Dynamical flavor symmetry breaking by a magnetic field in 2 + 1 dimensions

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It is shown that in 2 + 1 dimensions a constant magnetic field is a strong catalyst of dynamical flavor symmetry breaking, leading to generating a fermion dynamical mass even at the weakest attractive interaction between fermions. The essence of this effect is that in a magnetic field, in 2 + 1 dimensions, the dynamics of fermion pairing is essentially one dimensional. The effect is illustrated in the Nambu-Jona-Lasinio model in a magnetic field. The low-energy effective action in this model is derived and the thermodynamic properties of the model are considered. The relevance of this effect for planar condensed matter systems and for (3 + 1)-dimensional theories at high temperature is pointed out.

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I. INTRODUCTION

Recently there has been considerable interest in relativistic field models in 2 + 1 dimensions. In addition to the fact that the sophisticated dynamics of these models is interesting in itself, the models also serve as effective theories for the description of long wavelength excitations in planar condensed matter systems [1,2]. Also, their dynamics imitates the dynamics of (3 + 1)-dimensional theories at high temperature.

In this paper, we will show that a constant magnetic field acts as a strong catalyst of dynamical flavor symmetry breaking (generating fermion masses) in 2 + 1 dimensions. We will in particular show that there is a striking similarity between the role of the magnetic field in (2+1)-dimensional models and the role of the Fermi surface in the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [3]: both of them enhance the interactions of fermions in the infrared region (at small momenta) thus leading to generating a fermion dynamical mass (energy gap in the fermion spectrum) even at the weakest attractive interaction between fermions.

We note that necessity of a supercritical dynamics (with an effective coupling constant g being larger than a critical value $g_c > 0$) for generating fermion dynamical masses is a common feature of the dynamics in 3 + 1 and 2 + 1 dimensions [4]. As will be shown in this paper, in 2 + 1 dimensions, a magnetic field reduces the value of the critical coupling to zero. We note that the fact that a constant magnetic field enhances fermion dynamical masses in the Nambu–Jona-Lasinio (NJL) model [5] has already been pointed out in Ref. [6]. However, what we will show is not just that the magnetic field enhances the dynamical mass created by the strong (supercritical) NJL interaction but that in 2 + 1 dimensions, it catalyzes generating the mass even at the weakest attractive interaction. The essence of this effect is that in a magnetic field, in 2 + 1 dimensions, the dynamics of fermion pairing (relating essentially to fermions at the lowest Landau level) is one dimensional (see Sec. II).

We stress that this effect is universal, i.e., model independent, in 2 + 1 dimensions. This point may be important in connection with consideration of this effect in such condensed matter phenomena as the quantum Hall effect [1] and high temperature superconductivity [2]. Another, potentially interesting, application for this effect may be in (3+1)-dimensional theories at high temperature (quark-gluon plasma in a magnetic field, for example); indeed, at high temperature, their dynamics effectively reduces to the dynamics of (2 + 1)-dimensional theories.

As a soluble example we shall consider the NJL model in a magnetic field, in the leading order in $1/N$ expansion. We shall derive the low-energy effective action in the model and also study its thermodynamic properties.

The paper is organized as follows. In Secs. II and III we consider the problem of a relativistic fermion in a magnetic field in 2+1 dimensions. We show that the roots of the fact that a magnetic field is a strong catalyst of dynamical flavor symmetry breaking in 2 + 1 dimensions are actually in this problem. In Secs. IV–VII we study the NJL model in a magnetic field in 2 + 1 dimensions. We derive the low-energy effective action and determine the spectrum of long wavelength collective excitations in this model. In Sec. VIII we study the thermodynamic properties of the NJL model in a magnetic field. We show that there is a symmetry restoring phase transition at high temperature. In Sec. IX we conclude the main results of the paper. In Appendices A and B, some useful formulas and relations are derived. In Appendix C the reliability of the $1/N$ expansion in this model is discussed.

II. DYNAMICAL FLAVOR SYMMETRY BREAKING IN THE PROBLEM OF FERMIONS IN A CONSTANT MAGNETIC FIELD

In this section we will discuss the problem of relativistic fermions in a magnetic field in $2+1$ dimensions. We will show that the roots of the fact that a magnetic field is a strong catalyst of flavor symmetry breaking are actually in this dynamics, which plays here the role similar to that of the dynamics of the ideal Bose gas for an almost ideal Bose gas in the theory of superfluidity [4].

The Lagrangian density in the problem of a relativistic fermion in a constant magnetic field B takes the following form in $2+1$ dimensions:

$$\mathcal{L} = \frac{1}{2}\left[\bar{\Psi}, (i\tilde{\gamma}^\mu D_\mu - m)\Psi\right], \quad \mu = 0,1,2, \tag{1}$$

where the covariant derivative is

$$D_\mu = \partial_\mu - ieA_\mu^{\text{ext}}, \quad A_\mu^{\text{ext}} = -Bx_2\delta_{\mu 1}. \tag{2}$$

In $2+1$ dimensions, there are two inequivalent representations of the Dirac algebra:

$$\tilde{\gamma}^0 = \sigma_3, \quad \tilde{\gamma}^1 = i\sigma_1, \quad \tilde{\gamma}^2 = i\sigma_2, \tag{3}$$

and

$$\tilde{\gamma}^0 = -\sigma_3, \quad \tilde{\gamma}^1 = -i\sigma_1, \quad \tilde{\gamma}^2 = -i\sigma_2, \tag{4}$$

where σ_i are Pauli matrices.

Let us begin by considering the representation (3). The energy spectrum in the problem (1) depends on the sign of eB; let us first assume that $eB > 0$. Then, the energy spectrum takes the form (to be concrete, we assume that $m \geq 0$) [7]

$$\begin{aligned} E_0 &= \omega_0 = m, \\ E_n &= \pm\omega_n = \pm\sqrt{m^2 + 2|eB|n}, \quad n = 1,2,\ldots \end{aligned} \tag{5}$$

(the Landau levels).

The general solution is

$$\Psi(x) = \sum_{n,p} a_{np}u_{np}(x) + \sum_{n,p} b_{np}^\dagger v_{n-p}(x), \tag{6}$$

where

$$\begin{aligned} u_{0p} &= \frac{1}{(lL_1)^{1/2}}\exp(-i\omega_0 t + ikx_1)\begin{bmatrix} w_0(\xi) \\ 0 \end{bmatrix}, \\ u_{np} &= \frac{1}{(lL_1)^{1/2}}\exp(-i\omega_n t + ikx_1)\frac{1}{\sqrt{2\omega_n}}\begin{bmatrix} \sqrt{\omega_n + m}w_n(\xi) \\ -i\sqrt{\omega_n - m}w_{n-1}(\xi) \end{bmatrix}, \quad n \geq 1, \\ v_{np} &= \frac{1}{(lL_1)^{1/2}}\exp(i\omega_n t + ikx_1)\frac{1}{\sqrt{2\omega_n}}\begin{bmatrix} \sqrt{\omega_n - m}w_n(\xi) \\ i\sqrt{\omega_n + m}w_{n-1}(\xi) \end{bmatrix}, \quad n \geq 1. \end{aligned} \tag{7}$$

Here $w_n(\xi) = (\pi^{1/2}2^n n!)^{-1/2}e^{-\xi^2/2}H_n(\xi)$, $H_n(\xi)$ are Hermite polynomials, $l \equiv |eB|^{-1/2}$ is the magnetic length, $k = 2\pi p/L_1$ ($p = 0, \pm1, \pm2, \ldots$), L_1 is the size in the x_1 direction, and $\xi \equiv x_2/l + kl$. As $L_1 \to \infty$, the density of the states at each level n is $|eB|/2\pi$ [7].

Thus the lowest Landau level with $n = 0$ is special: while at $n \geq 1$, there are solutions corresponding to both fermion ($E_n = \omega_n$) and antifermion ($E_n = -\omega_n$) states, the solution with $n = 0$ describes only fermion states.

As $eB \to -eB < 0$, the solution becomes

$$\Psi(x) = \sum_{n,p} a_{np}v_{n-p}^c(x) + \sum_{np} b_{np}^\dagger u_{np}^c(x), \tag{8}$$

where the charge conjugate spinors v^c and u^c are $v^c = \tilde{\gamma}_2\bar{v}^T$, $u^c = \tilde{\gamma}_2\bar{u}^T$. Therefore at $eB < 0$, the lowest Landau level with $n = 0$ describes antifermion states.

If we used the representation (4) for Dirac's matrices, the general solution would be given by Eq. (6) with $u_{np}(x)$, $v_{np}(x)$ being substituted by $(-1)^n v_{n-p}(-x), (-1)^n u_{n-p}(-x)$ [the factor $(-1)^n$ is introduced here for convenience]:

$$\Psi(x) = \sum_{n,p} c_{np}(-1)^n v_{n-p}(-x) + \sum_{n,p} d_{np}^\dagger(-1)^n u_{np}(-x). \tag{9}$$

We note that the mass term in the Lagrangian density (1) violates parity defined by

$$P: \Psi(x^0, x^1, x^2) \to \sigma_1\Psi(x^0, -x^1, x^2). \tag{10}$$

However, if one uses the four-component fermions [8], connected with a four-dimensional (reducible) representation of Dirac's matrices,

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \\ \gamma^2 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \end{aligned} \tag{11}$$

the mass term in the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\bar{\Psi}, (i\gamma^\mu D_\mu - m)\,\Psi\right] \tag{12}$$

preserves parity defined now as

$$P : \Psi(x^0, x^1, x^2) \to \frac{1}{i}\gamma^3\gamma^1\Psi(x^0, -x^1, x^2), \tag{13}$$

where the Dirac matrix γ^3 is

$$\gamma^3 = i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{14}$$

The important point is that the Lagrangian density (12) with $m = 0$ is invariant under the U(2) (flavor) transformations with the generators

$$T_0 = I,\;\; T_1 = \gamma_5,\;\; T_2 = \frac{1}{i}\gamma^3,\;\; T_3 = \gamma^3\gamma^5, \tag{15}$$

where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{16}$$

The mass term breaks this symmetry down to the U(1)×U(1) with the generators T_0 and T_3.

We note that the four-component fermions appear in low-energy effective actions describing planar condensed matter systems with two sublattices [2]. Actually, usually they appear in the actions without the mass term, and the important problem is to establish a criterion of dynamical flavor symmetry breaking which may occur as a result of interaction between fermions [4,9–16]. As was already indicated in Sec. I, dynamical flavor symmetry breaking in $2+1$ dimensions usually takes place only at a rather strong effective coupling between fermions.

Let us now show that at $m = 0$ and $B \neq 0$, the dynamical breakdown of the U(2) flavor symmetry takes place already in the theory (12), even without any additional interaction between fermions. In order to prove this, we will show that in the limit $m \to 0$, the flavor condensate $\langle 0|\bar{\Psi}\Psi|0\rangle$ is nonzero: $\langle 0|\bar{\Psi}\Psi|0\rangle = -|eB|/2\pi$.

The condensate $\langle 0|\bar{\Psi}\Psi|0\rangle$ is expressed through the fermion propagator $S(x,y) = \langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle$:

$$\langle 0|\bar{\Psi}\Psi|0\rangle = -\lim_{x\to y}\mathrm{tr}S(x,y). \tag{17}$$

The propagator S is calculated [following the Schwinger (proper time) approach [17]] in Appendix A. It is

$$S(x,y) = \exp\left(ie\int_y^x A_\lambda^{\mathrm{ext}} dz^\lambda\right)\tilde{S}(x-y), \tag{18}$$

$$\begin{aligned}\tilde{S}(x) = \int_0^\infty &\frac{ds}{8(\pi s)^{3/2}}\exp\left[-i\left(\frac{\pi}{4} + sm^2\right)\right]\\ &\times\exp\left[-\frac{i}{4s}\left(x_\nu C^{\nu\mu}x_\mu\right)\right]\\ &\times\left[\left(m + \frac{1}{2s}\gamma^\mu C_{\mu\nu}x^\nu - \frac{e}{2}\gamma^\mu F_{\mu\nu}^{\mathrm{ext}}x^\nu\right)\right.\\ &\times\left.\left(esB\cot(eBs) - \frac{es}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}^{\mathrm{ext}}\right)\right],\end{aligned} \tag{19}$$

where $C^{\mu\nu} = g^{\mu\nu} + ((F^{ext})^2)^{\mu\nu}[1 - eBs\cot(eBs)]/B^2$, $F_{\mu\nu}^{\mathrm{ext}} = \partial_\mu A_\nu^{\mathrm{ext}} - \partial_\nu A_\mu^{\mathrm{ext}}$ with A_μ^{ext} given in Eq. (2). The integral in Eq. (18) is calculated along the straight line.

The Fourier transform $\tilde{S}(k) = \int d^3x e^{ikx}\tilde{S}(x)$ is

$$\begin{aligned}\tilde{S}(k) = \int_0^\infty &ds\exp\left[-ism^2 + isk_0^2 - is\boldsymbol{k}^2\frac{\tan(eBs)}{eBs}\right]\\ &\times\{[\hat{k} + m + (k^2\gamma^1 - k^1\gamma^2)\tan(eBs)]\\ &\times[1 + \gamma^1\gamma^2\tan(eBs)]\}\,.\end{aligned} \tag{20}$$

Transferring this expression into Euclidean space ($k^0 \to ik_3, s \to -is$), we find

$$\begin{aligned}\tilde{S}_E(k) = -i\int_0^\infty &ds\exp\left[-s\left(m^2 + k_3^2 + \boldsymbol{k}^2\frac{\tanh(eBs)}{eBs}\right)\right]\\ &\times\left[\left(-k_\mu\gamma_\mu + m + \frac{1}{i}(k_2\gamma_1 - k_1\gamma_2)\tanh(eBs)\right)\right.\\ &\times\left.\left(1 + \frac{1}{i}\gamma_1\gamma_2\tanh(eBs)\right)\right]\end{aligned} \tag{21}$$

($\gamma_3 = -i\gamma^0, \gamma_1 \equiv \gamma^1, \gamma_2 \equiv \gamma^2$ are anti-Hermitian matrices).

From Eqs. (17), (18), and (21) we find the expression for the condensate:

$$\begin{aligned}\langle 0|\bar{\Psi}\Psi|0\rangle &= -\frac{i}{(2\pi)^3}\mathrm{tr}\int d^3k\tilde{S}_E(k)\\ &= -\lim_{\Lambda\to\infty}\lim_{m\to 0}\frac{4m}{(2\pi)^3}\int d^3k\int_{1/\Lambda^2}^\infty ds\exp\left[-s\left(m^2 + k_3^2 + \boldsymbol{k}^2\frac{\tanh(eBs)}{eBs}\right)\right]\\ &= -\lim_{\Lambda\to\infty}\lim_{m\to 0}\frac{m}{2\pi^{3/2}}\int_{1/\Lambda^2}^\infty ds e^{-sm^2}(s^{-1/2})(eB)\coth(eBs)\\ &= -\lim_{\Lambda\to\infty}\lim_{m\to 0}\frac{m}{2\pi^{3/2}}\left[\pi^{1/2}|eB|\frac{1}{m} + O\left(\frac{1}{\Lambda}\right)\right]\\ &= -\frac{|eB|}{2\pi},\end{aligned} \tag{22}$$

where Λ is an ultraviolet cutoff.

Thus in a constant magnetic field, spontaneous breakdown of the flavor U(2) symmetry takes place even though fermions do not acquire mass ($m = 0$). Note that in $3+1$ dimensions the result would be $\langle 0|\bar{\Psi}\Psi|0\rangle \sim m \ln m \to 0$ as $m \to 0$. Therefore, this is a specific (2+1)-dimensional phenomenon.

What is the physical basis of this phenomenon? In order to answer this question, we note that a singular $1/m$ behavior of the integral in Eq. (22) is formed at large, $s \to \infty$, distances (s is the proper time coordinate). Actually one can see from Eq. (22) that the magnetic field effectively removes the two space dimensions in the infrared region thus reducing the dynamics to a one-dimensional dynamics which has much more severe infrared singularities. From this viewpoint, the action of the magnetic field in the present problem is similar to that of the Fermi surface in the BCS theory [3].

This point is intimately connected with the form of the energy spectrum of fermions in a constant magnetic field. Equations (5), (11), and (12) imply that for the four-component fermions, the energy spectrum is

$$\begin{aligned} E_0 &= \pm\omega_0 = \pm m, \\ E_n &= \pm\omega_n = \pm\sqrt{m^2 + 2|eB|n}, \quad n \geq 1. \end{aligned} \tag{23}$$

The density of the states with the energy $\pm\omega_0 = \pm m$ is $|eB|/2\pi$, and it is $|eB|/\pi$ at $n \geq 1$. As $m \to 0$, the energy E_0 goes to zero and therefore there is the infinite vacuum degeneracy in this case. The value of the condensate (22) is equal to the density of the states at the lowest Landau level. This implies that spontaneous flavor symmetry breaking is intimately connected with the dynamics of fermions at this level. In particular, since this dynamics, described by one continuous variable $k_3 = -ik^0$, is one dimensional, we get a simple explanation of the one-dimensional character of the dynamics of flavor symmetry breaking in this problem. More precisely, the situation is the following. Using the identity $\tanh(x) = 1 - 2\exp(-2x)/[1+\exp(-2x)]$ and the relation [18]

$$(1-z)^{-(\alpha+1)} \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n^\alpha(x) z^n, \tag{24}$$

where $L_n^\alpha(x)$ are the generalized Laguerre polynomials, the propagator $\tilde{S}_E(k)$ can be decomposed over the Landau level poles [19]:

$$\tilde{S}_E(k) = -i \exp\left(-\frac{\mathbf{k}_\perp^2}{|eB|}\right) \sum_{n=0}^{\infty} (-1)^n \frac{D_n(eB,k)}{k_3^2 + m^2 + 2|eB|n}, \tag{25}$$

with

$$\begin{aligned} D_n(eB,k) = (m - k_3\gamma_3)\Big[&[1 - i\gamma_1\gamma_2 \mathrm{sgn}(eB)] L_n\left(2\frac{\mathbf{k}_\perp^2}{|eB|}\right) \\ &-[1 + i\gamma_1\gamma_2 \mathrm{sgn}(eB)] L_{n-1}\left(2\frac{\mathbf{k}_\perp^2}{|eB|}\right)\Big] \\ &+4(k_1\gamma_1 + k_2\gamma_2) L_{n-1}^1\left(2\frac{\mathbf{k}_\perp^2}{|eB|}\right), \end{aligned} \tag{26}$$

where $L_n \equiv L_n^0$ and $L_{-1}^\alpha(x) = 0$ by definition. Then Eq. (25) implies that as $m \to 0$, the condensate appears due to the lowest Landau level:

$$\langle 0|\bar{\Psi}\Psi|0\rangle \simeq -\frac{m}{2\pi^3} \int d^3k \frac{\exp\left(-\mathbf{k}_\perp^2/|eB|\right)}{k_3^2 + m^2} = -\frac{|eB|}{2\pi}. \tag{27}$$

We would like to note that in $3+1$ dimensions, the dynamics at the lowest Landau level, described by two continuous variables k_3 and k_4, is two dimensional.[1] Actually, the lowest Landau level pole in $\tilde{S}_E(k)$ is in $3+1$ dimensions:

$$\tilde{S}^0(k) = -i \exp\left(-\frac{\mathbf{k}_\perp^2}{|eB|}\right) \frac{m - k_4\gamma_4 - k_3\gamma_3}{k_4^2 + k_3^2 + m^2} (1 - i\gamma_1\gamma_2). \tag{28}$$

The contribution of this level to the condensate is now

$$\langle 0|\bar{\Psi}\Psi|0\rangle \simeq -|eB| \frac{m}{4\pi^2}\left(\ln\frac{\Lambda^2}{m^2} + O(m^0)\right) \stackrel{m\to 0}{\to} 0; \tag{29}$$

i.e., there is no spontaneous flavor symmetry breaking in a magnetic field in $3+1$ dimensions.

In the next section, we will discuss aspects of spontaneous flavor symmetry breaking for $(2+1)$-dimensional fermions in a magnetic field in more detail.

III. MORE ABOUT DYNAMICAL FLAVOR SYMMETRY BREAKING FOR FERMIONS IN A MAGNETIC FIELD

As was shown in the preceding section, the flavor condensate $\langle 0|\bar{\Psi}\Psi|0\rangle$ is nonzero as the fermion mass m goes to zero. Although usually this fact is considered as a firm signature of spontaneous flavor (or chiral) symmetry breaking, the following questions may arise in this case.

(a) Unlike the conventional spontaneous flavor (chiral) symmetry breaking, the dynamical mass of fermions equals zero in this problem. Is spontaneous flavor symmetry breaking "real" in this case?

(b) The vacuum $|0\rangle$ was defined as $\lim_{m\to 0} |0\rangle_m$ of the vacuum $|0\rangle_m$ in the theory with $m \neq 0$. The vacuum

[1] The fact that a magnetic field reduces the effective dimension of the dynamics of the fermion pairing by two units was observed earlier in the theory of superconductivity [20].

$|0\rangle$ corresponds to a particular filling of the lowest Landau level. Indeed, at $m \neq 0$, in the vacuum $|0\rangle_m$, the states with $E_0 = m > 0$ are empty and the states with $E_0 = -m$ are filled, i.e., the vacuum $|0\rangle = \lim_{m\to 0}|0\rangle_m$ is annihilated by all the operators a_{0p}, d_{0p} and a_{np}, b_{np}, c_{np}, d_{np} $(n \geq 1)$. On the other hand, at $m = 0$, there is the infinite degeneracy of the vacuum in this problem, connected with different fillings of the lowest Landau level. Why should one choose the filling leading to the vacuum $|0\rangle$? And is there a filling of the lowest Landau level leading to the ground state which is invariant under the flavor U(2)? One might think that the latter possibility would imply that spontaneous flavor symmetry breaking can be avoided.

In this section we will show that there is a genuine realization of the spontaneous breakdown of the flavor symmetry in the present problem. More precisely, we shall show that this phenomenon satisfies all the criteria of the spontaneous symmetry breaking phenomenon established by Haag long ago [21]. We will also discuss such related questions as the status of Nambu-Goldstone (NG) modes and induced quantum numbers [22,23] in this problem.

Let us begin by constructing the charge operators $Q_i = 1/2\int d^2x[\Psi^\dagger(x), T_i\Psi(x)]$ of the flavor U(2) group. By using Eqs.(6), (7), and (9), we find

$$Q_0 = \sum_p \left(a^\dagger_{0p}a_{0p} - d^\dagger_{0-p}d_{0-p}\right) + \sum_{n=1}^{\infty}\sum_p \left(a^\dagger_{np}a_{np} - b^\dagger_{np}b_{np} + c^\dagger_{np}c_{np} - d^\dagger_{np}d_{np}\right),$$

$$Q_1 = i\sum_p \left(a^\dagger_{0p}d^\dagger_{0-p} - d_{0-p}a_{0p}\right) + i\sum_{n=1}^{\infty}\sum_p \left(a^\dagger_{np}c_{np} - c^\dagger_{np}a_{np} + b^\dagger_{np}d_{np} - d^\dagger_{np}b_{np}\right), \tag{30}$$

$$Q_2 = \sum_p \left(a^\dagger_{0p}d^\dagger_{0-p} + d_{0-p}a_{0p}\right) + \sum_{n=1}^{\infty}\sum_p \left(a^\dagger_{np}c_{np} + c^\dagger_{np}a_{np} + b^\dagger_{np}d_{np} + d^\dagger_{np}b_{np}\right),$$

$$Q_3 = \frac{|eB|}{2\pi}S + \sum_p \left(a^\dagger_{0p}a_{0p} + d^\dagger_{0-p}d_{0-p}\right) + \sum_{n=1}^{\infty}\sum_p \left(a^\dagger_{np}a_{np} - b^\dagger_{np}b_{np} - c^\dagger_{np}c_{np} + d^\dagger_{np}d_{np}\right),$$

where $a_{np}, c_{np}, (b_{np}, d_{np})$ are annihilation operators of fermions (antifermions) from the nth Landau level and $S = L_1L_2$ is the two-dimensional volume. Now we can construct a set of the degenerate vacua

$$|\theta_1,\theta_2\rangle = \exp(iQ_1\theta_1 + iQ_2\theta_2)|0\rangle\ , \tag{31}$$

where, we recall, the vacuum $|0\rangle = \lim_{m\to 0}|0\rangle_m$ is annihilated by all the operators a_{np}, b_{np}, c_{np}, and d_{np}. As one can see from Eq. (30), the crucial point for the existence of the continuum set of the degenerate vacua is the first sum, over the states at the lowest Landau level, in the charges Q_1 and Q_2.

The presence of such a set of the degenerate vacua is a signal of the spontaneous breakdown, U(2)→ U(1)×U(1). Note that the vacua $|\theta_1,\theta_2\rangle$ can also be constructed by replacing the mass term $m\bar\Psi\Psi$ by $m\bar\Psi_{\theta_1,\theta_2}\Psi_{\theta_1,\theta_2}$, where $\Psi_{\theta_1,\theta_2} = \exp(iQ_1\theta_1 + iQ_2\theta_2)\Psi$, and then performing the limit $m \to 0$. Again, this is a standard way of constructing degenerate vacua in the case of spontaneous breakdown of a symmetry.

One can check that different vacua $|\theta_1,\theta_2\rangle$ become orthogonal as size L_1 in the x_1 direction goes to infinity. For example,

$$|\langle 0,\theta_2|0,\theta_2'\rangle| = \prod_p |\cos\theta|$$
$$= \exp(L_1\int dk \ln|\cos\theta|), \quad \theta = \theta_2' - \theta_2,$$

and at $\theta \neq 0$ or π, it goes to zero as $L_1 \to \infty$ (the vacuum $|0,\theta_2+\pi\rangle = -|0,\theta_2\rangle$). It also goes to zero as the maximum momentum $|k_{\max}| = \Lambda$ (Λ is an ultraviolet cutoff) goes to infinity. As usual, this point reflects the fact that spontaneous symmetry breaking occurs only in a system with an infinite number of degrees of freedom. One can check that in this case all states (and not just vacua) from different Fock spaces $F_{\{\theta_1\theta_2\}}$, defined by different vacua $|\theta_1,\theta_2\rangle$, are orthogonal. That is, different vacua $|\theta_1,\theta_2\rangle$ define nonequivalent representations of canonical commutation relations.

On the other hand, taking the ground state

$$|\Omega\rangle = C\int d\mu(\theta_1,\theta_2,\theta_3)|\theta_1,\theta_2\rangle, \tag{32}$$

where $d\mu$ is the invariant measure of SU(2) and C is a normalization constant, we are led to the vacuum $|\Omega\rangle$ which is a singlet with respect to the flavor U(2). In fact, the set of the vacua $\{|\theta_1,\theta_2\rangle\}$ can be decomposed in irreducible representations of SU(2):

$$\{|\theta_1,\theta_2\rangle\} = \{|\Omega^{(i)}\rangle\} \; . \qquad (33)$$

Why should we consider the vacua $|\theta_1,\theta_2\rangle$ instead of the vacua $|\Omega^{(i)}\rangle$?

To answer this question, we consider, following Haag [21], the clusterization property of Green's functions. It means the following. Let us consider a Green's function

$$G^{(n+k)} = \left\langle 0 \left| T \prod_{i=1}^{n} A_i(x_i) \prod_{j=1}^{k} B_j(y_j) \right| 0 \right\rangle , \qquad (34)$$

where $A_i(x_i)$, $B_j(y_j)$ are some local operators. The clusterization property implies that when $r_{ij}^2 \to \infty$ [$r_{ij}^2 = (\mathbf{x}_i - \mathbf{y}_j)^2$] for all i and j, the Green's function then factorizes as follows:

$$G^{(n+k)} \to \left\langle 0 \left| T \prod_{i=1}^{n} A_i(x_i) \right| 0 \right\rangle \left\langle 0 \left| T \prod_{j=1}^{k} B_j(y_j) \right| 0 \right\rangle . \qquad (35)$$

The physical meaning of this property is clear: clusterization implies the absence of instantaneous long-range correlations in the system, so that the dynamics in two distant spatially separated regions are independent.

The clusterization property takes place for all the vacua $|\theta_1,\theta_2\rangle$. The simplest way to show this is to note that the vacuum $|\theta_1,\theta_2\rangle$ appears in the limit $m \to 0$ from the vacuum in the system with the mass term $m\bar{\Psi}_{\theta_1\theta_2}\Psi_{\theta_1\theta_2}$. Since at $m \neq 0$, the vacuum in this system is unique, the clusterization is valid at *every* value of $m \neq 0$. Therefore, it is also valid in the limit $m \to 0$, as far as the Green's functions exist in this limit. In connection with that, we would like to note that, in thermodynamic limit $L_1, L_2 \to \infty$, the vacuum $|\theta_1,\theta_2\rangle$ is the only normalizable and translation invariant state in the Fock space $F_{\theta_1\theta_2}$. To show this, let us introduce the operators $a_n(k) = (L_1/2\pi)^{1/2} a_{np}$, $b_n(k) = (L_1/2\pi)^{1/2} b_{np}$, $c_n(k) = (L_1/2\pi)^{1/2} c_{np}$, and $d_n(k) = (L_1/2\pi)^{1/2} d_{np}$, where $k = 2\pi p/L_1$. They satisfy the commutation relations $[a_n(k), a_{n'}^\dagger(k')] = \delta_{nn'}\delta(k-k')$, etc. Therefore, though states of the form $\prod_i a_0^\dagger(k_i) \prod_j d_0^\dagger(k_j)|\theta_1,\theta_2\rangle$ have zero energy, they are not normalizable and, at $\sum_i k_i + \sum_j k_j \neq 0$, not translation invariant.

On the other hand, the clusterization property is not valid for all Green's functions on the vacua $|\Omega^{(i)}\rangle$. As an example, consider the Green's function

$$G^{(4)} = \langle \Omega | T[\bar{\Psi}(x_1)\Psi(x_2)][\bar{\Psi}(y_1)\Psi(y_2)] | \Omega \rangle , \qquad (36)$$

where $|\Omega\rangle$ is the vacuum singlet (32). Since the bilocal operator $\bar{\Psi}(x_1)\Psi(x_2)$ is assigned to the triplet of SU(2), the clusterization property would imply that

$$G^{(4)} \to \langle \Omega | T[\bar{\Psi}(x_1)\Psi(x_2)] | \Omega \rangle \langle \Omega | [\bar{\Psi}(y_1)\Psi(y_2)] | \Omega \rangle \to 0 \qquad (37)$$

as $r_{ij}^2 = (\mathbf{x}_i - \mathbf{y}_j)^2 \to \infty$. However, since

$$\langle \Omega | T[\bar{\Psi}(x_1)\Psi(x_2)] | \Omega^{(3)} \rangle \neq 0,$$
$$\langle \Omega^{(3)} | T[\bar{\Psi}(y_1)\Psi(y_2)] | \Omega \rangle \neq 0, \qquad (38)$$

where $|\Omega^{(3)}\rangle$ is a state from the vacuum triplet, we see that $G^{(4)}$ does not vanish as $r_{ij}^2 \to \infty$.

Thus the clusterization property does not take place for the $|\Omega^{(i)}\rangle$ vacua.

This is a common feature of the systems with spontaneous continuous symmetry breaking [4,21]: an orthogonal set of vacua can either be labeled by the continuous parameters $\{\theta_i\}$, connected with the generators Q_i of the broken symmetry, or it can be decomposed in irreducible representations of the initial group. However, the latter vacua do not satisfy the clusterization property.

All the Fock spaces $F_{\{\theta_1\theta_2\}}$ yield physically equivalent descriptions of the dynamics: in the space $F_{\{\theta_1\theta_2\}}$, the SU(2) spontaneously breaks down to $\mathrm{U}_{\{\theta_1\theta_2\}}(1)$, where the $\mathrm{U}_{\{\theta_1\theta_2\}}(1)$ symmetry is connected with the generator $Q_3^{\{\theta_1\theta_2\}} = \exp(iQ_1\theta_1 + iQ_2\theta_2)Q_3\exp(-iQ_1\theta_1 - iQ_2\theta_2)$. Are there NG modes in the present system? To answer this question, let us consider the thermodynamic limit $L_1, L_2 \to \infty$. One can see that in every Fock space $F_{\{\theta_1\theta_2\}}$, with the vacuum $|\theta_1,\theta_2\rangle$, there are a lot of "excitations" with nonzero momentum k and zero energy E created by the operators $a_0^\dagger(k)$ and $d_0^\dagger(k)$. However, there are no genuine (i.e., with a nontrivial dispersion law) NG modes: the energy E is $E \equiv 0$ at the lowest Landau level. Since the Lorentz symmetry is broken by a magnetic field, this point does not contradict the Goldstone's theorem.[2] This of course does not imply that the existence of NG modes is incompatible with a magnetic field: the situation is model dependent. As will be shown in Secs. IV–VII, even the weakest attractive interaction in the problem of $(2+1)$-dimensional fermions in a magnetic field is enough to "resurrect" the genuine NG modes. The key point for their existence is that the flavor condensate $\langle 0|\bar{\Psi}\Psi|0\rangle$ and the NG modes are neutral, and the translation symmetry in neutral channels is not violated by a magnetic field (see the next section). We shall also see that the "excitations" from the lowest Landau level (with quantum numbers of the NG modes) in the problem of free fermions in a magnetic field can be interpreted as "remnants" of the genuine NG modes in the limit when the interaction between fermions is being switched off. Moreover, we shall see in Sec. V that the vacua $|\theta_1,\theta_2\rangle$ constructed above yield a very good approximation for the vacua of systems with weakly interacting fermions in a magnetic field (in fact, it appears that the role of the vacua $|\theta_1,\theta_2\rangle$ is the same as that of the θ vacuum of the ideal Bose gas for an almost ideal Bose gas in the theory of superfluidity [4]).

In conclusion, let us discuss the phenomenon of induced quantum numbers [22,23] in this problem. As it follows from Eq. (30), the vacuum $|\theta_1,\theta_2\rangle$ is an eigenstate of the density operator $\rho_3^{\{\theta_1\theta_2\}} = \lim_{S\to\infty} Q_3^{\{\theta_1\theta_2\}}/S$ with a nonzero value:

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[2] As to a nonrelativistic analogue of the Goldstone's theorem, it has been proved only for translation invariant systems with finite range interactions [24]. Systems in a magnetic field do not satisfy this condition.

$$\rho_3^{\{\theta_1\theta_2\}}|\theta_1,\theta_2\rangle = \frac{|eB|}{2\pi}|\theta_1,\theta_2\rangle. \quad (39)$$

Thus, there is the induced quantum number of the operator $\rho_3^{\{\theta_1\theta_2\}}$ in the $|\theta_1\theta_2\rangle$ vacuum.[3] This fact is intimately connected with the phenomenon of spontaneous flavor symmetry breaking in this problem. Indeed, since $Q_3^{\{\theta_1\theta_2\}}$ is one of the generators of the non-Abelian SU(2) symmetry, its vacuum eigenvalue would be equal to zero if the symmetry were exact and the vacuum were assigned to the singlet (trivial) representation of SU(2). This is in contrast with the case of Abelian U(1) symmetry: since U(1) has an infinite number of one-dimensional representations, the vacuum can be an eigenstate of the charge density $\rho = \lim_{S\to\infty} Q/S$ with an arbitrary eigenvalue in that case.

Note that, since the SU(2) is spontaneously broken here, it is appropriate to redefine the generator of the exact $U_{\{\theta_1\theta_2\}}(1)$ symmetry as $\tilde{Q}_3^{\{\theta_1\theta_2\}}=Q_3^{\{\theta_1\theta_2\}} - |eB|S/2\pi$.

IV. THE NAMBU–JONA-LASINIO MODEL IN A MAGNETIC FIELD: GENERAL CONSIDERATION

In this and the following four sections, we shall consider the NJL model in (2 + 1) dimensions. This model gives a clear illustration of the general fact that a constant magnetic field is a strong catalyst of generating a fermion dynamical mass in 2 + 1 dimensions.

Let us consider the (2 + 1)-dimensional NJL model invariant under the U(2) flavor transformations:

$$\mathcal{L} = \frac{1}{2}\left[\bar{\Psi}, i\gamma^\mu D_\mu \Psi\right] + \frac{G}{2}\left[(\bar{\Psi}\Psi)^2 + (\bar{\Psi}i\gamma_5\Psi)^2 + (\bar{\Psi}\gamma_3\Psi)^2\right], \quad (40)$$

where D_μ is the covariant derivative (2) and fermion fields carry an additional "color," index $\alpha = 1, 2, \ldots, N$. This theory is equivalent to a theory with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\bar{\Psi}, i\gamma^\mu D_\mu \Psi\right] - \bar{\Psi}\left(\sigma + \gamma^3\tau + i\gamma^5\pi\right)\Psi - \frac{1}{2G}\left(\sigma^2 + \pi^2 + \tau^2\right). \quad (41)$$

The Euler-Lagrange equations for the auxiliary fields σ, τ, and π take the form of constraints:

$$\sigma = -G(\bar{\Psi}\Psi), \quad \tau = -G(\bar{\Psi}\gamma^3\Psi), \quad \pi = -G(\bar{\Psi}i\gamma^5\Psi). \quad (42)$$

The Lagrangian density (41) reproduces Eq. (40) upon application of the constraints (42).

The effective action for the composite fields is expressed through the path integral over fermions:

$$\Gamma(\sigma,\tau,\pi) = -\frac{1}{2G}\int d^3x(\sigma^2+\tau^2+\pi^2) + \tilde{\Gamma}(\sigma,\tau,\pi), \quad (43)$$

$$\exp(i\tilde{\Gamma}) = \int [d\Psi][d\bar{\Psi}]\exp\left\{\frac{i}{2}\int d^3x[\bar{\Psi},\{i\gamma^\mu D_\mu - (\sigma+\gamma^3\tau+i\gamma^5\pi)\}\Psi]\right\} = \exp\{\mathrm{Tr}\ln[i\gamma^\mu D_\mu - (\sigma+\gamma^3\tau+i\gamma^5\pi)]\}, \quad (44)$$

i.e.,

$$\tilde{\Gamma}(\sigma,\tau,\pi) = -i\mathrm{Tr}\ln\left[i\gamma^\mu D_\mu - (\sigma+\gamma^3\tau+i\gamma^5\pi)\right]. \quad (45)$$

As $N \to \infty$, the path integral over the composite (auxiliary) fields is dominated by stationary points of the action: $\delta\Gamma/\delta\sigma = \delta\Gamma/\delta\tau = \delta\Gamma/\delta\pi = 0$. We will analyze the dynamics in this limit by using the expansion of the action Γ in powers of derivatives of the composite fields.

Is the $1/N$ expansion reliable in this problem? This question appears naturally since, as was emphasized in Sec. II, a magnetic field reduces the dimension of the dynamics of the fermion pairing by two units. If such a reduction took place for the whole dynamics (and not just for that of the fermion pairing), the $1/N$ perturbative expansion would be unreliable. In particular, the contribution of the NG modes in the gap equation, in next-to-leading order in $1/N$, would lead to infrared divergences. Just such a situation takes place in the (1+1)-dimensional Gross-Neveu model with a continuous chiral symmetry [25]. This phenomenon reflects the Mermin-Wagner-Coleman theorem [26] forbidding spontaneous breakdown of continuous symmetries in space dimensions lower than two.

Fortunately, as will be shown in Appendix C, this is not the case in the present problem. The central point is that condensate $\langle 0|\bar{\Psi}\Psi|0\rangle$ and the NG modes are neutral in this problem. As we shall see in Sec. VI and Appendix C, this is reflected in the structure of the propagator of the NG modes: unlike the fermion propagator, it has a genuine (2 + 1)-dimensional structure. As a result, their contribution to the dynamics does not lead to infrared divergences, and the $1/N$ expansion is reliable in this problem. This point is intimately connected with the status of the space-translation symmetry in a constant magnetic field. In the gauge (2), the translation symmetry along the x_2 direction is broken (though it can be restored by applying a certain gauge transformation). Therefore, the momentum k_2 is a bad quantum number for fermions and all other charged states [see Eqs. (6) and (7)]. However, for neutral states, both the momenta k_1 and k_2 of their center of mass are conserved quantum numbers (this property is gauge invariant) [27]. In order to show this fact in the gauge (2), let us introduce the following operators describing space translations in first quantized theory:

[3]Note that this fact agrees with the consideration in Ref. [23].

$$\hat{P}_{x_1} = \frac{1}{i}\frac{\partial}{\partial x_1}, \qquad \hat{P}_{x_2} = \frac{1}{i}\frac{\partial}{\partial x_2} + \hat{Q}Bx_1, \tag{46}$$

where $\hat{Q}$ is the charge operator. One can easily check that these operators commute with the Hamiltonian of the Dirac equation in a constant magnetic field. Also, the commutator $[\hat{P}_{x_1}, \hat{P}_{x_2}]$ is

$$[\hat{P}_{x_1}, \hat{P}_{x_2}] = -i\hat{Q}B. \tag{47}$$

Therefore, the commutator equals zero for neutral states, and both the momenta k_1 and k_2 can be used to describe the dynamics of the center of mass of neutral states. As we shall see, this point is important for providing the $(2+1)$-dimensional character of this dynamics.

V. THE NJL MODEL IN A MAGNETIC FIELD: THE EFFECTIVE POTENTIAL

We begin the calculation of Γ by calculating the effective potential V. Since V depends only on the SU(2)-invariant $\rho^2 = \sigma^2 + \tau^2 + \pi^2$, it is sufficient to consider a configuration with $\tau = \pi = 0$ and σ independent of x. So now $\tilde{\Gamma}(\sigma)$ is

$$\tilde{\Gamma}(\sigma) = -i\mathrm{Tr}\ln(i\hat{D} - \sigma) = -i\ln\mathrm{Det}(i\hat{D} - \sigma), \tag{48}$$

where $\hat{D} \equiv \gamma^\mu D_\mu$. Since

$$\mathrm{Det}(i\hat{D} - \sigma) = \mathrm{Det}[\gamma^5(i\hat{D} - \sigma)\gamma^5] = \mathrm{Det}(-i\hat{D} - \sigma), \tag{49}$$

we find that

$$\tilde{\Gamma}(\sigma) = -\frac{i}{2}\mathrm{Tr}[\ln(i\hat{D} - \sigma) + \ln(-i\hat{D} - \sigma)] \\ = -\frac{i}{2}\mathrm{Tr}\ln(\hat{D}^2 + \sigma^2). \tag{50}$$

Therefore, $\tilde{\Gamma}(\sigma)$ can be expressed through the following integral over the proper time s:

$$\tilde{\Gamma}(\sigma) = -\frac{i}{2}\mathrm{Tr}\ln(\hat{D}^2 + \sigma^2) \\ = \frac{i}{2}\int d^3x \int_0^\infty \frac{ds}{s}\mathrm{tr}\langle x|e^{-is(\hat{D}^2+\sigma^2)}|x\rangle, \tag{51}$$

where

$$\hat{D}^2 = D_\mu D^\mu - \frac{ie}{2}\gamma^\mu\gamma^\nu F^{\mathrm{ext}}_{\mu\nu} \\ = D_\mu D^\mu + ie\gamma^1\gamma^2 B. \tag{52}$$

The matrix element $\langle x|e^{-is(\hat{D}^2+\sigma^2)}|y\rangle$ can be calculated by using the Schwinger approach [17]. It is

$$\langle x|e^{-is(\hat{D}^2+\sigma^2)}|y\rangle = e^{-is\sigma^2}\langle x|e^{-isD_\mu D^\mu}|y\rangle \\ \times[\cos(eBs) + \gamma^1\gamma^2\sin(eBs)] \\ = \frac{e^{-i\frac{\pi}{4}}}{8(\pi s)^{3/2}}e^{-i(s\sigma^2 - S_{\mathrm{cl}})} \\ \times[eBs\cot(eBs) + \gamma^1\gamma^2 eBs], \tag{53}$$

where

$$S_{\mathrm{cl}} = e\int_y^x A^{\mathrm{ext}}_\lambda dz^\lambda - \frac{1}{4s}(x-y)_\nu\Bigg(g^{\nu\mu} + \frac{(F^2_{\mathrm{ext}})^{\nu\mu}}{B^2} \\ \times[1 - eBs\cot(eBs)]\Bigg)(x-y)_\mu. \tag{54}$$

Here the integral $\int_y^x A^{\mathrm{ext}}_\lambda dz^\lambda$ is taken along the straight line.

Substituting Eq. (53) in Eq. (51), we find

$$\tilde{\Gamma}(\sigma) = \frac{iNe^{-i\frac{\pi}{4}}}{4\pi^{3/2}}\int d^3x\int_0^\infty \frac{ds}{s^{5/2}}e^{-is\sigma^2}eBs\cot(eBs). \tag{55}$$

Therefore the effective potential is

$$V(\rho) = \frac{\rho^2}{2G} + \tilde{V}(\rho) \\ = \frac{\rho^2}{2G} + \frac{N}{4\pi^{3/2}}\int_{1/\Lambda^2}^\infty \frac{ds}{s^{5/2}}e^{-s\rho^2}eBs\coth(eBs), \tag{56}$$

where $\rho^2 = \sigma^2+\tau^2+\pi^2$, and now we introduced explicitly the ultraviolet cutoff Λ.

By using the integral representation for the generalized Riemann zeta function ζ [18],

$$\int_0^\infty ds s^{\mu-1}e^{-\beta s}\coth s = \Gamma(\mu)\left[2^{1-\mu}\zeta\left(\mu, \frac{\beta}{2}\right) - \beta^{-\mu}\right], \tag{57}$$

which is valid at $\mu > 1$, and analytically continuing this representation to $\mu = -\frac{1}{2}$, we can rewrite Eq. (56) as

$$V(\rho) = \frac{N}{\pi}\Bigg[\frac{\Lambda}{2\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{g} - 1\right)\rho^2 \\ -\frac{\sqrt{2}}{l^3}\zeta\left(-\frac{1}{2}, \frac{(\rho l)^2}{2} + 1\right) - \frac{\rho}{2l^2}\Bigg] + O(1/\Lambda), \tag{58}$$

where the magnetic length l is $l = |eB|^{-1/2}$ and here we introduce the dimensionless coupling constant

$$g \equiv N\frac{\Lambda}{\pi}G. \tag{59}$$

We recall that $\zeta(\mu, \frac{\beta}{2})$ is defined as

$$\zeta\left(\mu, \frac{\beta}{2}\right) = \sum_{n=0}^\infty \frac{1}{(n + \frac{\beta}{2})^\mu} \tag{60}$$

at $\mu > 1$ [18].

Let us now analyze the gap equation $dV/d\rho = 0$. It is

$$\frac{\Lambda\rho}{\pi g} = \frac{\rho}{2\pi^{3/2}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{3/2}} e^{-s\rho^2} eBs \coth(eBs), \qquad (61)$$

which can be rewritten as

$$2\Lambda l \left(\frac{1}{g} - \frac{1}{\sqrt{\pi}}\right) \rho = \frac{1}{l} + \sqrt{2}\rho\zeta\left(\frac{1}{2}, 1 + \frac{\rho^2 l^2}{2}\right) + O(1/\Lambda). \qquad (62)$$

As $B \to 0$, we recover the known gap equation [14]:

$$\rho^2 = \rho\Lambda\left(\frac{1}{\sqrt{\pi}} - \frac{1}{g}\right). \qquad (63)$$

It admits a nontrivial solution only if the coupling constant g is supercritical, $g > g_c = \sqrt{\pi}$ [as Eq. (41) implies, a solution to the gap equation, $\sigma = \bar{\sigma}$, coincides with the fermion dynamical mass, $\bar{\sigma} = m_{\rm dyn}$]. We will show that the magnetic field changes the situation dramatically: at $B \neq 0$, a nontrivial solution exists at all $g > 0$. The reason for this is that the magnetic field enhances the interaction in the infrared region (large s): at $B \neq 0$, the integral in Eq. (61) becomes proportional to $1/\rho$ as $\rho \to 0$.

We shall first consider the case of subcritical g, $g < g_c = \sqrt{\pi}$, which in turn can be divided into two subcases: (a) $g \ll g_c$ and (b) $g \to g_c - 0$ (near critical g). Assuming that $|\bar{\sigma} l| \ll 1$ at $g \ll g_c$, we find, from Eq. (62),

$$m_{\rm dyn} \equiv \bar{\sigma} \simeq \frac{|eB| g\sqrt{\pi}}{2\Lambda(g_c - g)}. \qquad (64)$$

Since Eq. (64) implies that the condition $|\bar{\sigma} l| \ll 1$ fulfills at all g satisfying $(g_c - g) \gg \frac{|eB|^{1/2}}{\Lambda}$, the relation (64) is actually valid in that whole region.

Note the following interesting point. Equation (42) implies that $m_{\rm dyn} = \langle 0|\sigma|0\rangle = -\pi g/N\Lambda\langle 0|\bar{\Psi}\Psi|0\rangle$. From here and Eq. (64) we find that the condensate $\langle 0|\bar{\Psi}\Psi|0\rangle$ is $\langle 0|\bar{\Psi}\Psi|0\rangle = -N|eB|/2\pi$ in leading order in g; *i.e.*, it coincides with the value of the condensate calculated in the problem of free fermions in a magnetic field [see Eq. (22)]. This point implies that at small g (weakly interacting fermions) the $|\theta_1, \theta_2\rangle$ vacua constructed in Sec. III are good trial states for the vacua of the problem with interacting fermions. This point also explains why the dynamical mass $m_{\rm dyn}$ in this problem is an analytic function of g at $g = 0$: indeed, the condensate already exists at $g = 0$.

At $g_c - g \lesssim \sqrt{|eB|}/\Lambda$, introducing the scale $m^* = \Lambda(1/g - 1/g_c)$, we get the equation

$$2m^* l = \frac{1}{|\sigma| l} + \sqrt{2}\zeta\left(\frac{1}{2}, \frac{(\sigma l)^2}{2} + 1\right) \qquad (65)$$

which implies that in the near critical region, $m_{\rm dyn}$ is

$$m_{\rm dyn} = \bar{\sigma} \sim |eB|^{1/2}. \qquad (66)$$

Thus in the scaling region, with $g_c - g \lesssim \sqrt{|eB|}/\Lambda$, the cutoff disappears from the observable $m_{\rm dyn}$. This agrees with the well-known fact that the critical value $g_c = \sqrt{\pi}$ is an ultraviolet stable fixed point at leading order in $1/N$ [14]. The relation (66) can be considered as a scaling law in the scaling region.

In the supercritical region, at $g > g_c$, the analytic expression for $m_{\rm dyn}$ can be obtained at weak $|eB|$, satisfying the condition $\sqrt{|eB|}/m_{\rm dyn}^{(0)} \ll 1$, where $m_{\rm dyn}^{(0)}$ is the solution of the gap equation (63) with $B = 0$. Then, using the asymptotic formula [18]

$$\zeta(z, q) \underset{q \to \infty}{\simeq} \frac{1}{(z-1)q^{z-1}}\left[1 + \frac{z-1}{2q} + \cdots\right], \qquad (67)$$

we find, from Eq. (62),

$$m_{\rm dyn} = \bar{\sigma} = m_{\rm dyn}^{(0)}\left[1 + \frac{(eB)^2}{12(m_{\rm dyn}^{(0)})^4}\right]; \qquad (68)$$

i.e., $m_{\rm dyn}$ increases with B. The numerical study of Eq. (62) shows that $m_{\rm dyn}$ increases with B at all values of g and B.

A striking fact is that, unlike the gap equation (63) with $B = 0$, the gap equation with $B \neq 0$ does not have the trivial solution $\sigma = 0$. Indeed, Eq. (56) implies that $dV/d\sigma|_{\sigma=0} = d\tilde{V}/d\sigma|_{\sigma=0}$, and then we find from Eqs. (41) and (44) that

$$\frac{d\tilde{V}}{d\sigma}|_{\sigma=0} = \langle 0|\bar{\Psi}\Psi|0\rangle|_{g=0} = -N\frac{|eB|}{2\pi} \neq 0 \qquad (69)$$

[see Eq. (22)]. Thus, despite the spontaneous character of the U(2) symmetry breakdown, there is no trivial solution (stable or unstable) in the magnetic field at all values of g.

VI. THE NJL MODEL IN A MAGNETIC FIELD: THE KINETIC TERM IN THE EFFECTIVE ACTION

Let us now consider the kinetic term $\mathcal{L}_k$ in the effective action (43).

The U(2) symmetry implies that the general form of $\mathcal{L}_k$ is

$$\mathcal{L}_k = N\frac{F_1^{\mu\nu}}{2}(\partial_\mu \rho_j \partial_\nu \rho_j) + N\frac{F_2^{\mu\nu}}{\rho^2}(\rho_j \partial_\mu \rho_j)(\rho_i \partial_\nu \rho_i), \qquad (70)$$

where $\boldsymbol{\rho} = (\sigma, \tau, \pi)$ and $F_1^{\mu\nu}$, $F_2^{\mu\nu}$ are functions of $\rho^2 = \sigma^2 + \tau^2 + \pi^2$. To find the functions $F_1^{\mu\nu}$, and $F_2^{\mu\nu}$, one can use different methods. We used the method of Ref. [28]. The derivation of $\mathcal{L}_k$ is considered in Appendix A. Here we shall present the final results.

The functions $F_1^{\mu\nu}$ and $F_2^{\mu\nu}$ take the form $F_1^{\mu\nu} = g^{\mu\nu} F_1^{\mu\mu}$, $F_2^{\mu\nu} = g^{\mu\nu} F_2^{\mu\mu}$ where

$$F_1^{00} = \frac{l}{8\pi}\left(\frac{1}{\sqrt{2}}\zeta\Big(\frac{3}{2},\frac{(\rho l)^2}{2}+1\Big)+(\rho l)^{-3}\right),$$
$$F_1^{11} = F_1^{22} = \frac{1}{4\pi\rho},$$
$$F_2^{00} = -\frac{l}{16\pi}\left(\frac{(\rho l)^2}{2\sqrt{2}}\zeta\Big(\frac{5}{2},\frac{(\rho l)^2}{2}+1\Big)+(\rho l)^{-3}\right),$$
$$F_2^{11} = F_2^{22} = \frac{l}{8\pi}\left[\frac{(\rho l)^4}{\sqrt{2}}\zeta\Big(\frac{3}{2},\frac{(\rho l)^2}{2}+1\Big) + \sqrt{2}(\rho l)^2\zeta\Big(\frac{1}{2},\frac{(\rho l)^2}{2}+1\Big)+2\rho l-(\rho l)^{-1}\right] \quad (71)$$

(we recall that the magnetic length $l \equiv |eB|^{-1/2}$).

We would like to emphasize that, as follows from Eq. (71), the propagator of the NG modes in leading order in $1/N$ has a genuine $(2+1)$-dimensional form. We shall see in Appendix C that this fact is crucial for providing the reliability of the $1/N$ expansion in this problem (physical reasons for the $(2+1)$-dimensional character of the dynamics of the neutral NG bosons are considered in the next section).

Now, knowing the effective potential and the kinetic term, we can define the energy spectrum (dispersion law) of the collective excitations σ and τ,π.

VII. THE NJL MODEL IN A MAGNETIC FIELD: THE SPECTRUM OF THE COLLECTIVE EXCITATIONS

We begin by considering the spectrum of the collective excitations in the subcritical, $g < g_c$, region.

At $g_c - g \gg \sqrt{|eB|}/\Lambda$ [where $|\bar{\sigma}l| \ll 1$, see Eq. (64)], we find from Eqs. (70) and (71) the dispersion law for the τ and π NG (gapless) modes:

$$E_{\tau,\pi} \simeq \sqrt{2}(\bar{\sigma}l)(\boldsymbol{k}^2)^{1/2} = \frac{gg_c|eB|^{1/2}}{\sqrt{2}\Lambda(g_c-g)}(\boldsymbol{k}^2)^{1/2} \quad (72)$$

[see Eq. (64)]. As the interaction is switched off, $g \to 0$, their velocity, $v = gg_c(g_c-g)^{-1}\sqrt{|eB|/2\Lambda^2}$, becomes zero, and we return to the dynamics with spontaneous flavor symmetry breaking but without genuine NG modes discussed in Sec. III.

In order to define the "mass" (energy gap) M_σ of the σ mode, we note that

$$\left.\frac{d^2V}{d\sigma^2}\right|_{\sigma=\bar{\sigma}} = N\frac{\bar{\sigma}^2 l}{\pi^{3/2}}\int_0^\infty ds\sqrt{s}\exp[-(\bar{\sigma}l)^2 s]\coth s = N\frac{\bar{\sigma}^2 l}{2\pi}\left[\frac{1}{\sqrt{2}}\zeta\Big(\frac{3}{2},\frac{(\bar{\sigma}l)^2}{2}+1\Big)+(\bar{\sigma}l)^{-3}\right] \quad (73)$$

[see Eq. (56)]. Then we find from Eqs. (70), (71), and (73) that

$$M_\sigma^2 \simeq \frac{8\sqrt{2}(g_c-g)}{gg_c\zeta(\frac{3}{2})}\Lambda|eB|^{1/2} \quad (74)$$

at $g_c - g \gg \sqrt{|eB|}/\Lambda$. As $g \to 0$, the σ mode decouples ($M_\sigma \to \infty$).

Thus the dynamics in the problem of a relativistic fermion in an external magnetic field emerges from this model in the limit when the interaction between fermions is switched off. The attractive ($g > 0$) interaction "resurrects" the NG modes and they acquire a velocity $v \sim g$.

Let us now consider the near critical region with $g_c - g \lesssim \sqrt{|eB|}/\Lambda$. From Eqs. (70) and (71), we find that

$$E_{\tau,\pi} = f(\bar{\sigma}l)(\boldsymbol{k}^2)^{1/2}, \quad (75)$$

where

$$f(\bar{\sigma}l) = \left(\frac{2}{\bar{\sigma}l}\right)^{1/2}\left(\frac{1}{\sqrt{2}}\zeta\Big(\frac{3}{2},\frac{(\bar{\sigma}l)^2}{2}+1\Big)+(\bar{\sigma}l)^{-3}\right)^{-1/2}. \quad (76)$$

Since in this near critical (scaling) region the parameter $\bar{\sigma}$ is $\bar{\sigma} \sim |eB|^{1/2} = l^{-1}$, we conclude that the cutoff Λ disappears from the observables E_τ and E_π in the scaling region.

In the same way, we find from Eqs. (70), (71), and (73) that

$$M_\sigma^2 \sim |eB| \quad (77)$$

in the scaling region.

Let us turn to the supercritical region with $g > g_c$. The analytic expressions for $E_{\tau,\pi}$ and M_σ^2 can be obtained for small $|eB| = l^{-2}$, satisfying the condition $|\bar{\sigma}l| \gg 1$. Then, using the asymptotic formula (67) for zeta functions, we find, from Eqs. (70), (71), and (73),

$$E_{\tau,\pi} = \left(1-\frac{1}{8(\bar{\sigma}l)^4}\right)(\boldsymbol{k}^2)^{1/2}, \quad (78)$$

$$M_\sigma^2 = 6\bar{\sigma}^2\left(1-\frac{3}{4}\frac{1}{(\bar{\sigma}l)^2}\right), \quad (79)$$

where $\bar{\sigma}$ is given in Eq. (68). These relations show that the magnetic field leads to decreasing both the velocity of the NG modes (it becomes less than 1) and the mass (energy gap) of the σ mode.

Let us indicate the following interesting point intimately connected with the $(2+1)$-dimensional character of the dynamics of the neutral NG modes. The $(2+1)$-dimensional character is reflected in that the velocity $\mathbf{v}_{\pi,\tau} = \partial E_{\pi,\tau}/\partial\mathbf{k}$ is not zero. As follows from Eqs. (72), (75), and (78), the velocity $\mathbf{v}_{\pi,\tau}$ decreases with $m_{\text{dyn}} = \bar{\sigma}$ and becomes zero [i.e., the dynamics becomes $0+1$ dimensional] when $m_{\text{dyn}} \to 0$, i.e., when the interaction is switched off ($g \to 0$). The reason for this is clear: since at $g = 0$ the energy of the neutral system made up of a fermion and an antifermion from the lowest Landau level is identically zero, its velocity is also zero. This fact in turn reflects the point that the motion of charged fermions in the x_1-x_2 plane is restricted by a magnetic field. On the other hand, at $g > 0$, there are genuine neutral NG bound states (with the bound energy $\Delta E_{\pi,\tau} \equiv 2m_{\rm dyn} - E_{\pi,\tau}|_{k=0} = 2m_{\rm dyn}$). Since the motion of the center of the mass of *neutral* bound states is not restricted by a magnetic field, their dynamics is 2 + 1 dimensional.

Let us now discuss the continuum limit $\Lambda \to \infty$ in more detail. As is known, at $B = 0$, in this model, an interacting continuum theory appears only at the critical value $g = g_c = \sqrt{\pi}$ (the continuum theory is trivial at $g < g_c$) [4,14–16]. Therefore, since at $g < g_c$, in the continuum limit, there is no attractive interaction between fermions, it is not surprising that at $g < g_c$, the dynamical mass $m_{\rm dyn} \sim g|eB|/\Lambda$ disappears as $\Lambda \to \infty$.

At $B = 0$, the continuum theory is in the symmetric phase at $g \to g_c - 0$ and in the broken phase at $g \to g_c + 0$. On the other hand, as follows from our analysis, in a magnetic field, it is in the broken phase both at $g \to g_c - 0$ and $g \to g_c + 0$ (though the dispersion relations for fermions and collective excitations ρ are different at $g \to g_c - 0$ and at $g \to g_c + 0$).

Up to now we have considered four-component fermions. In the case of two-component fermions, the effective potential V_2 is $V_2(\sigma) = V(\sigma)/2$, where $V(\sigma)$ is defined in Eqs. (56) and (58). However, the essential new point is that there is no continuous [U(2)] symmetry (and therefore NG modes) in this case. As in the case of four-component fermions, in an external magnetic field, the dynamical fermion mass (now breaking parity) is generated at any positive value of the coupling constant g.

The NJL model illustrates the general phenomenon in 2 + 1 dimensions: in the infrared region, a magnetic field reduces the dynamics of fermion pairing to one-dimensional dynamics (at the lowest Landau level), thus catalyzing the generation of a dynamical mass for fermions. A concrete sample of dynamical symmetry breaking is of course different in different models.

VIII. THERMODYNAMIC PROPERTIES OF THE NJL MODEL IN A MAGNETIC FIELD

In this section, we will study the thermodynamic properties of the NJL model in a magnetic field. In particular, we will show that there is a symmetry restoring phase transition at high temperature.

Our goal is to determine the thermodynamic (effective) potential in the NJL model in a magnetic field. Although we are mostly interested in studying the system at finite temperature T and zero chemical potential μ (i.e., at equal densities of fermions and antifermions), we shall derive the effective potential $V_{\beta,\mu}(\sigma)$ (at the leading order in $1/N$) at arbitrary values of $\beta = 1/T$ and μ.

Since in the leading order in $1/N$, the effective potential $V(\sigma) \equiv V_{\beta,\mu}(\sigma)|_{\beta=\infty,\ \mu=0}$ is given by a sum of one-(fermion) loop diagrams, the thermodynamic potential is, in this approximation,

$$\begin{aligned} V_{\beta,\mu}(\sigma) &= V(\sigma) + \tilde{V}_{\beta,\mu}(\sigma) \\ &= \frac{N}{\pi}\left[\frac{\Lambda}{2\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{g}-1\right)\sigma^2 - \frac{\sqrt{2}}{l^3}\zeta\Big(-\frac{1}{2},\frac{(\sigma l)^2}{2}+1\Big) - \frac{\sigma}{2l^2}\right] \\ &\quad -N\frac{|eB|}{2\pi\beta}\left\{\ln(1+e^{-\beta(\sigma-\mu)}) + 2\sum_{k=1}^{\infty}\ln\left(1+e^{-\beta\left(\sqrt{\sigma^2+\frac{2k}{l^2}}-\mu\right)}\right) + (\mu\to-\mu)\right\} \end{aligned} \tag{80}$$

[see Eq. (58)]. Here the sum is taken over all the fermion and antifermion (with $\mu \to -\mu$) Landau levels; the factor $|eB|/2\pi$ describes the degeneracy of each level. The relation (80) is derived (in the framework of the imaginary time formalism [29]) in Appendix B. We also show there that it can be rewritten as

$$V_{\beta,\mu}(\sigma) = \frac{\sigma^2}{2G} + \frac{N}{4\pi^{3/2}l^3}\int_0^\infty \frac{dt}{t^{3/2}}e^{-(tl^2\sigma^2)}\coth t \times\Theta_4\left[\frac{i}{2}\mu\beta|\frac{i}{4\pi t}\left(\frac{\beta}{l}\right)^2\right], \tag{81}$$

where Θ_4 is the fourth Jacobian theta function [18].

Henceforth we will consider the case of zero chemical potential corresponding to equal densities of fermions and antifermions in the system. In this case, the thermodynamic potential $V_\beta \equiv V_{\beta,\mu}|_{\mu=0}$ is

$$\begin{aligned} V_\beta(\sigma) = \frac{N}{\pi}\Bigg[&\frac{\Lambda}{2\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{g}-1\right)\sigma^2 \\ &-\frac{\sqrt{2}}{l^3}\zeta\Big(-\frac{1}{2},\frac{(\sigma l)^2}{2}+1\Big) - \frac{\sigma}{2l^2}\Bigg] \\ &-N\frac{|eB|}{\pi\beta}\Bigg[\ln(1+e^{-\beta\sigma}) \\ &+2\sum_{k=1}^{\infty}\ln\left(1+e^{-\beta\sqrt{\sigma^2+\frac{2k}{l^2}}}\right)\Bigg]. \end{aligned} \tag{82}$$

We solved numerically the gap equation

$$\frac{dV_\beta(\sigma)}{d\sigma} = 0. \tag{83}$$

The main result is that at $T = T_c \sim m_{\rm dyn}$ (actually at $T = T_c \simeq m_{\rm dyn/2}$), there is a symmetry restoring (second order) phase transition (see Figs. 1 and 2). The phase diagram in the $B - T$ plane is shown in Figs. 3 and 4.

We recall that there cannot be spontaneous breakdown of a continuous symmetry at finite ($T > 0$) temperature in 2 + 1 dimensions [the Mermin-Wagner-Coleman (MWC) theorem [26]]. This happens because at nonzero temperature the dynamics of a zero mode in (2 + 1)-dimensional field theories is two dimensional. As a result, strong fluctuations of would-be NG modes lead to vanishing order parameter connected with a spontaneous breakdown of a continuous symmetry. In the NJL model with a finite temperature (both at $B = 0$ and in a magnetic field), the MWC theorem manifests itself only beyond the leading order in $1/N$. One plausible possibility of what happens at $T \neq 0$ beyond the leading order in $1/N$ is the following. The dynamics of the zero mode in this model is essentially equivalent to that of the SU(2) σ model in two-dimensional Euclidean space. As is known, the SU(2) symmetry is exact in the latter model and, as a result, the would-be NG bosons become massive excitations [30]. Therefore it seems plausible that in the (2+1)-dimensional NJL model in a magnetic field, the SU(2) symmetry will be restored at any finite temperature, and the dynamically generated mass $m_{\rm dyn}$ of fermions will disappear.

The question whether this, or another, scenario is realized at finite temperature in this model deserves further study.

IX. CONCLUSION

The main result of this paper is that a magnetic field is a strong catalyst, generating a fermion mass (energy

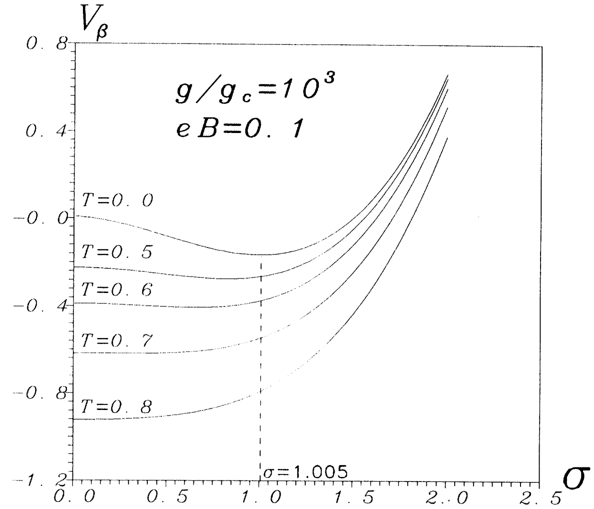

FIG. 2. The thermodynamic potential V_β as a function of σ at different temperatures at supercritical g: $g/g_c = 10^3$. All quantities are measured in $\mu \equiv \Lambda/g_c$ units: $V_\beta \to \pi V_\beta/\mu^3$, $eB \to eB/\mu^2$, $\sigma \to \sigma/\mu$.

gap), in $2+1$ dimensions. It would be worth considering the present effect in (2 + 1)-dimensional effective theories describing high temperature superconductivity and the quantum Hall effect where a magnetic field is an important ingredient of the dynamics. In connection with this, we note that in some models of high temperature superconductivity of Ref. [2], the energy gap in the electron spectrum results from electron hole (i.e., fermion-antifermion rather than fermion-fermion) pairing. Also, using the four-component spinors in these models reflects the presence of two sublattices in high temperature su-

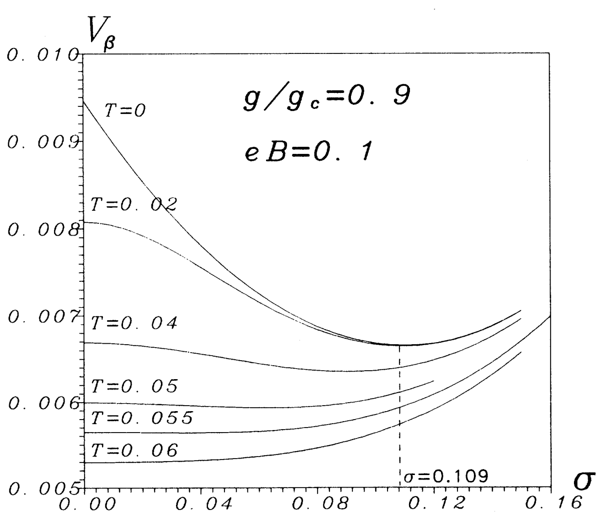

FIG. 1. The thermodynamic potential V_β as a function of σ at different temperatures at subcritical g: $g/g_c = 0.9$. All quantities are measured in $\mu \equiv \Lambda/g_c$ units: $V_\beta \to \pi V_\beta/\mu^3$, $eB \to eB/\mu^2$, $\sigma \to \sigma/\mu$.

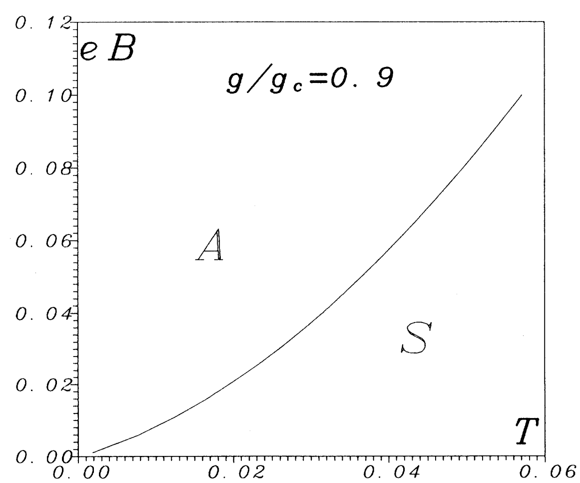

FIG. 3. The critical line in the $eB - T$ plane separating the asymmetric (A) and symmetric (S) phases at subcritical g: $g/g_c = 0.9$. All quantities are measured in $\mu \equiv \Lambda/g_c$ units: $eB \to eB/\mu^2$, $T \to T/\mu$.

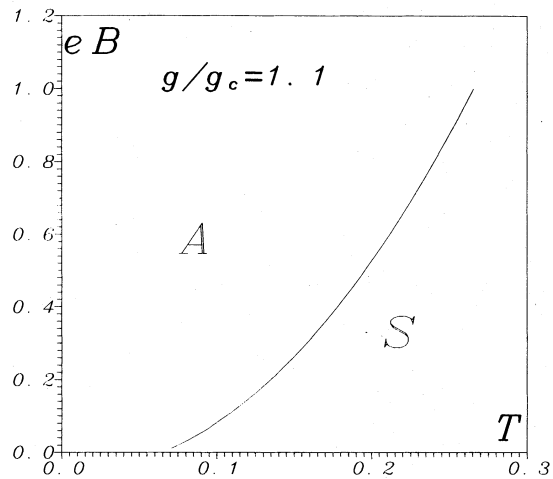

FIG. 4. The critical line in the $eB-T$ plane separating the asymmetric (A) and symmetric (S) phases at supercritical g: $g/g_c = 1.1$. All quantities are measured in $\mu = \Lambda/g_c$ units: $eB \to eB/\mu^2$, $T \to T/\mu$.

perconductors. Another, potentially interesting, application of the present effect may be in $(3+1)$-dimensional field theories at high temperature. Since at high temperature, their dynamics effectively reduces to that of $(2+1)$-dimensional theories, it might happen that in a magnetic field, at high temperature, fermions (quarks in quark-gluon plasma, for example) acquire a dynamical mass and NG excitations appear.

It would be interesting to check the realization of this effect in $(2+1)$-dimensional theories in lattice computer simulations. Note that the recent computer simulations of the $(2+1)$-dimensional NJL model [16] show that the $1/N$ expansion is quite reliable, at least at $N \geq 12$.

The essence of the present effect is that in a constant magnetic field, the dynamics of fermion pairing is one dimensional: the pairing takes place essentially for fermions at the (degenerate) lowest Landau level. This implies the universal character of this effect in $2+1$ dimensions.

In this paper, we considered the dynamics in the presence of a constant magnetic field only. It would be interesting to extend this analysis to the case of inhomogeneous electromagnetic fields in $2+1$ dimensions. In connection with this we note that the present effect is intimately connected with the fact that in $2+1$ dimensions, the massless Dirac equation in a constant magnetic field admits an infinite number of normalized solutions with $E = 0$ (zero modes); more precisely, the density of such solutions is finite. One may expect that the same effect will take place for any electromagnetic field configuration in which the density of zero modes is finite. As we have known recently, the program of the derivation of a low energy effective action in $(2+1)$-dimensional QED in external electromagnetic fields has been developed in Ref. [31].

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APPENDIX A

In this appendix, we derive the expressions for the fermion propagator and for the kinetic term $\mathcal{L}_k$ in the effective action.

In the coordinate space, the fermion propagator is

$$S(x,y) = (i\hat{D}+m)_x \left\langle x \left| \frac{-i}{m^2+\hat{D}^2} \right| y \right\rangle$$
$$= (i\hat{D}+m)_x \int_0^\infty ds \langle x|\exp[-is(m^2+\hat{D}^2)]|y\rangle, \tag{A1}$$

where $\hat{D} \equiv \gamma^\mu D_\mu$ and D_μ is the covariant derivative in Eq. (2).

The matrix element $\langle x|e^{-is(m^2+\hat{D}^2)}|y\rangle$ can be calculated by using the Schwinger (proper time) approach [17]. It is

$$\langle x|e^{-is(m^2+\hat{D}^2)}|y\rangle = \frac{e^{-i\frac{\pi}{4}}}{8(\pi s)^{3/2}} e^{i[S_{\rm cl}-sm^2]} \times \Big(eBs\cot(eBs) + \gamma^1\gamma^2 eBs\Big), \tag{A2}$$

where

$$S_{\rm cl} = e\int_y^x A_\lambda^{\rm ext} dz^\lambda - \frac{1}{4s}(x-y)_\nu \left[g^{\nu\mu} + \frac{((F^{\rm ext})^2)^{\mu\nu}}{B^2} \times \Big(1 - eBs\cot(eBs)\Big)\right](x-y)_\mu. \tag{A3}$$

Here the integral is calculated along the straight line.

From Eqs. (A1) and (A2), we find the expression (18) for the fermion propagator.

Let us now consider the derivation of the kinetic term (70) in the low-energy effective action:

$$\mathcal{L}_k = N\frac{F_1^{\mu\nu}}{2}(\partial_\mu\rho_j\partial_\nu\rho_j) + N\frac{F_2^{\mu\nu}}{\rho^2}(\rho_j\partial_\mu\rho_j)(\rho_i\partial_\nu\rho_i), \tag{A4}$$

where $\boldsymbol{\rho} = (\sigma,\tau,\pi)$ and $F_1^{\mu\nu}$, $F_2^{\mu\nu}$ depend on the U(2)-invariant $\rho^2 = \sigma^2+\tau^2+\pi^2$. The definition $\Gamma_k = \int d^3x \mathcal{L}_k$

and Eq. (A4) imply that the form of the functions $F_1^{\mu\nu}$, $F_2^{\mu\nu}$ is determined from the equations:

$$N^{-1}\frac{\delta^2\Gamma_k}{\delta\sigma(x)\delta\sigma(0)}\bigg|_{\substack{\sigma=\text{const}\\ \tau=\pi=0}} = -(F_1^{\mu\nu}+2F_2^{\mu\nu})\bigg|_{\substack{\sigma=\text{const}\\ \tau=\pi=0}}\partial_\mu\partial_\nu\delta^3(x), \quad \text{(A5)}$$

$$N^{-1}\frac{\delta^2\Gamma_k}{\delta\pi(x)\delta\pi(0)}\bigg|_{\substack{\sigma=\text{const}\\ \tau=\pi=0}} = -F_1^{\mu\nu}\bigg|_{\substack{\sigma=\text{const}\\ \tau=\pi=0}}\cdot\partial_\mu\partial_\nu\delta^3(x). \quad \text{(A6)}$$

Here Γ_k is the part of the effective action (43) containing terms with two derivatives. Equation (43) implies that $\Gamma_k=\tilde\Gamma_k$. Therefore we find from Eq. (A6) that

$$\begin{aligned} F_1^{\mu\nu} &= -\frac{N^{-1}}{2}\int d^3x x^\mu x^\nu\frac{\delta^2\tilde\Gamma_k}{\delta\pi(x)\delta\pi(0)}\\ &= -\frac{N^{-1}}{2}\int d^3x x^\mu x^\nu\frac{\delta^2\tilde\Gamma}{\delta\pi(x)\delta\pi(0)} \end{aligned} \quad \text{(A7)}$$

(henceforth we shall not write explicitly the condition $\tau=\pi=0, \sigma=\text{const}$). Taking into account the definition of the fermion propagator,

$$iS^{-1}=i\hat D-\sigma, \quad \text{(A8)}$$

we find from Eq. (45) that

$$\begin{aligned}\frac{\delta^2\tilde\Gamma}{\delta\pi(x)\delta\pi(0)} &= -i\text{tr}\Big(S(x,0)i\gamma^5S(0,x)i\gamma^5\Big)\\ &= -i\text{tr}\Big(\tilde S(x,0)i\gamma^5\tilde S(0,x)i\gamma^5\Big)\\ &= -i\int\frac{d^3kd^3q}{(2\pi)^6}e^{iqx}\text{tr}\Big(\tilde S(k)i\gamma^5\tilde S(k+q)i\gamma^5\Big)\end{aligned} \quad \text{(A9)}$$

[the functions $\tilde S(x)$ and $\tilde S(k)$ are given in Eqs. (18)–(20)]. Therefore,

$$F_1^{\mu\nu}=-\frac{iN^{-1}}{2}\int\frac{d^3k}{(2\pi)^3}\text{tr}\left(\tilde S(k)i\gamma^5\frac{\partial^2\tilde S(k)}{\partial k_\mu\partial k_\nu}i\gamma^5\right). \quad \text{(A10)}$$

In the same way, we find that

$$\begin{aligned}F_2^{\mu\nu} &= -\frac{iN^{-1}}{4}\int\frac{d^3k}{(2\pi)^3}\text{tr}\Big(\tilde S(k)\frac{\partial^2\tilde S(k)}{\partial k_\mu\partial k_\nu}\Big)-\frac{1}{2}F_1^{\mu\nu}\\ &= -\frac{iN^{-1}}{4}\int\frac{d^3k}{(2\pi)^3}\text{tr}\left(\tilde S(k)\frac{\partial^2\tilde S(k)}{\partial k_\mu\partial k_\nu}\right.\\ &\qquad\left.-\tilde S(k)i\gamma^5\frac{\partial^2\tilde S(k)}{\partial k_\mu\partial k_\nu}i\gamma^5\right).\end{aligned} \quad \text{(A11)}$$

Taking into account the expression for $\tilde S(k)$ in Eq. (20) (with $m=\sigma$), we get

$$\begin{aligned}\frac{\partial^2\tilde S(k)}{\partial k^0\partial k^0} &= 2il^4\int_0^\infty dtt\exp\left((R(t))\{\sigma(1+\eta\gamma^1\gamma^2T)+3k^0\gamma^0(1+\eta\gamma^1\gamma^2T)-k^i\gamma^i(1+T^2)\right.\\ &\quad+2itl^2(k^0)^2\sigma(1+\eta\gamma^1\gamma^2T)+2itl^2(k^0)^3\gamma^0(1+\eta\gamma^1\gamma^2T)-2itl^2(k^0)^2(k^i\gamma^i)(1+T^2)\},\end{aligned} \quad \text{(A12)}$$

$$\begin{aligned}\frac{\partial^2\tilde S(k)}{\partial k^j\partial k^j} &= -2il^4\int_0^\infty dtT\exp\left(R(t)\right)\{\sigma(1+\eta\gamma^1\gamma^2T)-k^i\gamma^i(1+T^2)-2k^j\gamma^j(1+T^2)\\ &\quad+k^0\gamma^0(1+\eta\gamma^1\gamma^2T)-2iTl^2(k^j)^2\sigma(1+\eta\gamma^1\gamma^2T)\\ &\quad-2iTl^2(k^j)^2k^0\gamma^0(1+\eta\gamma^1\gamma^2T)+2iTl^2(k^j)^2k^i\gamma^i(1+T^2)\}\end{aligned} \quad \text{(A13)}$$

($i,j=1,2$; there is no summation over j), where

$$\begin{aligned}&\eta=\text{sgn}(eB), \quad T=\tan t,\\ &R(t)=-it(\sigma l)^2+it(k^0)^2-il^2\mathbf{k}^2T\ .\end{aligned} \quad \text{(A14)}$$

Equations (20), (A10), and (A11) imply that nondiagonal terms $F_1^{\mu\nu}$ and $F_2^{\mu\nu}$ are equal to zero. The diagonal terms are determined from Eqs. (20), (A10)–(A13), after rather long, although routine, calculations:

$$\begin{aligned}F_1^{00} &= \frac{l}{12\pi^{3/2}}\int_0^\infty d\tau\frac{\sqrt\tau}{\sinh\tau}e^{-(\sigma l)^2\tau}\Big[(\sigma l)^2\tau\cosh\tau\\ &\qquad+\frac{3}{2}\cosh\tau+\frac{\tau}{\sinh\tau}\Big]\\ &= \frac{l}{8\pi}\left(\frac{1}{\sqrt2}\zeta\Big(\frac{3}{2},\frac{(\sigma l)^2}{2}+1\Big)+(\sigma l)^{-3}\right),\end{aligned} \quad \text{(A15)}$$

$$F_2^{00} = -\frac{l(\sigma l)^2}{12\pi^{3/2}}\int_0^\infty d\tau \tau^{3/2} e^{-(\sigma l)^2\tau}\coth\tau$$
$$= -\frac{l}{16\pi}\left(\frac{(\sigma l)^2}{2\sqrt{2}}\zeta\Big(\frac{5}{2},\frac{(\sigma l)^2}{2}+1\Big)+(\sigma l)^{-3}\right), \quad \text{(A16)}$$

$$F_1^{11} = F_1^{22} = \frac{1}{4\pi\sigma}, \quad \text{(A17)}$$

$$F_2^{11} = F_2^{22}$$
$$= \frac{l(\sigma l)^2}{4\pi^{3/2}}\int_0^\infty d\tau \tau^{-1/2}e^{-(\sigma l)^2\tau}\coth\tau(1-\tau\coth\tau)$$
$$= \frac{l}{8\pi}\left(\frac{(\sigma l)^4}{\sqrt{2}}\zeta\Big(\frac{3}{2},\frac{(\sigma l)^2}{2}+1\Big)\right.$$
$$\left.+\sqrt{2}(\sigma l)^2\zeta\Big(\frac{1}{2},\frac{(\sigma l)^2}{2}+1\Big)+2\sigma l-(\sigma l)^{-1}\right). \quad \text{(A18)}$$

Here [in addition to Eq. (57)] the following relations were used [18]:

$$\int_0^\infty \frac{\tau^{\mu-1}e^{-\beta\tau}}{\sinh^2\tau}d\tau = 2^{1-\mu}\Gamma(\mu)\left[2\zeta\left(\mu-1,\frac{\beta}{2}\right)\right.$$
$$\left.-\beta\zeta\left(\mu,\frac{\beta}{2}\right)\right], \quad \mu > 2, \quad \text{(A19)}$$

$$\int_0^\infty \tau^{\mu-1}e^{-\beta\tau}\coth^2\tau d\tau = \beta^{-\mu}\Gamma(\mu)+\int_0^\infty \frac{\tau^{\mu-1}e^{-\beta\tau}}{\sinh^2\tau}d\tau,$$
$$\mu > 2, \quad \text{(A20)}$$

$$\int_0^\infty \frac{\tau^{\mu-1}e^{-\beta\tau}\coth\tau}{\sinh^2\tau}d\tau = \frac{\mu-1}{2}\int_0^\infty \frac{\tau^{\mu-2}e^{-\beta\tau}}{\sinh^2\tau}d\tau$$
$$-\frac{\beta}{2}\int_0^\infty \frac{\tau^{\mu-1}e^{-\beta\tau}}{\sinh^2\tau}d\tau, \quad \mu > 3. \quad \text{(A21)}$$

APPENDIX B

In this appendix we shall derive the thermodynamic potential $V_{\beta,\mu}$ in the NJL model (40); here $\beta = 1/T$ is an inverse temperature and μ is a chemical potential.

As is well known [29], the partition function

$$Z_{\beta,\mu} = \mathrm{Tr}[\exp(-\beta H')] \quad \text{(B1)}$$

is expressed through a path integral over fields of a system (here $H' = H - \mu\int\bar{\Psi}\gamma^0\Psi d^2x$, H is the Hamiltonian of the system). In the NJL model (40), (41), the path integral is

$$Z_{\beta,\mu} = \int[d\Psi][d\bar{\Psi}][d\sigma][d\tau][d\pi]\exp\left\{i\int_0^{-i\beta}dt\int d^2x\right.$$
$$\left.\times\left[\bar{\Psi}(iS^{-1}+\mu\delta^0)\Psi-\frac{1}{2G}\rho^2\right]\right\}, \quad \text{(B2)}$$

where $\rho^2 = \sigma^2+\tau^2+\pi^2$, S is the fermion propagator (18) with m replaced by $\sigma+\gamma^3\tau+i\gamma^5\pi$, and while the fermion fields satisfy the antiperiodic boundary conditions

$$\Psi|_{t=0} = -\Psi|_{t=-i\beta}, \quad \bar{\Psi}|_{t=0} = -\bar{\Psi}|_{t=-i\beta}, \quad \text{(B3)}$$

the boson fields satisfy the periodic boundary conditions.

In order to calculate the thermodynamic potential $V_{\beta,\mu}(\rho)$, it is sufficient to consider configurations with $\tau = \pi = 0$ and $\sigma = \text{const}$. Then the potential is defined as

$$\exp\left\{-\beta V_{\beta,\mu}\left[\int d^2x\right]\right\} = \int[d\Psi][d\bar{\Psi}]$$
$$\times\exp\left\{i\int_0^{-i\beta}dt\int d^2x\right.$$
$$\left.\times\left[\bar{\Psi}(iS^{-1}+\mu\gamma^0)\Psi-\frac{1}{2G}\sigma^2\right]\right\}. \quad \text{(B4)}$$

At the leading order in $1/N$, this potential defines the thermodynamic properties of the system.

As is known [29], in the formalism of the imaginary time, the thermodynamic potential $V_{\beta,\mu}$ can be obtained from the representation for the effective potential V, at $T = 0$ and $\mu = 0$, by replacing

$$\int\frac{d^3k}{(2\pi)^3}\to\frac{i}{\beta}\sum_{n=-\infty}^{+\infty}\int\frac{d^2k}{(2\pi)^2},$$
$$k^0\to i\omega_n+\mu; \quad \omega_n = \frac{\pi}{\beta}(2n+1) \quad \text{(B5)}$$

[$\omega_n = \frac{\pi}{\beta}(2n+1)$ follows from the antiperiodic conditions (B3)]. Then, using the representation for the effective potential in Sec. IV and the expression (20) for the fermion propagator, we get

$$V_{\beta,\mu}(\sigma) = \frac{\sigma^2}{2G}+\frac{N}{2\pi\beta l^2}\int_0^\infty\frac{dt}{t}e^{-tl^2(\sigma^2-\mu^2)}\coth t$$
$$\times\Theta_2\Big(2\pi t\frac{\mu l^2}{\beta}|4i\pi t\frac{l^2}{\beta^2}\Big), \quad \text{(B6)}$$

where

$$\Theta_2(u|\tau) = 2\sum_{n=0}^\infty e^{i\pi\tau(n+\frac{1}{2})^2}\cos\Big((2n+1)u\Big) \quad \text{(B7)}$$

is the second Jacobian theta function [18].

By using the identity [18]

$$\Theta_2(u|\tau) = \left(\frac{i}{\tau}\right)^{1/2}e^{-i\frac{u^2}{\pi\tau}}\Theta_4\left(\frac{u}{\tau}\Big|-\frac{1}{\tau}\right), \quad \text{(B8)}$$

where

$$\Theta_4(u|\tau) = 1+2\sum_{n=1}^\infty(-1)^n e^{i\pi n^2\tau}\cos(2nu) \quad \text{(B9)}$$

is the fourth Jacobian theta function, one can rewrite the relation (B6) as

$$V_{\beta,\mu}(\sigma) = \frac{\sigma^2}{2G} + \frac{N}{4\pi^{3/2}l^3}\int_0^\infty \frac{dt}{t^{3/2}} e^{-tl^2\sigma^2} \coth t\Theta_4\left[\frac{i}{2}\mu\beta|\frac{i}{4\pi t}\left(\frac{\beta}{l}\right)^2\right] \tag{B10}$$
$$= V(\sigma) + \frac{N}{2\pi^{3/2}l^3}\int_0^\infty \frac{dt}{t^{3/2}}\coth t\sum_{n=1}^{\infty}(-1)^n\cosh(\mu\beta\, n)\cdot\exp\left[-\left(t\sigma^2 l^2 + \frac{\beta^2 n^2}{4tl^2}\right)\right],$$

where $V(\sigma)$ is the effective potential (56).

Thus we have derived the representation (81) for the thermodynamic potential. Let us show that it is equivalent to the representation (80).

By using the series

$$\coth t = 1 + 2\sum_{m=1}^{\infty} e^{-2tm}, \tag{B11}$$

the expression for $\tilde{V}_{\mu,\beta} = V_{\mu,\beta} - V$ in Eq. (B10) can be rewritten as

$$\tilde{V}_{\mu,\beta} = \frac{N}{\pi l^2\beta}\sum_{n=1}^{\infty}(-1)^n\frac{\cosh(\mu\beta n)}{n}\left[e^{-\beta\sigma n} + 2\sum_{m=1}^{\infty}\exp\left(-\beta\sigma n\sqrt{1+\frac{2m}{(\sigma l)^2}}\right)\right]. \tag{B12}$$

Here we also used the relations [18]:

$$\int_0^\infty dx x^{\nu-1}\exp\left(-\frac{\beta}{x} - \gamma x\right) = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}),$$
$$K_{-\frac{1}{2}}(z) = K_{\frac{1}{2}}(z) = (\frac{\pi}{2z})^{1/2}e^{-z}, \tag{B13}$$

where $K_\nu(z)$ is a modified Bessel function.

Since

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^{\alpha n}+e^{-\alpha n}}{n}e^{-\beta n} = -\ln(1+e^{-2\beta} + 2e^{-\beta}\cosh\alpha), \tag{B14}$$

we find that

$$\tilde{V}_{\mu,\beta} = -\frac{N}{2\pi\beta l^2}\Big\{\ln[1+e^{-2\beta\sigma}+2e^{-\beta\sigma}\cosh(\mu\beta)] + 2\sum_{m=1}^{\infty}\ln\left[1+e^{-2\beta\sigma\sqrt{1+\frac{2m}{(\sigma l)^2}}} + 2e^{-\beta\sigma\sqrt{1+\frac{2m}{(\sigma l)^2}}}\cosh(\mu\beta)\right]\Big\}. \tag{B15}$$

It is now easy to check that the expression for the thermodynamic potential $V_{\beta,\mu} = V + \tilde{V}_{\beta,\mu}$ coincides with that in Eq. (80).

APPENDIX C

In this appendix we analyze the next-to-leading order in $1/N$ expansion in the $(2+1)$-dimensional NJL model at zero temperature. Our main goal is to show that the propagator of the neutral NG bosons π and τ have a $(2+1)$-dimensional structure in this approximation and that [unlike the $(1+1)$-dimensional Gross-Neveu model [25]] the $1/N$ expansion is reliable in this model.

A review of the $1/N$ expansion in $(2+1)$-dimensional four-fermion interaction models can be found in Ref. [14]. For our purposes, it is sufficient to know that this perturbative expansion is given by Feynman diagrams with the vertices and the propagators of fermions and composite particles σ, π, and τ calculated in leading order in $1/N$. In leading order, the fermion propagator is given in Eqs. (18)–(21). As follows from Eq. (41), the Yukawa coupling of fermions with σ, τ, and π is $g_Y = 1$ in this approximation. The inverse propagators of σ, τ, and π are [14,28]

$$D_{\boldsymbol{\rho}}^{-1}(x) = N\left(\frac{\Lambda}{g\pi}\delta^3(x) + i\mathrm{tr}[S(x,0)T_{\boldsymbol{\rho}}S(0,x)T_{\boldsymbol{\rho}}]\right), \tag{C1}$$

where $\boldsymbol{\rho} = (\sigma,\tau,\pi)$ and $T_\sigma = 1$, $T_\tau = \gamma^3$, and $T_\pi = i\gamma^5$. Here $S(x,0)$ is the fermion propagator (18) with the mass $m_{\rm dyn} = \bar{\sigma}$ defined from the gap equation (62). For completeness, we write down the explicit expression for the Fourier transform of the propagators of the NG bosons:

$$D_\tau^{-1}(k) = D_\pi^{-1}(k)$$
$$= \frac{N}{4\pi^{3/2}l}\int_0^1 du\int_0^\infty \frac{ds\sqrt{s}}{\sinh s}\exp[-s(\sigma l)^2]$$
$$\times\Bigg[(1-\exp[R(s,u)])\left((\sigma l)^2\cosh s + \frac{1}{\sinh s} - \frac{\cosh s}{s}\right) - 2(lk_0)^2(1-u^2)$$
$$\times\cosh s\exp[R(s,u)] + 3l^2\mathbf{k}^2\exp[R(s,u)]$$
$$\times\left(\cosh su - u\sinh su\coth s + \frac{2}{3\sinh^2 s}(\cosh su - \cosh s)\right)\Bigg], \tag{C2}$$

where

$$R(s,u) \overset{def}{=} \frac{s}{4}(lk_0)^2(1-u^2) - \frac{l^2\mathbf{k}^2}{2}\frac{\cosh s - \cosh su}{\sinh s}. \tag{C3}$$

Actually, for our purposes, we need to know the form of these propagators at small momenta only. We find, from Eqs. (70), and (71):

$$D_\tau(k) = D_\pi(k) = -\frac{4\pi\bar{\sigma}}{N} f^2(\bar{\sigma}l)[k_0^2 - f^2(\bar{\sigma}l)\mathbf{k}^2]^{-1}, \tag{C4}$$

where

$$f(\bar{\sigma}l) = \left(\frac{2}{\bar{\sigma}l}\right)^{1/2}\left[\frac{1}{\sqrt{2}}\zeta\left(\frac{3}{2},\frac{(\bar{\sigma}l)^2}{2}+1\right) + (\bar{\sigma}l)^{-3}\right]^{-1/2} \tag{C5}$$

[see Eq. (76)].

The crucial point for us is that, because of the dynamical mass $m_{\rm dyn}$, the fermion propagator is soft in the infrared region [see Eq. (25)] and that the propagators of the τ and π (C4) have a $(2+1)$-dimensional form in the infrared region [as follows from Eqs. (70) and (71) the propagator of σ has of course also a $(2+1)$-dimensional form].

Let us begin by considering the next-to-leading corrections in the effective potential. The diagram which contributes to the effective potential in this order is shown in Fig. 5(a). Because of the structure of the propagators pointed out above, there are no infrared divergences in this contribution to the potential. [Note that this is in contrast with the Gross-Neveu model: because of a $(1+1)$-dimensional form of the propagators of the NG bosons, this contribution is logarithmically divergent in the infrared region in that model, i.e., the $1/N$ expansion is unreliable in that case.] Therefore the diagram in Fig. 5(a) leads to a finite, order 1, correction to the potential V (we recall that the leading contribution in V is of order N). As a result, at sufficiently large values of N, the gap equation in next-to-leading order in $1/N$ in this model admits a nontrivial solution $\bar{\rho} \neq 0$. Since the potential depends only on the radial variable ρ, the angular variables θ and φ [$\boldsymbol{\rho} = (\rho\cos\theta, \rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi)$], connected with the τ and π, appear in the effective Lagrangian only through their derivatives. This in turn implies that the τ and π retain to be gapless NG modes in the next-to-leading order in $1/N$.

Let us now consider the next-to-leading corrections to the propagators of these NG modes. First of all, note that in a constant magnetic field, the propagator of a neutral local field $\varphi(x)$, $D_\varphi(x,y)$, is translation invariant, i.e., it depends on $(x-y)$. This immediately follows from the fact that the operators of space translations (46) take the canonical form for neutral fields (the operator of time translations is $i\partial/\partial t$ for both neutral and charged fields in a constant magnetic field). The diagrams contributing to the propagators of the NG modes in this order are shown in Fig. 5(b). Because of the dynamical mass $m_{\rm dyn}$ in the fermion propagator, this contribution is analytic at $k_\mu = 0$. Since at large N the gap equation has a nontrivial solution in this approximation, there is no contribution of $O(k^0) \sim$ const in the inverse propagators of τ and π. Therefore the first term in the momentum expansion of this contribution has the form $C_1k_0^2 - C_2\mathbf{k}^2$, where C_1 and C_2 are functions of $\bar{\sigma}l$, *i.e.* the propagators take the following form in this approximation:

$$D_\tau(k) = D_\pi(k) \overset{k\to 0}{=} -\frac{4\pi\bar{\sigma}}{N} f^2(\bar{\sigma}l)\left[\left(1 - \frac{1}{N}\tilde{C}_1(\bar{\sigma}l)\right)k_0^2 - \left(f^2(\bar{\sigma}l) - \frac{1}{N}\tilde{C}_2(\bar{\sigma}l)\right)\mathbf{k}^2\right]^{-1}. \tag{C6}$$

[see Eq. (C4)].

Because of the same reasons, there are also no infrared divergences either in the fermion propagator [see Fig. 5(c)] or in the Yukawa vertices [see Fig. 5(d)] in this order. Therefore at sufficiently large values of N, the results retain essentially the same as in leading order in $1/N$.

We believe that there should not be any principal obstacle to extend this analysis for all orders in $1/N$.

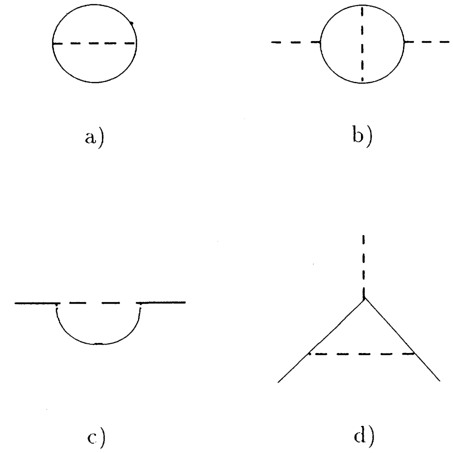

FIG. 5. Diagrams in next-to-leading order in $1/N$. A solid line denotes the fermion propagator and a dashed line denotes the propagators of σ, τ, and π in leading order in $1/N$.

[1] R. Jackiw, Phys. Rev. D **29**, 2375 (1984); I. Affleck, Nucl. Phys. **B265**, 409 (1986); I.V. Krive and A.S. Rozhavskii, Usp. Fiz. Nauk **152**, 33 (1987) [Sov. Phys. Usp. **30**, 370 (1987)].

[2] A. Kovner and B. Rosenstein, Phys. Rev. B **42**, 4748 (1990); G.W. Semenoff and L.C.R. Wijewardhana, Phys. Rev. D **45**, 1342 (1992); N. Dorey and N.E. Mavromatos, Nucl. Phys. **B368**, 614 (1992); R. MacKenzie, P.K. Pan-

igrahi, and S. Sakhi, Phys. Rev. B **48**, 3892 (1993).
[3] J. Bardeen, L.N. Cooper, and J.R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[4] V.A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993).
[5] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[6] I.V. Krive and S.A. Naftulin, Phys. Rev. D **46**, 2737 (1992).
[7] A.I. Akheizer and V.B. Berestetsky, *Quantum Electrodynamics* (Interscience, New York, 1965).
[8] T. Appelquist, M. Bowick, D. Karabali, and L.C.R. Wijewardhana, Phys. Rev. D **33**, 3704 (1986).
[9] T. Appelquist, D. Nash, and L.C.R. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).
[10] E. Dagotto, J.B. Kogut, and A. Kocić, Phys. Rev. Lett. **62**, 1083 (1989); Nucl. Phys. **B334**, 279 (1990); S. Hands and J.B. Kogut, *ibid.* **B335**, 455 (1990).
[11] D. Atkinson, P.W. Johnson, and P. Maris, Phys. Rev. D **42**, 602 (1990); K.-I. Kondo and H. Nakatani, Mod. Phys. Lett. A **5**, 407 (1990); Prog. Theor. Phys. **87**, 193 (1992).
[12] M.R. Pennington and D. Walsh, Phys. Lett. B **253**, 246 (1991); R. Pisarski, Phys. Rev. D **44**, 1866 (1991).
[13] M.C. Diamantini, G.W. Semenoff, and P. Sodano, Phys. Rev. Lett. **70**, 3438 (1993).
[14] B. Rosenstein, B.J. Warr, and S.H. Park, Phys. Rep. **205**, 59 (1991).
[15] K.I. Kondo, M. Tanabashi, and K. Yamawaki, Prog. Theor. Phys. **89**, 1249 (1993).
[16] S. Hands, A. Kocić, and J.B. Kogut, Ann. Phys. (N.Y.) **224**, 29 (1993).
[17] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[18] I.S. Gradshtein and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, Orlando, 1980).
[19] A. Chodos, K. Everding, and D.A. Owen, Phys. Rev. D **42**, 2881 (1990).
[20] P. Lee and S. Shenoy, Phys. Rev. Lett. **28**, 1025 (1972); see also Z. Tesanovic, Physica **C220**, 303 (1994) for an update and the list of references.
[21] R. Haag, Nuovo Cimento **25**, 287 (1962).
[22] See the first paper in Ref. 1; A.J. Niemi and G.W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983); Phys. Rep. **135**, 99 (1986); D. Boyanovsky, R. Blankenbecler, and R. Yahalom, Nucl. Phys. **B270**, 483 (1986).
[23] A. Kovner, B. Rosenstein, and D. Eliezer, Nucl. Phys. **B350**, 235 (1991); A. Kovner and D. Eliezer, Int. J. Mod. Phys. A **7**, 2775 (1992).
[24] N.N. Bogolyubov, Physica **26**, 1 (1960); R.V. Lange, Phys. Rev. **146**, 301 (1966).
[25] D.J. Gross and A. Neveu, Phys. Rev. D **10**, 3225 (1974); E. Witten, Nucl. Phys. **B145**, 110 (1978).
[26] N.D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
[27] J. Zak, Phys. Rev. **134**, A1602 (1964); J.E. Avron, I.W. Herbst, and B. Simon, Ann. Phys. **114**, 431 (1978).
[28] V.P. Gusynin and V.A. Miransky, Mod. Phys. Lett. A **6**, 2443 (1991); Sov. Phys. JETP **74**, 216 (1992); V.A. Miransky, Int. J. Mod. Phys. A **8**, 135 (1993).
[29] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
[30] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. **131B**, 121 (1983).
[31] D. Cangemi, E. D'Hoker, and G. Dunne, Phys. Rev. D **51**, R2513 (1995).