_2) \to {}^1D_2(\nu) \to {}^1P_1(\sigma) + \gamma(\mu)$$
$$\to [\psi(\rho) + \pi^0] + \gamma(\mu)$$
$$\to [e^-(\kappa_1) + e^+(\kappa_2)] + \pi^0 + \gamma(\mu) ,$$

where the Greek symbols after the particle symbols represent their helicities except for the stationary ${}^1D_2$ resonance in which case the symbol ν represents the z component of the angular momentum. We choose the Z axis along the direction of ${}^1P_1$. The X and Y axes of the right-handed coordinate system may be chosen according to the convenience of the experimentalist.

The probability amplitude for the above cascade process can be written (within constants) as a product of the matrix elements for the individual processes. So we write the probability amplitude as

$$T^{\kappa_1\kappa_2,\mu}_{\lambda_1\lambda_2} = \sum_{\nu}^{-2\to+2}\sum_{\sigma,\rho}^{-1,0,+1} {}_{{}^1D_2}\langle {}^1D_2(\nu)|B|\bar{p}(\lambda_1)p(\lambda_2)\rangle_{{}^1D_2}$$
$$\times {}_{{}^1D_2}\langle {}^1P_1(\sigma)\gamma(\mu)|A|{}^1D_2(\nu)\rangle_{{}^1D_2}\, {}_{{}^1D_2}\langle \psi(\rho)\pi^0|E|{}^1P_1(\sigma)\rangle_{{}^1D_2}$$
$$\times {}_{{}^1D_2}\langle e^-(\kappa_1)e^+(\kappa_2)|C|\psi(\rho)\rangle_{{}^1D_2} . \quad (1)$$

Only the helicities of the initial and the final particles are observed. We sum over the helicities of the unobserved particles in Eq. (1). The subscript ${}^1D_2$ attached to the bra or the ket vector indicates that each individual matrix element is evaluated in the ${}^1D_2$ rest frame. The symbol B, A, E, and C represent the appropriate transition operators. Except for the last matrix element $\langle e^-(\kappa_1)e^+(\kappa_2)|C|\psi(\rho)\rangle$ they are all equal to the matrix elements evaluated in the rest frame of the decaying particles (or the created particle in the case of ${}^1D_2$ formation) in $\bar{p}p$ collisions.

We should also clarify what we mean by the two particle helicity states which are *not* defined in their c.m. frame. For example, the two-particle state $|\bar{e}(\kappa_1)e^+(\kappa_2)\rangle_{{}^1D_2}$ defined in the ${}^1D_2$ rest frame, which is not the c.m. frame of the e^+e^- system, has the following meaning. First, construct the two-particle helicity state $|e^-(\kappa_1)e^+(\kappa_2)\rangle_\psi$ in the c.m. frame of e^+e^- system or the ψ rest frame according to the usual conventions [5] with e^- and e^+ having exactly equal and opposite momenta and helicities κ_1 and κ_2, respectively. Then,

$$|e^-(\kappa_1)e^+(\kappa_2)\rangle_{{}^1D_2} = U_\Lambda({}^1D_2,\psi)|e^-(\kappa_1)e^+(\kappa_2)\rangle_\psi , \quad (2)$$

where $U_\Lambda(A,B)$ is the unitary operator corresponding to the Lorentz transformation $\Lambda(A,B)$ which takes the system from the Lorentz frame where B is at rest to the Lorentz frame where A is at rest. It is important to clarify this point since in general e^- and e^+ do not have definite helicities κ_1 and κ_2 in the ${}^1D_2$ rest frame. A similar meaning also holds for the two-particle state $|\psi(\rho)\pi^0\rangle_{{}^1D_2}$.

Let us now consider these matrix elements one by one. First,

$${}_{{}^1D_2}\langle {}^1D_2(\nu)|B|\bar{p}(\lambda_1)p(\lambda_2)\rangle_{{}^1D_2} = \langle 2\nu|B|p(\theta,\phi);\lambda_1\lambda_2\rangle , \quad (3)$$

where $p(\theta,\phi)$ is the magnitude of the c.m. momentum of $\bar{p}$ which is taken to be in the direction (θ,ϕ). We are choosing the Z axis along the direction of ${}^1P_1$ in the ${}^1D_2$ rest frame. The X and Y axes are otherwise arbitrary. Using the usual expansion [5,6] of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states we find

$${}_{{}^1D_2}\langle {}^1D_2(\nu)|B|\bar{p}(\lambda_1)p(\lambda_2)\rangle_{{}^1D_2} = \left(\frac{5}{4\pi}\right)^{1/2} B_{\lambda_1\lambda_2} D^2_{\nu\lambda}(\phi,\theta,-\phi) , \quad (4)$$

where

$$\lambda = \lambda_1 - \lambda_2 . \quad (5)$$

The angular-momentum helicity amplitudes $B_{\lambda_1\lambda_2}$ are not all independent because of C and P invariance [5]:

$$B_{\lambda_1\lambda_2} \overset{C}{=} B_{\lambda_2\lambda_1} \overset{P}{=} -B_{-\lambda_1-\lambda_2} . \quad (6)$$

Because of Eq. (6),

$$B_{+-} = B_{-+} = 0 . \quad (7)$$

We will call

$$B_{++} = -B_{--} = \frac{1}{\sqrt{2}}B_0 . \quad (8)$$

Next we consider the matrix element:

$${}_{{}^1D_2}\langle {}^1P_1(\sigma)\gamma(\mu)|A|{}^1D_2(\nu)\rangle_{{}^1D_2}$$
$$= \langle p_1(0,0,0);\sigma\mu|A|2\nu\rangle$$
$$= \left(\frac{5}{4\pi}\right)^{1/2} D^{2*}_{\nu,\sigma-\mu}(0,0,0)A_{\sigma\mu}$$
$$= \left(\frac{5}{4\pi}\right)^{1/2} \delta_{\nu,\sigma-\mu}A_{\sigma\mu} , \quad (9)$$

C invariance is trivially satisfied in this process. By P invariance [5],

$$A_{\sigma\mu} = A_{-\sigma-\mu} . \quad (10)$$

There will be three independent A amplitudes. We call them

$$A_\nu = A_{\nu-1,-1} = A_{-\nu+1,1} \quad (\nu = 0,1,2) . \tag{11}$$

The matrix element for the process ${}^1P_1(\sigma) \to \psi(\rho)+\pi^0$ in the ${}^1D_2$ and the ${}^1P_1$ rest frames are equal. That is,

$$\begin{aligned}
{}_{{}^1D_2}\langle\psi(\rho)\pi^0|E|{}^1P_1(\sigma)\rangle_{{}^1D_2}
&= {}_{{}^1P_1}\langle\psi(\rho)\pi^0|U_\Lambda^\dagger({}^1D_2,{}^1P_1)EU_\Lambda({}^1D_2,{}^1P_1){}^1P_1(\sigma)\rangle_{{}^1P_1} \\
&= {}_{{}^1P_1}\langle\psi(\rho)\pi^0|E|{}^1P_1(\sigma)\rangle_{{}^1P_1} . \qquad (12)
\end{aligned}$$

In Eq. (12), we used the fact that the transition operator E is invariant under Lorentz transformations: namely,

$$U_\Lambda^\dagger E U_\Lambda = E . \tag{13}$$

Using Eq. (12), we can now write

$${}_{{}^1D_2}\langle\psi(\rho)\pi^0|E|{}^1P_1(\sigma)\rangle_{{}^1D_2} = \langle p'(\theta',\psi');\rho_0|E|1\sigma\rangle , \tag{14}$$

where p' is the magnitude of the ψ momentum directed along (θ',ϕ') direction in the $\psi\pi^0$ c.m. frame. Again, using the expansion [5] of the two-particle helicity state in the c.m. frame in terms of the angular-momentum states, we can write

$${}_{{}^1D_2}\langle\psi(\rho)\pi^0|E|{}^1P_1(\sigma)\rangle_{{}^1D_2} = \left(\frac{3}{4\pi}\right)^{1/2} D^{1*}_{\sigma\rho}(\phi',\theta',-\phi')E_\rho . \tag{15}$$

Because of parity invariance the angular-momentum helicity amplitudes E_ρ satisfy the condition

$$E_\rho = E_{-\rho} . \tag{16}$$

For the matrix element of the final process $\psi \to e^+e^-$, the situation is more involved. We have

$$\begin{aligned}
{}_{{}^1D_2}\langle e^-(\kappa_1)e^+(\kappa_2)|\mathcal{C}|\psi(\rho)\rangle_{{}^1D_2} &= {}_\psi\langle e^-(\kappa_1)e^+(\kappa_2)|U_\Lambda^\dagger({}^1D_2,\psi)\mathcal{C}U_\Lambda({}^1D_2,{}^1P_1)U_\Lambda({}^1P_1,\psi)|\psi(\rho)\rangle_\psi \\
&= {}_\psi\langle e^-(\kappa_1)e^+(\kappa_2)|U_\Lambda^\dagger({}^1D_2,\psi)\mathcal{C}U_\Lambda({}^1D_2,\psi)U_\Lambda^\dagger({}^1D_2,\psi)U_\Lambda({}^1D_2,{}^1P_1)U_\Lambda({}^1P_1,\psi)|\psi(\rho)\rangle_\psi \\
&= {}_\psi\langle e^-(\kappa_1)e^+(\kappa_2)|\mathcal{C}U_\Lambda^\dagger({}^1D_2,\psi)U_\Lambda({}^1D_2,{}^1P_1)U_\Lambda({}^1P_1,\psi)|\psi(\rho)\rangle_\psi . \qquad (17)
\end{aligned}$$

In the first equality of Eq. (17), we made use of the fact that the single-particle state $|\psi(\rho)\rangle_{{}^1D_2}$ was also part of the two-particle state of Eqs. (12) and (14). It was obtained by successively performing two unitary operations corresponding to the two Lorentz transformations, the first taking the ψ state from its rest frame to the ${}^1P_1$ rest frame and the second taking it from the ${}^1P_1$ rest frame to the ${}^1D_2$ rest frame. In the last equality of Eq. (17) we now make use of the fact that

$$U_\Lambda({}^1D_2,{}^1P_1)U_\Lambda({}^1P_1,\psi) = U_\Lambda({}^1D_2,\psi)U_{R_W} , \tag{18}$$

where U_{R_W} is the unitary operator corresponding to a pure rotation, usually called "Wigner rotation." Using Eq. (18) and the unitarity of U_Λ, Eq. (17) now leads to

$$\begin{aligned}
{}_{{}^1D_2}\langle e^-(\kappa_1)e^+(\kappa_2)|\mathcal{C}|\psi(\rho)\rangle_{{}^1D_2}
&= {}_\psi\langle e^-(\kappa_1)e^+(\kappa_2)|\mathcal{C}U_{R_W}|\psi(\rho)\rangle_\psi \\
&= {}_\psi\langle e^-(\kappa_1)e^+(\kappa_2)|U_{R_W}U_{R_W}^\dagger\mathcal{C}U_{R_W}|\psi(\rho)\rangle_\psi \\
&= {}_\psi\langle e^-(\kappa_1)e^+(\kappa_2)|U_{R_W}\mathcal{C}|\psi(\rho)\rangle_\psi \qquad (19)
\end{aligned}$$

since

$$U_{R_W}^\dagger\mathcal{C}U_{R_W} = \mathcal{C} . \tag{20}$$

Using the expansion [5] of the two-particle helicity state in terms of the angular-momentum states, we can write the right-hand side of Eq. (19), as

$$\begin{aligned}
{}_\psi\langle e^-(\kappa_1)e^+(\kappa_2)|U_{R_W}\mathcal{C}|\psi(\rho)\rangle_\psi
&= \left(\frac{3}{4\pi}\right)^{1/2} D^{1*}_{\rho\kappa}(R_W^{-1}\hat{e}_\psi)\mathcal{C}_{\kappa_1,\kappa_2} \\
&= \left(\frac{3}{4\pi}\right)^{1/2} D^{1*}_{\rho\kappa}(\phi'',\theta'',-\phi'')\mathcal{C}_{\kappa_1\kappa_2} , \qquad (21)
\end{aligned}$$

where

$$\kappa = \kappa_1 - \kappa_2 \tag{22}$$

and $\hat{e}_\psi$ is a unit vector in the direction e^- in the ψ rest frame and R_W is the (3×3) rotation matrix and $\mathcal{C}_{\kappa_1\kappa_2}$ are the angular-momentum helicity amplitudes.

The Wigner rotated unit vector $R_W^{-1}\hat{e}_\psi$ can be obtained in the following way. Let R represent the (4×4) matrix whose spatial part gives the (3×3) matrix R_W mentioned above. Then, from the definition of U_{R_W} in Eq. (18).

$$R = \Lambda^{-1}({}^1D_2,\psi)\Lambda({}^1D_2,{}^1P_1)\Lambda({}^1P_1,\psi) , \tag{23}$$

where the Λ's are the (4×4) Lorentz transformation matrices. Now we note that the electron is highly relativistic in the ψ rest frame and its four-momentum vector p_{e_ψ} can be represented to a very good approximation as

$$p_{e_\psi} = \frac{M_\psi}{2}(1,\hat{e}_\psi) , \tag{24}$$

$$\begin{aligned}
R^{-1}p_{e_\psi} &= \Lambda^{-1}({}^1P_1,\psi)\Lambda^{-1}({}^1D_2,{}^1P_1)\Lambda({}^1D_2,\psi)p_{e_\psi} \\
&= \Lambda^{-1}({}^1P_1,\psi)\Lambda^{-1}({}^1D_2,{}^1P_1)\Lambda({}^1D_2,\psi) \\
&\quad \times\Lambda^{-1}({}^1D_2,\psi)p_{e_D} \\
&= \Lambda^{-1}({}^1P_1,\psi)\Lambda^{-1}({}^1D_2,{}^1P_1)p_{e_D} , \qquad (25)
\end{aligned}$$

where p_{e_D} is the four-momentum of e^- in the ${}^1D_2$ rest frame:

$$p_{e_D} = E_{e_D}(1,\hat{e}_D) , \tag{26}$$

where $\hat{e}_D$ is a unit vector in the direction of e^- three-momentum in the ${}^1D_2$ rest frame. So,

$$
\begin{aligned}
R^{-1}p_{e_\psi} &= \frac{M_\psi}{2}(1, R_W^{-1}\hat{e}_\psi) \\
&= \Lambda^{-1}({}^1P_1, \psi)\Lambda^{-1}({}^1D_2, {}^1P_1)p_{e_D} \\
&= \Lambda^{-1}({}^1P_1, \psi)\Lambda^{-1}({}^1D_2, {}^1P_1)E_{e_D}(1, \hat{e}_D) \; . \qquad (27)
\end{aligned}
$$

The spatial part of Eq. (27) gives, within a normalization factor, the Wigner rotated unit vector $\tilde{\hat{e}} = R_W^{-1}\hat{e}_\psi(\theta'', \phi'')$ in terms of the angles $(\tilde{\theta}'', \tilde{\psi}'')$ measured in the ${}^1D_2$ frame.

Because of the C and P invariance of the transition operator $\mathcal{C}$, the angular-momentum helicity amplitudes $\mathcal{C}_{\kappa_1\kappa_2}$ of Eq. (21) satisfy the constraint relations

$$\mathcal{C}_{\kappa_1\kappa_2} \overset{C}{=} \mathcal{C}_{\kappa_2\kappa_1} \overset{P}{=} \mathcal{C}_{-\kappa_1-\kappa_2} \; . \qquad (28)$$

The independent $\mathcal{C}$ amplitudes are $\mathcal{C}_0$ and $\mathcal{C}_1$:

$$\mathcal{C}_0 = \mathcal{C}_{++} = \mathcal{C}_{--} \; , \qquad (29)$$

$$\mathcal{C}_1 = \mathcal{C}_{+-} = \mathcal{C}_{-+} \; . \qquad (30)$$

If e^+e^- is created through a virtual photon, in the high-energy limit of the electron, $\mathcal{C}_0$ can be neglected compared to $\mathcal{C}_1$.

Using Eqs. (3), (9), (15), and (21) in Eq. (1), we obtain

$$
\begin{aligned}
T_{\lambda_1\lambda_2}^{\kappa_1\kappa_2,\mu} = \sum_{\nu}^{-2\to+2} \sum_{\sigma,\rho}^{-1,0,+1} \frac{15}{(4\pi)^2} B_{\lambda_1\lambda_2} D^2_{\nu\lambda}(\phi,\theta,-\phi) \\
\times \delta_{\nu,\sigma-\mu} A_{\sigma\mu} D^{1*}_{\sigma\rho}(\phi',\theta',-\phi') E_\rho \\
\times D^{1*}_{\rho\kappa}(\phi'',\theta'',-\phi'')\mathcal{C}_{\kappa_1\kappa_2} . \qquad (31)
\end{aligned}
$$

We should note that the angles (θ', ϕ') (the direction of ψ) and (θ'', ϕ'') (the direction of $R_W^{-1}\hat{e}_\psi$) are measured in the rest frame of ${}^1P_1$ and ψ, respectively. Later we will relate them to the corresponding angles $(\tilde{\theta}', \tilde{\phi}')$ and $(\tilde{\theta}'', \tilde{\phi}'')$ measured in the ${}^1D_2$ rest frame.

When $\bar{p}$ and p are unpolarized, the normalized function describing the combined angular distribution of γ, π^0, and e^- can be written as

$$
\begin{aligned}
W(\theta,\phi;\theta',\phi';\theta'',\phi'') \\
= N\frac{1}{4} \sum_{\lambda_1\lambda_2}^{+1/2,-1/2} \sum_{\kappa_1\kappa_2}^{+1/2,-1/2} \sum_{\mu}^{+1,-1} T_{\lambda_1\lambda_2}^{\kappa_1\kappa_2,\mu} T_{\lambda_1\lambda_2}^{\kappa_1\kappa_2,\mu*} \; , \qquad (32)
\end{aligned}
$$

where N is a normalization constant so chosen that W integrated over all the angles will give the value one. After we substitute Eq. (31) into Eq. (32) the various sums have to be performed. Before we do the sums we make use of the Clebsch-Gordon series relation for the D^J functions, namely,

$$
\begin{aligned}
D^{j_1}_{m_1m_2} D^{j_2}_{m_1'm_2'} \\
= \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle j_1j_2m_1m_1'|J, m_1+m_1'\rangle \\
\times \langle j_1j_2m_2m_2'|J, m_2+m_2'\rangle D^J_{m_1+m_1', m_2+m_2'} \qquad (33)
\end{aligned}
$$

and the relation

$$D^{j*}_{m_1m_2} = (-1)^{m_1-m_2} D^j_{-m_1-m_2} \; . \qquad (34)$$

Then we see that the various sums in Eq. (32) factor out, or in other words, the angular distribution function W becomes a product of four sums, one involving λ_1 and λ_2, a second involving κ_1 and κ_2, a third involving ρ, ρ', and a fourth involving σ, σ' and μ. The sums over λ_1, λ_2 and κ_1, κ_2 are trivial. The sums over the indices ρ, ρ' and σ, σ' and μ are performed after we make the following change of variables:

$$
\begin{aligned}
d &= \rho - \rho' \; , \\
s &= \rho + \rho' \; , \\
d' &= \sigma - \sigma' \; , \\
s' &= \sigma + \sigma' \; . \qquad (35)
\end{aligned}
$$

We now notice that the terms for negative d, d' are the complex conjugates of those with positive d, d'. After a lengthy algebra we finally obtain the following expression for the normalized angular distribution:

$$
\begin{aligned}
W(\theta,\phi;\theta',\phi';\theta'',\phi'') = \frac{1}{4(4\pi)^3} \sum_{L_1}^{0,2,4} \beta_{L_1} \sum_{L_2}^{0,1,2} \sum_{L_3}^{0,2} \gamma_{L_3} \sum_{d'}^{0\to\mathrm{Min}(L_1,L_2)} \alpha_{d'}^{L_1L_2} \\
\times \sum_{d}^{0\to\mathrm{Min}(L_3,L_2)} \mathcal{E}_d^{L_3L_2} \mathcal{Y}_{d'd}^{L_1L_2L_3}(\theta,\phi;\theta',\phi';\theta'',\phi'') \; , \qquad (36)
\end{aligned}
$$

where

$$\beta_{L_1} = \sqrt{5}\langle 2200|L_10\rangle |B_0|^2 \; , \qquad (37)$$

$$\gamma_{L_3} = -\sqrt{3}\sum_{\kappa}^{0,1} |\mathcal{C}_\kappa|^2 (-1)^\kappa \langle 11\kappa-\kappa|L_30\rangle \; , \qquad (38)$$

$$
\begin{aligned}
\alpha_{d'}^{L_1L_2} = \sqrt{15}\left(1 - \frac{\delta_{d'0}}{2}\right) \sum_{s'}^{d',d'+2,\ldots,4-d'} \left[A_{(s'+d')/2}A^*_{(s'-d')/2} + (-1)^{L_2}A^*_{(s'+d')/2}A_{(s'-d')/2}\right] \\
\times \left\langle 22; \frac{s'+d'}{2}, -\frac{s'-d''}{2} \middle| L_1d' \right\rangle \left\langle 11; \left(\frac{s'+d'-2}{2}\right), -\left(\frac{s'-d'-2}{2}\right) \middle| L_2d' \right\rangle \; , \qquad (39)
\end{aligned}
$$

$$\mathcal{E}_d^{L_3L_2} = 6\left(1-\frac{\delta_{d0}}{2}\right)\sum_s^{[1-(-1)^d]/2,[1-(-1)^d]/2+2,\ldots,2-d}\left(1-\frac{\delta_{s0}}{2}\right)\left[E_{(s+d)/2}E^*_{(s-d)/2}+(-1)^{L_2}E^*_{(s+d)/2}E_{(s-d)/2}\right]$$
$$\times\left\langle 11;\frac{s+d}{2},\frac{-(s-d)}{2}\middle|L_3d\right\rangle\left\langle 11;\frac{s+d}{2},\frac{-(s-d)}{2}\middle|L_2d\right\rangle . \quad (40)$$

In the above equations,

$$B_0 = \sqrt{2}B_{++} = -\sqrt{2}B_{--} ,$$
$$C_\kappa = C_{\kappa_1-\kappa_2} = \sqrt{2}C_{\kappa_1\kappa_2} ,$$
$$A_\nu = A_{\nu-1,1} = A_{-\nu+1,1} . \quad (41)$$

We also use the normalizations

$$|B_0|^2 = |C_0|^2+|C_1|^2 = |E_0|^2+2|E_1|^2 = |A_0|^2+|A_1|^2+|A_2|^2 = 1 . \quad (42)$$

Finally, the angular function $\mathcal{Y}_{d'd}^{L_1L_2L_3}$ is defined as

$$\mathcal{Y}_{d'd}^{L_1L_2L_3} = [(D^{L_1}_{d'0}D^{L_2^*}_{d'd}D^{L_3^*}_{d0}+D^{L_1^*}_{d'0}D^{L_2}_{d'd}D^{L_3}_{d0})+(-1)^{L_2}(D^{L_1}_{d'0}D^{L_2^*}_{d',-d}D^{L_3^*}_{-d,0}+D^{L_1^*}_{d'0}D^{L_2}_{d',-d}D^{L_3}_{-d,0})] . \quad (43)$$

The arguments of the Wigner functions D^{L_1}, D^{L_2}, and D^{L_3} are $(\phi,\theta,-\phi)$, $(\phi',\theta',-\phi')$, and $R_W^{-1}\hat{e}_\psi = (\phi'',\theta'',\phi'')$, respectively.

The angles (θ',ϕ') and (θ'',ϕ'') are measured in the ${}^1P_1$ and in the ψ rest frames, respectively. Their relations to the angles $(\tilde\theta',\tilde\phi')$ and $(\tilde\theta'',\tilde\phi'')$ measured in the ${}^1D_2$ rest frame are given below:

$$\phi' = \tilde\phi' , \quad (44)$$

$$\cos\theta' = \frac{(\cos^2\tilde\theta'-1)(\beta_2/\beta_1)+\cos\tilde\theta'\sqrt{1-\beta_2^2}\sqrt{1-(\beta_2/\beta_1)^2+\cos^2\tilde\theta'[(\beta_2/\beta_1)^2-\beta_2^2]}}{(1-\beta_2^2\cos^2\tilde\theta')} \quad (45)$$

Since $0\le\theta'\le\pi$, $\sin\theta'$ has to be positive and so it will be given by the positive square root

$$\sin\theta' = +\sqrt{1-\cos^2\theta'} , \quad (46)$$

where $\cos\theta'$ is given by Eq. (45). In Eq. (45), β_1 is the parameter v/c of ψ in the ${}^1P_1$ rest frame and β_2 is v/c of ${}^1P_1$ in the ${}^1D_2$ rest frame. If M_P and M_D denote the masses of the ${}^1P_1$ and the ${}^1D_2$ states, respectively, simple relativistic kinematics gives

$$\beta_1 = \frac{\{[M_P^2-(M_\psi+M_\pi)^2][M_P^2-(M_\psi-M_\pi)^2]\}^{1/2}}{[M_P^2+(M_\psi^2-M_\pi^2)]} ,$$
$$\beta_2 = \frac{M_D^2-M_P^2}{M_D^2+M_P^2} . \quad (47)$$

If the direction of the unit vector $\tilde{\hat{e}} = R_W^{-1}\hat{e}_\psi$ (where $\hat{e}_\psi$ is the unit vector in the direction of the electron momentum in the ψ frame) is given by the spherical polar angles θ'',ϕ'', then these angles are related to the corresponding angles $(\tilde\theta',\tilde\phi'')$ measured in the ${}^1D_2$ rest frame by the relations

$$\cos\phi'' = \frac{1}{\eta'}[\gamma_2\beta_2\sin\theta'+\cos\theta'\cos\phi'\sin\tilde\theta''\cos\tilde\phi'' + \cos\theta'\sin\phi'\sin\tilde\theta''\sin\tilde\phi''-\sin\theta'\cos\tilde\theta''\gamma_2] , \quad (48)$$

$$\sin\phi'' = \frac{1}{\eta'}[\cos\phi'\sin\tilde\theta''\sin\tilde\phi''-\sin\phi'\sin\tilde\theta''\cos\tilde\phi''] , \quad (49)$$

$$\cos\theta'' = [-\gamma_1\gamma_2(\beta_1+\beta_2\cos\theta') +\gamma_1(\sin\theta'\cos\phi'\sin\tilde\theta''\cos\tilde\phi''+\sin\theta'\sin\phi'\sin\tilde\theta'' \times\sin\tilde\phi'')+\gamma_1\gamma_2(\beta_1\beta_2+\cos\theta')\cos\tilde\theta'']\frac{1}{\eta} , \quad (50)$$

$$\sin\theta'' = +\sqrt{1-\cos^2\theta''} = \frac{\eta'}{\eta} , \quad (51)$$

where

$$\eta' = [(\gamma_2\beta_2\sin\theta'+\cos\theta'\cos\phi'\sin\tilde\theta''\cos\tilde\phi'' +\cos\theta'\sin\phi'\sin\tilde\theta''\sin\tilde\phi''-\sin\theta'\cos\tilde\theta''\gamma_2)^2 +(\cos\phi'\sin\tilde\theta''\sin\tilde\phi''-\sin\phi'\sin\tilde\theta''\cos\tilde\phi'')^2]^{1/2} , \quad (52)$$

$$\eta = [\gamma_1\gamma_2(1+\beta_1\beta_2\cos\theta')-\gamma_1\beta_1(\sin\theta'\cos\phi'\sin\tilde\theta''\cos\tilde\phi'' +\sin\theta\ \sin\phi\ \sin\tilde\theta''\sin\tilde\phi'') -\gamma_1\gamma_2(\beta_2+\beta_1\cos\theta')\cos\tilde\theta''] . \quad (53)$$

The constants γ_i $(i=1,2)$ are related to β_i $(i=1,2)$ by the relations

$$\gamma_i = \frac{1}{\sqrt{1-\beta_i^2}} . \quad (54)$$

Substituting Eqs. (47) into Eq. (54) gives

$$\gamma_1 = \frac{M_P^2 + M_\psi^2}{2M_P M_\psi} ,$$
$$\gamma_2 = \frac{M_D^2 + M_P^2}{2M_D M_P} . \quad (55)$$

It is also useful to note that, in Eq. (43),

$$D^L_{M0} = \left(\frac{4\pi}{2L+1}\right)^{1/2} Y^*_{LM} . \quad (56)$$

Finally, the explicit expressions for the nonzero coefficients in Eq. (36) are

$$\beta_0 = 1 ,$$
$$\beta_2 = -(\tfrac{10}{7})^{1/2} ,$$
$$\beta_4 = 3(\tfrac{2}{7})^{1/2} , \quad (57)$$
$$\gamma_0 = 1 ,$$
$$\gamma_2 = \frac{1}{\sqrt{2}}(1 - 3|\mathcal{C}_0|^2) \simeq \frac{1}{\sqrt{2}} , \quad (58)$$
$$\mathcal{E}_0^{00} = 1, \quad \mathcal{E}_0^{L_3 1} = 0 ,$$
$$\mathcal{E}_0^{02} = -\sqrt{2}(|E_0|^2 - |E_1|^2) ,$$
$$\mathcal{E}_0^{20} = -\sqrt{2}(|E_0|^2 - |E_1|^2) ,$$
$$\mathcal{E}_1^{21} = 6i\,\mathrm{Im}(E_1 E_0^*) ,$$
$$\mathcal{E}_0^{22} = 2(|E_0|^2 + \tfrac{1}{2}|E_1|^2) ,$$
$$\mathcal{E}_1^{22} = 6\,\mathrm{Re}(E_1 E_0^*) ,$$
$$\mathcal{E}_2^{22} = 6|E_1|^2 , \quad (59)$$
$$\alpha_0^{00} = 1, \quad \alpha_0^{L_1 1} = 0 ,$$
$$\alpha_0^{02} = \frac{1}{\sqrt{2}}(|A_0|^2 - 2|A_1|^2 + |A_2|^2) ,$$
$$\alpha_0^{20} = -(\tfrac{10}{7})^{1/2}(|A_0|^2 + \tfrac{1}{2}|A_1|^2 - |A_2|^2) ,$$
$$\alpha_1^{21} = i(\tfrac{15}{7})^{1/2}[\mathrm{Im}(A_1 A_0^*) + \sqrt{6}\,\mathrm{Im}(A_2 A_1^*)] ,$$
$$\alpha_0^{22} = -(\tfrac{5}{7})^{1/2}[|A_0|^2 - |A_1|^2 - |A_2|^2] ,$$
$$\alpha_1^{22} = -(\tfrac{15}{7})^{1/2}[\mathrm{Re}(A_1 A_0^*) - \sqrt{6}\,\mathrm{Re}(A_2 A_1^*)] ,$$
$$\alpha_2^{22} = 2(\tfrac{30}{7})^{1/2}\mathrm{Re}(A_2 A_0^*) ,$$
$$\alpha_0^{40} = 3\left(\tfrac{2}{7}\right)^{1/2}[|A_0|^2 - \tfrac{2}{3}|A_1|^2 + \tfrac{1}{6}|A_2|^2] ,$$
$$\alpha_1^{41} = -i3\left(\frac{10}{7}\right)^{1/2}\left[\mathrm{Im}(A_1 A_0^*) - \frac{1}{\sqrt{6}}\mathrm{Im}(A_2 A_1^*)\right] ,$$
$$\alpha_0^{42} = 3\frac{1}{\sqrt{7}}\left[|A_0|^2 + \frac{4}{3}|A_1|^2 + \frac{1}{6}|A_2|^2\right] ,$$
$$\alpha_1^{42} = 3(\tfrac{10}{7})^{1/2}\left[\mathrm{Re}(A_1 A_0^*) + \frac{1}{\sqrt{6}}\mathrm{Re}(A_2 A_1^*)\right] ,$$
$$\alpha_2^{42} = 3\left(\tfrac{10}{7}\right)^{1/2}\mathrm{Re}(A_2 A_0^*) . \quad (60)$$

Equation (36) looks rather complicated because it gives the combined angular distribution function of the photon, of the ψ meson and of the electron. Since our result of Eq. (36) is expressed as a sum of products of the orthonormal Wigner D^J functions, we can obtain the α and the $\mathcal{E}$ coefficients as an integral of W over those orthonormal functions. Once these coefficients are known we can determine uniquely the magnitudes of all the helicity amplitudes, as well as the cosines of their relative phases. The sines of the relative phases are not determined uniquely. But if the sine of one of the phases is known, then the sines of the other phases can be calculated. The angles in the expression of Eq. (36) are not all measured in the ${}^1D_2$ rest frame or the $\bar{p}p$ c.m. frame. They are related to the angles measured in the $\bar{p}p$ c.m. frame through Eqs. (38)–(55). Even though these equations may look formidable, the two sets of angles can be related through a computer program, which is routinely done by the experimentalists.

The partially integrated angular distributions obtained from Eq. (36) will look a lot simpler and we can gain greater insight from them. We consider six cases of partially integrated angular distributions. From these alone, we can determine the magnitudes of all the helicity amplitudes, as well as the cosines of their relative phases. In addition, we can also obtain the products $\mathrm{Im}(E_1E_0^*)\mathrm{Im}(A_1A_0^*)$ and $\mathrm{Im}(E_1E_0^*)\mathrm{Im}(A_2A_1^*)$. So if we know the sine of one of the relative phases, we can get the sines of all the other relative phases. By considering the combined angular distribution of all three particles γ, ψ, and e^- we do not get any new information. But it will serve as a further check on the results we already obtained.

In deriving the partially integrated angular distributions, we make use of the results

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} D^{j*}_{mm'}(\alpha,\beta,\gamma) D^{j'}_{\mu\mu'}(\alpha,\beta,\gamma) \sin\beta d\beta = \frac{8\pi^2}{(2j+1)}\delta_{m\mu}\delta_{m'\mu'}\delta_{jj'} , \quad (61)$$

$$\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} D^{j}_{mm'}(\alpha,\beta,\gamma) \sin\beta d\beta = \frac{8\pi^2}{(2j+1)}\delta_{m0}\delta_{m'0}\delta_{j0} , \quad (62)$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} D^{L*}_{MM'}(\phi,\theta,-\phi) \sin\theta d\theta = \int_0^{2\pi} d\phi \int_0^{\pi} D^{L}_{MM'}(\phi,\theta,-\phi) \sin\theta d\theta = 2\pi\delta_{M-M',0}\int_{-1}^{+1} d^L_{MM'}(\theta)\sin\theta d\theta . \quad (63)$$

Case 1. We will integrate over (θ',ϕ') and (θ'',ϕ''). We are only observing the photon. Using Eqs. (61)–(63) and substituting for the α and the $\mathcal{E}$ coefficients the expressions given in Eqs. (59) and (60) we finally obtain

$$\tilde{W}(\theta,\phi)=\frac{1}{4\pi}\left[1+\left(\frac{10}{7}\right)\left(\frac{4\pi}{5}\right)^{1/2}\left(|A_0|^2+\frac{1}{2}|A_1|^2-|A_2|^2\right)Y_{20}(\theta)+\left(\frac{6}{7}\right)\sqrt{4\pi}\left(|A_0|^2-\frac{2}{3}|A_1|^2+\frac{1}{6}|A_2|^2\right)Y_{40}(\theta)\right], \quad (64)$$

where θ is the angle between the proton and the photon as measured in the $\bar{p}p$ c.m. frame which is the same as the lab frame. So once this angular distribution is measured, using the orthonormality of the spherical harmonics and the normalizations given by Eqs. (42), we can determine the magnitudes of the angular-momentum helicity amplitudes, $|A_0|$, $|A_1|$, and $|A_2|$.

Case 2. Here we will observe the angular distribution of ψ in the $^1D_2$ rest frame by determining the angular distribution of the total momentum of e^-e^+ in the $\bar{p}p$ c.m. frame. To get the theoretical expression for this angular distribution we integrate over (θ,ϕ) and (θ'',ϕ''). We finally obtain

$$\tilde{W}(\theta',\phi')=\frac{1}{4\pi}\left[1-(|A_0|^2-2|A_1|^2+|A_2|^2)\times(|E_0|^2-|E_1|^2)\left(\frac{4\pi}{5}\right)^{1/2}Y_{20}(\theta')\right], \quad (65)$$

where θ' is the angle between $^1P_1$ and ψ in the $^1P_1$ rest frame. This angle is related to the angle $\tilde{\theta}'$ measured in the $^1D_2$ rest frame through Eq. (45). Since $^1P_1$ and the photon γ move in opposite directions in the $^1D_2$ rest frame, $\tilde{\theta}'$ can be measured by measuring the angle between γ and the total momentum of e^+ and e^- in the $\bar{p}p$ c.m. frame. Since we already know $|A_0|$, $|A_1|$, and $|A_2|$, this angular distribution will enable us to calculate $|E_0|$ and $|E_1|$ through the normalization relation of Eqs. (42).

Case 3. Only the angular distribution of the electron with respect to the photon is observed. We integrate over (θ,ϕ) and (θ',ϕ'):

$$\tilde{W}(\theta'',\phi'')=\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta d\theta d\phi\,\sin\theta' d\theta' d\phi' \simeq\frac{1}{4\pi}\left[1-(|E_0|^2-|E_1|^2)\left(\frac{4\pi}{5}\right)^{1/2}Y_{20}(\theta'')\right], \quad (66)$$

where we neglected $|C_0|^2$ compared to $|C_1|^2$.

The angle θ'' which gives the "Wigner rotated" direction of e^- in the ψ rest frame is related to the angle $\tilde{\theta}''$ which gives the direction of e^- observed in the $^1D_2$ rest frame or the lab frame through Eqs. (48)–(53). We also notice that $(\pi-\tilde{\theta}'')$ is the angle between γ and e^- in the $^1D_2$ rest frame.

Case 4. Here we obtain the combined angular distribution of γ and e^- by integrating over (θ',ϕ'). The partially integrated angular distribution function $\tilde{\tilde{W}}(\theta,\phi;\theta'',\phi'')$ in this case is given by

$$\begin{aligned}\tilde{\tilde{W}}(\theta,\phi;\theta'',\phi'')&=\int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta' d\theta' d\phi'\\
&=\frac{1}{128\pi^2}\left\{8\left[\sqrt{4\pi}Y_{00}(\theta)+\frac{10}{7}\left(|A_0|^2+\frac{1}{2}|A_1|^2-|A_2|^2\right)\left(\frac{4\pi}{5}\right)^{1/2}Y_{20}(\theta)\right.\right.\\
&\left.+\frac{18}{7}\left(|A_0|^2-\frac{2}{3}|A_1|^2+\frac{1}{6}|A_2|^2\right)\left(\frac{4\pi}{9}\right)^{1/2}Y_{40}(\theta)\right]\\
&\times\left[\sqrt{4\pi}Y_{00}(\theta'')-(|E_0|^2-|E_1|^2)\left(\frac{4\pi}{5}\right)^{1/2}Y_{20}(\theta'')\right]\\
&+\frac{30\sqrt{3}}{7}[\mathrm{Im}(E_1E_0^*)(\mathrm{Im}A_1A_0^*+\sqrt{6}\,\mathrm{Im}A_2A_1^*)\\
&-\frac{1}{3}\mathrm{Re}(E_1E_0^*)(\mathrm{Re}A_1A_0^*-\sqrt{6}\,\mathrm{Re}A_2A_1^*)]\times\frac{8\pi}{5}\mathrm{Re}[Y_{21}^*(\theta,\phi)Y_1(\theta'',\phi'')]\\
&+\frac{54\sqrt{10}}{7}\left[\mathrm{Im}E_1E_0^*\left(\mathrm{Im}(A_1A_0^*)-\frac{1}{\sqrt{6}}\mathrm{Im}(A_2A_0^*)\right)\right.\\
&\left.-\frac{1}{3}\mathrm{Re}(E_1E_0^*)\left(\mathrm{Re}(A_1A_0^*)+\frac{1}{\sqrt{6}}\mathrm{Re}(A_2A_1^*)\right)\right]\times\frac{8\pi}{3\sqrt{5}}\mathrm{Re}[Y_{41}^*(\theta,\phi)Y_{21}(\theta'',\phi'')]\\
&+\left[\frac{-40\sqrt{6}}{7}|E_1|^2\mathrm{Re}(A_2A_0^*)\frac{8\pi}{5}\mathrm{Re}[Y_{22}^*(\theta,\phi)Y_{22}(\theta'',\phi'')]\right.\\
&\left.\left.+\frac{96\sqrt{2}\pi}{7}|E_1|^2\mathrm{Re}(A_2A_0^*)\mathrm{Re}[Y_{42}^*(\theta,\phi)Y_{22}(\theta'',\phi'')]\right]\right\}. \quad (67)\end{aligned}$$

The magnitudes, as well as the real parts of the products of the helicity amplitudes, can be obtained from other partially integrated angular distributions. So the measurement of this angular distribution will enable us to determine $\mathrm{Im}(E_1E_0^*)\mathrm{Im}(A_1A_0^*)$ and $\mathrm{Im}(E_1E_0^*)\mathrm{Im}(A_2A_1^*)$. We remind the reader again that θ is the angle between the photon and the proton as measured in the ${}^1D_2$ rest frame. The angle θ'' is related to the angle $(\pi - \tilde{\tilde{\theta}}\,'')$ between γ and e^-, measured in the ${}^1D_2$ rest frame, through Eqs. (48)–(63).

Case 5. Here we get the combined angular distribution of $\psi(\theta',\phi')$ and $e^-(\theta'',\phi'')$. We obtain

$$\begin{aligned}\tilde{\tilde{W}}(\theta',\phi';\theta'',\phi'') &= \int W(\theta,\phi;\theta'\phi';\theta'',\phi'')\sin\theta d\theta d\phi \\ &= \frac{1}{8\pi}\sum_{L_3}^{0,2}\gamma_{L_3}\sum_{L_2}^{0,2}\alpha_0^{0L_2}\sum_{d}^{0\to\mathrm{Min}(L_2L_3)}\varepsilon_d^{L_3L_2}\frac{(-1)^d}{\sqrt{(2L_2+1)(2L_3+1)}} \\ &\quad\times[Y^*_{L_2d}(\theta',\phi')Y_{L_3d}(\theta'',\phi'')+Y_{L_2d}(\theta',\phi')Y^*_{L_3d}(\theta'',\phi'')] \\ &= \frac{1}{8\pi}\Bigg[\frac{1}{2\pi}-\frac{1}{\sqrt{5\pi}}(|E_0|^2-|E_1|^2)Y_{20}(\theta'')-\frac{1}{\sqrt{5\pi}}(|A_0|^2-2|A_1|^2+|A_2|^2)(|E_0|^2-|E_1|^2)Y_{20}(\theta') \\ &\quad+\frac{2}{5}(|A_0|-2|A_1|^2+|A_2|^2)\Bigg\{\left(|E_0|^2+\frac{1}{2}|E_1|^2\right)Y_{20}(\theta')Y_{20}(\theta'') \\ &\quad-3\,\mathrm{Re}(E_1E_0^*)\mathrm{Re}[Y^*_{21}(\theta',\phi')Y_{21}(\theta'',\phi'')]+3|E_1|^2\mathrm{Re}[Y^*_{22}(\theta',\phi')Y_{22}(\theta'',\phi'')]\Bigg\}\Bigg]\,. \end{aligned} \tag{68}$$

Previous comments on the angles θ' and θ'' apply here also. By measuring combined angular distribution of ψ and e^- we can now determine $\mathrm{Re}(E_1E_0^*)$, in addition to the magnitudes of the helicity amplitudes. The angular distribution of $\psi(\theta',\phi')$ can be determined either by measuring the angular distribution of the total momentum of e^- and e^+ into which ψ decays or of the total momentum of the two γ's into which π^0 decays since π^0 and ψ have equal and opposite momenta in the ${}^1P_1$ rest frame.

Case 6. Here we get the combined angular distribution of $\gamma(\theta,\phi)$ and $\psi(\theta',\phi')$ by integrating over the directions (θ'',ϕ'') only. We obtain

$$\begin{aligned}\tilde{\tilde{W}}(\theta,\phi;\theta',\phi') &= \int W(\theta,\phi;\theta',\phi';\theta'',\phi'')\sin\theta'' d\theta'' d\phi'' \\ &= \frac{1}{8\pi}\sum_{L_1}^{0,2,4}\sum_{L_2}^{0,2}\sum_{d'}^{0\to\mathrm{Min}(L_1,L_2)}\beta_{L_1}\alpha_{d'}^{L_1L_2}\mathcal{E}_0^{0L_2}\frac{1}{\sqrt{(2L_1+1)(2L_2+1)}} \\ &\quad\times[Y^*_{L_1d'}(\theta,\phi)Y_{L_2d'}(\theta',\phi')+Y_{L_2d'}(\theta',\phi')+Y_{L_1d'}(\theta,\phi)Y^*_{L_2d'}(\theta',\phi')]\,. \end{aligned} \tag{69}$$

By substituting for the coefficients from Eq. (57)–(60), we finally obtain

$$\begin{aligned}\tilde{\tilde{W}}(\theta,\phi;\theta',\phi') &= \frac{1}{8\pi}\Bigg[\frac{1}{2\pi}-\frac{1}{\sqrt{5\pi}}\left(|A_0|^2-2|A_1|^2+|A_2|^2\right)\left(|E_0|^2-|E_1|^2\right)Y_{20}(\theta') \\ &\quad+\frac{10}{7}\frac{1}{\sqrt{5\pi}}\left(|A_0|^2+\frac{1}{2}|A_1|^2-|A_2|^2\right)Y_{20}(\theta) \\ &\quad+\frac{4}{7}(|E_1|^2-E_0|^2)\{(|A_0|^2-|A_1|^2-|A_2|^2)Y_{20}(\theta)Y_{20}(\theta') \\ &\quad+\sqrt{3}[\mathrm{Re}(A_1A_0^*)-\sqrt{6}\,\mathrm{Re}(A_2A_1^*)]\mathrm{Re}[Y^*_{21}(\theta,\phi)Y_{21}(\theta',\phi')] \\ &\quad-2\sqrt{6}\,\mathrm{Re}(A_2A_0^*)\mathrm{Re}[Y^*_{22}(\theta,\phi)Y_{22}(\theta',\phi')]\} \\ &\quad+\frac{6}{7\sqrt{\pi}}\left(|A_0|^2-\frac{2}{3}|A_1|^2+\frac{1}{6}|A_2|^2\right)Y_{40}(\theta) \\ &\quad+\frac{4}{\sqrt{35}}(|E_1|^2-|E_0|^2)\Bigg\{\frac{3}{\sqrt{7}}\left(|A_0|^2+\frac{4}{3}|A_1|^2+\frac{1}{6}|A_2|^2\right)Y_{40}(\theta)\,Y_{20}(\theta') \\ &\quad+\left(\frac{10}{7}\right)^{1/2}\left(\mathrm{Re}(A_1A_0^*)+\frac{1}{\sqrt{6}}\mathrm{Re}(A_2A_1^*)\right)\mathrm{Re}[Y^*_{41}(\theta,\phi)Y_{21}(\theta',\phi')] \\ &\quad+3\left(\frac{10}{7}\right)^{1/2}\mathrm{Re}(A_2A_0^*)\mathrm{Re}[Y^*_{42}(\theta,\phi)Y_{22}(\theta',\phi')]\Bigg\}\Bigg]\,. \end{aligned} \tag{70}$$

Again, θ represents the angle between p and γ in the $^1D_2$ rest frame. The angles $(\tilde{\theta}', \tilde{\phi}')$ are related to (θ', ϕ') through Eqs. (44)–(46). The angles $(\tilde{\theta}', \tilde{\phi}')$ represents the direction of ψ in the $^1D_2$ rest frame. For example, $(\pi - \tilde{\theta}')$ is the angle between γ and ψ in the $^1D_2$ rest frame or the $\bar{p}p$ c.m. frame. The direction of ψ can be obtained from the direction of the total momentum of e^+ and e^- or the direction of the total momentum of the two γ's into which π^0 decays. By measuring this angular distribution, one can determine the real parts of all $A_i A_j^*$ and from there the cosines of the relative phases of the helicity amplitudes A_i $(i = 0, 1, 2)$.

We can also obtain all the above information on the helicity amplitudes by looking at the combined angular distribution of γ, ψ, and e^- as given by Eq. (36). Using the orthonormality of the Wigner D^J functions, we can write the coefficient of the $Y_{d'd}^{L_1 L_2 L_3}$ angular function as

$$\beta_{L_1} \gamma_{L_3} \alpha_{d'}^{L_1 L_3} \mathcal{E}_d^{L_3 L_2} [1 + (-1)^{L_2} \delta_{d0}][1 + (-1)^{L_2} \delta_{d'0}] = (2L_1 + 1)(2L_2 + 1)(2L_3 + 1) \int W(\theta, \phi; \theta', \phi'; \theta'', \phi'') \mathcal{Y}_{d'd}^{*L_1 L_2 L_3} d\Omega d\Omega' d\Omega'' , \quad (71)$$

where $\mathcal{Y}_{d'd}^{L_1 L_2 L_3}$ is defined by Eq. (43). If the angular distribution W is determined at sufficiently large numbers of points the integral on the right-hand side can be performed numerically for all possible allowed values of L_1, L_2, L_3, d, and d'. A close examination of the expressions for β_{L_1}, γ_{L_3}, $\alpha_{d'}^{L_3 L_2}$ given by Eqs. (57)–(60) show that this will enable us to redetermine not only all the magnitudes of the helicity amplitudes and the cosines of all their relative phases, but also the products $\mathrm{Im}(E_1 E_0^*)\mathrm{Im}(A_1 A_0^*)$ and $\mathrm{Im}(E_1 E_0^*)\mathrm{Im}(A_2 A_1^*)$. They do not enable us to determine all the sines of the relative phases uniquely. But if the sine of the relative phase between any two is known by some other means, it enables us to determine the sines of the relative phases between any other two. All this information can also be obtained by measuring the six partially integrated angular distributions mentioned above. Moreover, the experimental verification of the partially integrated angular distribution functions in cases (1)–(6) is a confirmation of the discovery of the singlet D_2 state and its J^{PC} quantum numbers.

Finally, the $E3$, $M2$, and $E1$ multipole amplitudes in the decay $^1D_2 \to {}^1P_1 + \gamma$ are related to the helicity amplitudes A_0, A_1, and A_2 through an orthogonal transformation [6,8,9]:

$$A_\nu(\nu = 0, 1, 2) = \sum_{k=1}^{3} a_k \left(\frac{2k+1}{5} \right)^{1/2} \times \langle k1; -1, \nu | 2, \nu - 1 \rangle , \quad (72)$$

where a_1, a_2, and a_3 are the $E1$, $M2$, and the $E3$ multipole amplitudes, respectively. Since the transformation is orthogonal:

$$|a_1|^2 + |a_2|^2 + |a_3|^2 = |A_0|^2 + |A_1|^2 + |A_3|^2 = 1 . \quad (73)$$

In order to determine the multipole amplitudes a_k $(k = 1, 2, 3)$ uniquely, one should determine the amplitudes A_ν $(\nu = 0, 1, 2)$ uniquely. The relative phases of the multipole amplitudes are nontrivial. The potential model calculations [7] suggest that the multipole amplitudes in the $^1D_2 \to {}^1P_1 + \gamma$ transition are in general complex.

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