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Classical splitting of fundamental strings

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We find exact solutions of the string equations of motion and constraints describing the *classical* splitting of a string in two. We show that for the same Cauchy data the strings that split have a *smaller* action than the string without splitting. This phenomenon is already present in flat space-time. The mass, energy, and momentum carried out by the strings are computed. We show that the splitting solution describes a natural decay process of one string of mass M into two strings with a smaller total mass and some kinetic energy. The standard nonsplitting solution is contained as a particular case. We also describe the splitting of a closed string in the background of a singular gravitational plane wave, and show how the presence of the strong gravitational field increases (and amplifies by an overall factor) the negative difference between the action of the splitting and nonsplitting solutions.

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I. INTRODUCTION

A great amount of work has been devoted to string theory in past years. However, very little attention has been paid to the theory of fundamental strings as a classical theory. The interaction of strings through joining and splitting has been mostly considered at the *quantum* level, although self-intersections of classical cosmic strings have been studied in [1], and the classical string splitting for hadronic strings has been discussed in [2]. At the quantum level, the usual procedure to compute the quantum string scattering amplitudes is based on the evaluation of the correlation functions of vertex operators of functionals, which are constructed out of a particular type of solution of the classical string equations of motion and constraints: one in which the string propagates without splitting and sweeping a world sheet that has the topology of a cylinder or a strip [3]. In this paper, we will show that in addition to these classical solutions which are used to build vertex operators, there also exist *classical* solutions in which the string *splits.* More precisely, let us consider as our classical action the Polyakov action [4] (we set here $2\alpha' = 1$)

$$S = \frac{1}{2\pi}\int_{\rm WS} d\sigma\, d\tau \sqrt{-g}\, g^{ab} G_{AB}(X)\partial_a X^A \partial_b X^B, \quad (1.1)$$

where g_{ab} is the world sheet metric and G_{AB} is the background space-time metric. Since we are going to consider classical solutions, both g_{ab} and G_{AB} have Lorentzian signature. We will show that there exists stationary points of S which correspond to a classical splitting of one string in two. Moreover, we will show that for some fixed Cauchy data $X^A(\sigma,\tau_0)$ and $\dot{X}^A(\sigma,\tau_0)$, the solutions corresponding to a string that splits in two, have *smaller* action than the one in which the string does not split; i.e., the sum of the areas swept by the two pieces in which the string splits is smaller than the area swept out by the string that does not split. Of course, the existence of different kinds of solutions for the same Cauchy data is due to the fact that the world sheet metric is not fixed by any dynamical equation, and so we are free to choose it at will. Indeed, if we enforce the conformal gauge *globally* on our world sheet, we are left with a world sheet that has the topology of a cylinder (for closed strings), and therefore with a string that does not split. However, nothing prevents us from considering a different world sheet *topology*, and the interesting point is that the solutions so obtained describe strings that split and have a *smaller* action than the string that does not split. In order to show explicit solutions of this type, we give here some interesting examples, saving a more general discussion for future work. We will consider first an example in flat space-time, to show how the phenomenon of smaller action for the string that splits is already present in this case. Second, we will consider a closed string moving in the background given by a singular plane wave, and show how the presence of a strong gravitational field *increases* and *amplifies* the negative difference between the action for the string that splits and the action for the string that does not split. In fact, such a difference is amplified by an overall factor and becomes infinitely negative at the space-time singularity. The classical splitting string so-

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lution contains the standard (nonsplitting) solution as a particular case. We also show that the splitting solution describes a natural disintegration process of one string of mass M decaying into two pieces of a smaller total mass with some kinetic energy. In this sense, we want to stress that the splitting string solutions which we obtain are exact solutions of the D'Alambert equation (2.4) and constraints (2.5) (it must be noticed that $\partial_\sigma^2 X$ and $\partial_\tau^2 X$ exhibit equal Dirac δ singular terms for each of the two pieces in which the string splits).

It is also possible to construct string solutions where one string splits into more than two pieces.

II. STRING SPLITTING IN FLAT SPACE-TIME

Let us consider a closed string $X(\sigma,\tau)$ moving in a D-dimensional flat Minkowski space-time. In order to describe a string that splits, we choose a world sheet $\mathcal{M}$ with the topology of a pant, and call (σ_0,τ_0) the point at which the string breaks into two pieces. To construct a solution for the string with this world sheet topology, we consider in $\mathcal{M}$ the three regions I, II, and III given by

$$\begin{aligned} \mathrm{I} &\equiv \{(\sigma,\tau) : 0 \le \sigma < 2\pi, \tau_i \le \tau < \tau_0\}, \\ \mathrm{II} &\equiv \{(\sigma,\tau) : 0 \le \sigma < \sigma_0, \tau_0 < \tau \le \tau_f\}, \\ \mathrm{III} &\equiv \{(\sigma,\tau) : \sigma_0 \le \sigma < 2\pi, \tau \le \tau_f\}, \end{aligned} \tag{2.1}$$

and impose the continuity of the dynamical variables $X(\sigma,\tau)$ and $\dot X(\sigma,\tau)$ at the splitting world sheet time $\tau=\tau_0$:

$$X^{(\mathrm{I})}(\sigma,\tau_0) = \begin{cases} X^{(\mathrm{II})}(\sigma,\tau_0) & \text{if } 0\le\sigma<\sigma_0, \\ X^{(\mathrm{III})}(\sigma,\tau_0) & \text{if } \sigma_0\le\sigma<2\pi, \end{cases} \tag{2.2}$$

$$\dot X^{(\mathrm{I})}(\sigma,\tau_0) = \begin{cases} \dot X^{(\mathrm{II})}(\sigma,\tau_0) & \text{if } 0\le\sigma<\sigma_0, \\ \dot X^{(\mathrm{III})}(\sigma,\tau_0) & \text{if } \sigma_0\le\sigma<2\pi. \end{cases} \tag{2.3}$$

In each of the regions we are in the conformal gauge. Thus the equation of motion and the string constraints read the same for the three functions $X^{(\mathcal{J})}(\sigma,\tau)$, with $\mathcal{J} = \mathrm{I,II,III}$:

$$(\partial_\tau^2 - \partial_\sigma^2) X^{(\mathcal{J})} = 0, \tag{2.4}$$

$$(\dot X^{(\mathcal{J})} + X'^{(\mathcal{J})})^2 = 0. \tag{2.5}$$

(We shall use in this paper the symbol $\mathcal{J}$ to label the world-sheet regions I, II, and III.)

However, we must impose different boundary conditions for each of the functions $X^{(\mathcal{J})}$. Since we want to describe a closed string splitting into two closed strings, the appropriate boundary conditions are the periodicity conditions

$$X^{(\mathcal{J})}(\sigma+\lambda_{\mathcal{J}},\tau) = X^{(\mathcal{J})}(\sigma,\tau), \quad \mathcal{J} = \mathrm{I,II,III}, \tag{2.6}$$

where

$$\lambda_{\mathrm{I}} = 2\pi, \quad \lambda_{\mathrm{II}} = \sigma_0, \quad \lambda_{\mathrm{III}} = 2\pi - \sigma_0. \tag{2.7}$$

Of course, for the splitting to be possible, the string configuration $X^{(\mathrm{I})}(\sigma,\tau)$ with which we start must satisfy the consistency condition

$$X^{(\mathrm{I})}(0,\tau_0) = X^{(\mathrm{I})}(\sigma_0,\tau_0) = X^{(\mathrm{I})}(2\pi,\tau_0). \tag{2.8}$$

The general solution of Eqs. (2.4) with the periodic boundary conditions (2.6) is

$$X^{(\mathcal{J})}(\sigma,\tau) = \sum_{n=-\infty}^{\infty} X_n^{(\mathcal{J})}(\tau) \exp\left(i\frac{2\pi}{\lambda_{\mathcal{J}}} n\sigma\right), \tag{2.9}$$

where

$$X_0^{(\mathcal{J})} = q^{(\mathcal{J})} + p^{(\mathcal{J})}\tau \tag{2.10}$$

and

$$X_n^{(\mathcal{J})}(\tau) = A_n^{(\mathcal{J})} \exp\left(-i\frac{2\pi}{\lambda_{\mathcal{J}}} n\tau\right) + B_n^{(\mathcal{J})} \exp\left(i\frac{2\pi}{\lambda_{\mathcal{J}}} n\tau\right) \quad \text{for } n\neq 0, \tag{2.11}$$

where $\mathcal{J} = \mathrm{I,II,III}$ as defined above.

Now, in order to construct a solution of the equations of motion and constraints corresponding to a string that splits, we begin with a function $X^{(\mathrm{I})}(\sigma,\tau)$ that satisfies Eqs. (2.4) and the condition (2.8) at $\tau=\tau_0$, and then we construct $X^{(\mathrm{II})}$ and $X^{(\mathrm{III})}$ by determining their Fourier coefficients $X_n^{(\mathrm{II})}(\tau)$ and $X_n^{(\mathrm{III})}(\tau)$ through the matching conditions (2.2) and (2.3). Thus, from Eqs. (2.9), (2.2), and (2.3) we obtain

$$\begin{aligned} X_n^{(\mathrm{II})}(\tau_0) &= i \sum_{m=-\infty}^{\infty} \frac{1-e^{im\sigma_0}}{m\sigma_0 - 2\pi n} X_m^{(\mathrm{I})}(\tau_0), \\ X_n^{(\mathrm{II})}(\tau_0) &= i \sum_{m=-\infty}^{\infty} \frac{1-e^{im\sigma_0}}{m\sigma_0 - 2\pi n} \dot X_m^{(\mathrm{I})}(\tau_0), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} X_n^{(\mathrm{III})}(\tau_0) &= -i \exp\left(-i\frac{2\pi}{2\pi-\sigma_0} n\sigma_0\right) \sum_{m=-\infty}^{\infty} \frac{1-e^{im\sigma_0}}{m(2\pi-\sigma_0) - 2\pi n} X_m^{(\mathrm{I})}(\tau_0), \\ \dot X_n^{(\mathrm{III})}(\tau_0) &= -i \exp\left(-i\frac{2\pi}{2\pi-\sigma_0} n\sigma_0\right) \sum_{m=-\infty}^{\infty} \frac{1-e^{im\sigma_0}}{m(2\pi-\sigma_0) - 2\pi n} \dot X_m^{(\mathrm{I})}(\tau_0), \end{aligned} \tag{2.13}$$

so that Eqs. (2.10), (2.11), and their derivatives at $\tau = \tau_0$ determine the constant coefficients $(q,p,A_n,B_n)^{(\mathrm{II})}$ and $(q,p,A_n,B_n)^{(\mathrm{III})}$ in terms of the initial data $(q,p,A_n,B_n)^{(\mathrm{I})}$. In this way we obtain the solutions for the string pieces II and III from a general string solution I. This last must satisfy the consistency condition (2.8).

It is important also to notice that the constraints (2.5) are also satisfied by the strings $X^{(\mathrm{II})}$ and $X^{(\mathrm{III})}$. The matching conditions (2.2) and (2.3) imply that both $\dot{X}(\sigma,\tau)$ and $X'(\sigma,\tau)$ are continuous at $\tau = \tau_0$. Therefore

$$(\dot{X}^{(\mathrm{II})} \pm X'^{(\mathrm{II})})^2(\sigma,\tau_0) = (\dot{X}'^{(\mathrm{I})} \pm X'^{(\mathrm{I})})^2(\sigma,\tau_0) = 0, \tag{2.14}$$

$$(\dot{X}^{(\mathrm{III})} \pm X'^{(\mathrm{III})})^2(\sigma,\tau_0) = (\dot{X}^{(\mathrm{I})} \pm X'^{(\mathrm{I})})^2(\sigma,\tau_0) = 0.$$

Then, taking into account the equation of motion (2.4), it follows that the constraints hold for all $\tau \geq \tau_0$.

Let us now consider a particular example given by a circular string that winds r times upon itself:

$$\begin{aligned} T^{(\mathrm{I})} &= M\tau, \\ X^{(\mathrm{I})} &= \frac{M}{r}\sin r\tau\,\cos r\sigma, \\ X^{(\mathrm{I})} &= \frac{M}{r}\sin r\tau\,\sin r\sigma, \\ X^{i(\mathrm{I})} &= 0 \quad \text{for } i = 3,\ldots,D-1, \end{aligned} \tag{2.15}$$

which satisfy

$$X^{(\mathrm{I})}(\sigma,0) = 0 = Y^{(\mathrm{I})}(\sigma,0) = T^{(\mathrm{I})}(0) \tag{2.16}$$

and

$$\begin{aligned} \dot{X}^{(\mathrm{I})}(\sigma,0) &= M\,\cos r\sigma, \\ \dot{Y}^{(\mathrm{I})}(\sigma,0) &= M\,\sin r\sigma, \quad \dot{T}^{(\mathrm{I})}(0) = M. \end{aligned} \tag{2.17}$$

It is necessary to check that Eqs. (2.15) are a solution of Eq. (2.4) and satisfy the constraints (2.5).

In order to obtain a *splitting* string solution, let us choose without loss of generality $\tau_0 = 0$. [One can always replace τ by $\tau - \tau_0$ in Eqs. (2.15).] Then the splitting consistency condition(2.8) is trivially satisfied because at $\tau = 0$ the string (2.15) collapses to a point.

For the string coordinate T, the matching conditions (2.2) and (2.3) yield

$$T^{(\mathrm{II})} = M\tau, \; T^{(\mathrm{III})} = M\tau. \tag{2.18}$$

Next, to obtain $X^{(\mathrm{II})}(\sigma,\tau)$, $Y^{(\mathrm{II})}(\sigma,\tau)$, $X^{(\mathrm{III})}(\sigma,\tau)$, and $Y^{(\mathrm{III})}(\sigma,\tau)$, we observe that, from Eqs. (2.15) and (2.9),

$$\begin{aligned} T^{(\mathrm{I})}(0) &= T_0^{(\mathrm{I})}(0) = 0, \\ \dot{T}^{(\mathrm{I})}(0) &= \dot{T}_0^{(\mathrm{I})}(0) = M, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} & X_n^{(\mathrm{I})}(0) = Y_n^{(\mathrm{I})}(0) = 0, \\ & \dot{X}_r^{(\mathrm{I})}(0) = \dot{X}_{-r}^{(\mathrm{I})}(0) = \frac{1}{2}M, \\ & \dot{Y}_r^{(\mathrm{I})}(0) = -\dot{Y}_{-r}^{(\mathrm{I})}(0) = -\frac{i}{2}M, \\ & \dot{X}_n^{(\mathrm{I})}(0) = \dot{Y}_n^{(\mathrm{I})}(0) = 0 \quad \text{for } n \neq \pm r. \end{aligned} \tag{2.20}$$

Therefore, the matching equations (2.12) and (2.13) read, in this case,

$$\begin{aligned} X_n^{(\mathrm{II})}(0) &= 0 = Y_n^{(\mathrm{II})}(0), \\ X_n^{(\mathrm{III})}(0) &= 0 = Y_n^{(\mathrm{III})}(0), \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \dot{X}_n^{(\mathrm{II})}(0) &= M\phi_n(\sigma_0), \\ \dot{Y}_n^{(\mathrm{II})}(0) &= M\psi_n(\sigma_0), \\ \dot{X}_n^{(\mathrm{III})}(0) &= M\,\exp\left(-i\frac{2\pi}{2\pi-\sigma_0}n\sigma_0\right)\phi_n^*(2\pi-\sigma_0), \\ \dot{Y}_n^{(\mathrm{III})}(0) &= -M\,\exp\left(-i\frac{2\pi}{2\pi-\sigma_0}n\sigma_0\right)\psi_n^*(2\pi-\sigma_0), \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} \phi_n(\sigma_0) &= -\frac{r\sigma_0\sin r\sigma_0 + i2\pi n(1-\cos r\sigma_0)}{4\pi^2n^2 - r^2\sigma_0^2}, \\ \psi_n(\sigma_0) &= -\frac{r\sigma_0(1-\cos r\sigma_0) - i2\pi n\,\sin r\sigma_0}{4\pi^2n^2 - r^2\sigma_0^2}. \end{aligned} \tag{2.23}$$

Computing now the two sets of constants $(q,p,A_n,B_n)^{(\mathrm{II})}$ and $(q,p,A_n,B_n)^{(\mathrm{III})}$ from expressions (2.10) and (2.11), we obtain the time-dependent Fourier coefficients $X_n^{(\mathrm{II})}(\tau)$, $Y_n^{(\mathrm{II})}(\tau)$, $X_n^{(\mathrm{III})}(\tau)$, and $Y_n^{(\mathrm{III})}(\tau)$. Finally the two pieces in which the string splits read

$$\begin{aligned} X^{(\mathrm{II})}(\sigma,\tau) &= M\tau\frac{\sin r\sigma_0}{r\sigma_0} + MJ\frac{\sigma_0}{2\pi}\sum_{n\neq 0}\frac{\phi_n(\sigma_0)}{n}\sin\left[\frac{2\pi}{\sigma_0}n\tau\right]\exp\left(i\frac{2\pi}{\sigma_0}n\sigma\right), \\ Y^{(\mathrm{II})}(\sigma,\tau) &= M\tau\frac{1-\cos r\sigma_0}{r\sigma_0} + M\frac{\sigma_0}{2\pi}\sum_{n\neq 0}\frac{\psi_n(\sigma_0)}{n}\sin\left[\frac{2\pi}{\sigma_0}n\tau\right]\exp\left(i\frac{2\pi}{\sigma_0}n\sigma\right), \end{aligned} \tag{2.24}$$

$$X^{(\mathrm{III})}(\sigma,\tau) = -M\tau\frac{\sin r\sigma_0}{r(2\pi-\sigma_0)} + M\frac{2\pi-\sigma_0}{2\pi}\sum_{n\neq 0}\left\{\frac{1}{n}\sin\left[\frac{2\pi}{2\pi-\sigma_0}n\tau\right]\phi_n^*(2\pi-\sigma_0)\exp\left(i\frac{2\pi}{2\pi-\sigma_0}n(\sigma-\sigma_0)\right)\right\},$$

$$Y^{(\mathrm{III})}(\sigma,\tau) = -M\tau\frac{1-\cos r\sigma_0}{r(2\pi-\sigma_0)} - M\frac{2\pi-\sigma_0}{2\pi}\sum_{n\neq 0}\left\{\frac{1}{n}\sin\left[\frac{2\pi}{2\pi-\sigma_0}n\tau\right]\psi_n^*(2\pi-\sigma_0)\exp\left(i\frac{2\pi}{2\pi-\sigma_0}n(\sigma-\sigma_0)\right)\right\}. \quad (2.25)$$

Let us now discuss the properties of the splitting string solution given by Eqs. (2.15), (2.24), and (2.25). First, we notice that, for $n \neq 0$ and $\sigma_0 \to 2\pi$,

$$\phi_n(\sigma_0) \underset{\sigma_0\to 2\pi}{\to} \frac{1}{2}(\delta_{nr}+\delta_{n(-r)}),$$
$$\psi_n(\sigma_0) \underset{\sigma_0\to 2\pi}{\to} -\frac{i}{2}(\delta_{nr}+\delta_{n(-r)}). \quad (2.26)$$

Therefore,

$$X^{(\mathrm{II})}(\sigma,\tau) \underset{\sigma_0\to 2\pi}{\sim} \frac{M}{r}\sin r\tau\,\cos r\sigma = X^{(\mathrm{I})}(\sigma,\tau),$$
$$Y^{(\mathrm{II})}(\sigma,\tau) \underset{\sigma_0\to 2\pi}{\sim} \frac{M}{r}\sin r\tau\,\sin r\sigma = Y^{(\mathrm{I})}(\sigma,\tau). \quad (2.27)$$

Similarly for $\sigma_0 \to 0$,

$$X^{(\mathrm{III})}(\sigma,\tau) \underset{\sigma_0\to 0}{\sim} X^{(\mathrm{I})}(\sigma,\tau),$$
$$Y^{(\mathrm{III})}(\sigma,\tau) \underset{\sigma_0 J\to 0}{\sim} Y^{(\mathrm{I})}(\sigma,\tau). \quad (2.28)$$

Thus the splitting string solution given by Eqs. (2.24) and (2.25) gives the solution $X^{(\mathrm{I})}(\sigma,\tau)$ in the limits $\sigma_0 \to 2\pi$ and $\sigma_0 \to 0$ as it should be. In this sense, the splitting solution generalizes the standard string solution without splitting. Moreover, it can be checked from Eqs. (2.24) and (2.25) that the two string pieces II and III satisfy the (homogeneous) D'Alambert equation (2.4) and the constraints (2.5). It should be noticed that $X^{(\mathcal{J})}$ and $Y^{(\mathcal{J})}$ are continuous periodic functions with periods σ_0 and $2\pi-\sigma_0$; their first derivatives with respect to σ and τ have step discontinuities, and their second derivatives exhibit Dirac delta singularities. These singularities sit on the world sheet characteristics through the splitting point (σ_0, τ_0). However, the Dirac delta terms in the expressions for the second derivatives with respect to σ and τ are equal and cancel in the d'Alambertian.

The energy and momentum carried out by each of the strings $\mathcal{J}=$ I, II, and III is given by

$$E^{(\mathcal{J})} = \frac{1}{2\pi}\int_0^{\lambda_{\mathcal{J}}} d\sigma\,\dot{T}^{(\mathcal{J})},$$
$$P_X^{(\mathcal{J})} = \frac{1}{2\pi}\int_0^{\lambda_{\mathcal{J}}} d\sigma\,\dot{X}^{(\mathcal{J})}, \quad (2.29)$$
$$P_Y^{(\mathcal{J})} = \frac{1}{2\pi}\int_0^{\lambda_{\mathcal{J}}} d\sigma\,\dot{Y}^{(\mathcal{J})}.$$

Then, using the Fourier series expansions (2.15), (2.24), and (2.25) we obtain

$$(E,P_X,P_Y)^{(\mathrm{I})} = M(1,0,0),$$
$$(E,P_X,P_Y)^{(\mathrm{II})} = M\left(\frac{\sigma_0}{2\pi},\frac{\sin r\sigma_0}{2\pi r},\frac{1-\cos r\sigma_0}{2\pi r}\right), \quad (2.30)$$
$$(E,P_X,P_Y)^{(\mathrm{III})} = M\left(\frac{2\pi-\sigma_0}{2\pi},-\frac{\sin r\sigma_0}{2\pi r},-\frac{1-\cos r\sigma_0}{2\pi r}\right).$$

From Eqs. (2.30) we see that the energy momentum of the string before and after the splitting is conserved, as it should be. The masses of the three strings are given by

$$M_{\mathrm{I}} = M,$$
$$M_{\mathrm{II}} = \frac{M}{2\pi}\sqrt{\sigma_0^2-\frac{4}{r^2}\sin^2\frac{r\sigma_0}{2}}, \quad (2.31)$$
$$M_{\mathrm{III}} = \frac{M}{2\pi}\sqrt{(2\pi-\sigma_0)^2-\frac{4}{r^2}\sin^2\frac{r\sigma_0}{2}}.$$

Again, we see that

$$M_{\mathrm{II}} \underset{\sigma_0\to 2\pi}{\sim} M_{\mathrm{I}},$$
$$M_{\mathrm{III}} \underset{\sigma_0\to 0}{\sim} M_{\mathrm{I}}. \quad (2.32)$$

From Eqs. (2.31) it also follows that

$$M_{\mathrm{I}} \geq M_{\mathrm{II}} + M_{\mathrm{III}}. \quad (2.33)$$

This tells us that the classical splitting string solution describes a natural disintegration process, in which a string of mass M decays into two pieces with a smaller total mass and some kinetic energy, which depends on the point where the splitting takes place. Furthermore, one can see in Eqs. (2.30), (2.24), and (2.25) that the two outgoing pieces II and III go away one from each other with opposite momenta.

From Eqs. (2.32) it follows that the kinetic energy

$$K(\sigma_0) = M_{\mathrm{I}} - (M_{\mathrm{II}} + M_{\mathrm{III}}) \quad (2.34)$$

vanishes when

$$r\sigma_0 = 2l\pi, \quad l = 0,1,\ldots,r. \quad (2.35)$$

This corresponds to cutting the initial string $X^{(\mathrm{I})}$ into two pieces $X^{(\mathrm{II})}$ and $X^{(\mathrm{III})}$ which contain an integer number of turns: l and $r-l$, respectively. In this case, Eqs.

(2.30) tell us that the momentum of the two pieces vanishes, and moreover the series (2.24) and (2.25) sum up to $X^{(\mathrm{I})}(\sigma,\tau)$ and $Y^{(\mathrm{I})}(\sigma,\tau)$, respectively. That is, cutting the string (2.15) into two pieces which contain an integer number of circumferences, is equivalent to not cutting it at all. (In other words, a circular string $X^{(\mathrm{I})}$ wound r times is equivalent to two concentric strings $X^{(\mathrm{II})}$ and $X^{(\mathrm{III})}$ wound l and $r-l$ times, respectively.) And more generally, a circular string with $r=n$ turns is equivalent to r strings with $n=1$ turn each. Of course, this is due to the fact that, in this case, the periodicity conditions (2.6) that we are enforcing for $\tau>0$ are already present in $X^{(\mathrm{I})}(\sigma,\tau)$ for $\tau<0$.

From Eqs. (2.33), (2.34), and (2.35) it follows that in each of the r equal windings in σ described by the string I, there is a splitting point $\sigma_0^i, i=1,2,\ldots,r$ that maximizes the kinetic energy of the pieces II and III. That is, for each of the turns of the string upon itself, there is an intermediate point where the splitting of the string is most energetically favorable.

Also from Eq. (2.31) we see that the kinetic energy $K(\sigma_0)$ of the pieces II and III decreases with the growing of r. In face, $K(\sigma_0)\to 0$ for $r\to\infty$ and reaches its maximum value for $r=1$ and $\sigma_0=\pi$. The value of this maximum kinetic energy is

$$K_{\max}=M\left(1-\sqrt{1-\frac{4}{\pi^2}}\right)\simeq 0.229M. \tag{2.36}$$

So, the most energetically favorable case correspond to a string with $r=1$, which cuts into to equally long pieces. This also indicates that for a string with $r>1$ the most energetically favorable process is not the breaking of the string into two pieces, but the breaking into r pieces at the midpoints of each winding. Thus the fundamental case to be considered is that of a string with $r=1$ that cuts into two pieces.

Let us now discuss the string action. Let S_{I} be the area swept by the classical solution $X^{(\mathrm{I})}$ that does not split, when it evolves in τ from 0 to τ_f; and $S_{\mathrm{II}}, S_{\mathrm{III}}$ the areas swept by the two pieces of the splitting string evolving for the same τ interval. We are interested in comparing S_{I} with $S_{\mathrm{II}}+S_{\mathrm{III}}$, for long enough evolution time, i.e., $\tau_f\gg 2\pi$.

In flat space-time the string action (1.1) takes the form

$$S=\frac{1}{2\pi}\int_0^{\tau_f}d\tau\int_0^{2\pi}d\sigma(-\dot{X}^A\dot{X}_A+X'^AX'_A) \tag{2.37}$$

that using the constraints (2.5), can be rewritten as

$$S=\frac{1}{\pi}\int_0^{\tau_f}d\tau\int_0^{2\pi}d\sigma(\partial_\sigma X)^2. \tag{2.38}$$

Now for the string I, using Eq. (2.15) we have

$$S_{\mathrm{I}}=\frac{1}{\pi}\int_0^{\tau_f}d\tau\int_0^{2\pi}d\sigma M^2\sin^2 r\tau\sim M^2\tau_f\quad(\tau_f\to\infty). \tag{2.39}$$

For the string II, using Eqs. (2.18) and (2.24) we obtain

$$\begin{aligned}S_{\mathrm{II}}&=\frac{1}{\pi}\int_0^{\tau_f}d\tau\int_0^{\sigma_0}d\sigma\{(\partial_\sigma X^{(\mathrm{II})})^2+(\partial_\sigma Y^{(\mathrm{II})})^2\}\\&=M^2\frac{\sigma_0}{\pi}\sum_{n\neq0}[|\phi_n(\sigma_0)|^2+|\psi_n(\sigma_0)|^2]\\&\quad\times\int_0^{\tau_f}d\tau\sin^2\left(\frac{2\pi}{\sigma_0}n\tau\right).\end{aligned} \tag{2.40}$$

For large τ_f and $n\neq 0$,

$$\int_0^{\tau_f}d\tau\sin^2\left(\frac{2\pi}{\sigma_0}n\tau\right)\sim\frac{1}{2}\tau_f. \tag{2.41}$$

On the other hand,

$$\sum_{n\neq0}\left[|\phi_n(\sigma_0)|^2+|\psi_n(\sigma_0)|^2\right]=1-\frac{4}{r^2\sigma_0^2}\sin^2\left(\frac{r\sigma_0}{2}\right). \tag{2.42}$$

Thus, for large τ_f,

$$S_{\mathrm{II}}=M^2\tau_f\frac{\sigma_0}{2\pi}\left[1-\frac{4}{r^2\sigma_0^2}\sin^2\left(\frac{r\sigma_0}{2}\right)\right]. \tag{2.43}$$

Notice that when $\sigma_0\to 2\pi$, $S_{\mathrm{II}}\to S_{\mathrm{I}}$. In addition, the use of the large τ_f approximation deserves the following comment: one has to wait long enough time to appreciate the difference between the areas swept by the string I, and the strings II and III. In fact, as can be easily seen from Eqs. (2.24) and (2.25), at first order in the Taylor expansion in τ around $\tau=0$, $X^{(\mathrm{I})}$, $X^{(\mathrm{II})}$, and $X^{(\mathrm{III})}$ coincide, and, therefore, when $\tau_f\to 0$,

$$\frac{1}{\tau_f}(S_{\mathrm{I}}-S_{\mathrm{II}}-S_{\mathrm{III}})\underset{\tau_f\to0}{\frown}0. \tag{2.44}$$

Let us come back to the comparison between S_{I} and $S_{\mathrm{II}}+S_{\mathrm{III}}$, for large τ_f. First, the behavior of S_{III} is obtained from Eq. (2.43) through the replacement $\sigma_0\to 2\pi-\sigma_0$:

$$S_{\mathrm{III}}\sim M^2\tau_f\frac{2\pi-\sigma_0}{2\pi}\left[1-\frac{1}{r^2(2\pi-\sigma_0)^2}\sin^2\left(\frac{r\sigma_0}{2}\right)\right]. \tag{2.45}$$

Then

$$S_{\mathrm{II}}+S_{\mathrm{III}}$$

$$\sim M^2\tau_f\left\{1-\frac{2}{\pi r^2}\left(\frac{1}{\sigma_0}+\frac{1}{2\pi-\sigma_0}\right)\sin^2\left(\frac{r\sigma_0}{2}\right)\right\} \tag{2.46}$$

and from Eqs. (2.39) and (2.46) we obtain

$$\Delta S = (S_{\rm II} + S_{\rm III}) - S_{\rm I} = -M^2 \tau_f \frac{2}{\pi r^2} \left(\frac{1}{\sigma_0} + \frac{1}{2\pi - \sigma_0} \right) \sin^2 \frac{r\sigma_0}{2}, \quad (2.47)$$

which is a negative quantity. Therefore, the string that splits sweeps a smaller area than the string that does not split. From Eq. (2.47) it also follows that the decrease in area $|\Delta S|$ vanishes for $r\sigma_0 = 2l\pi$. This corresponds to the splitting into two pieces containing an integer number of circumferences, which is equivalent to nonsplitting.

For the fundamental case $r = 1$, the relative decrease in area is

$$\eta_1(\sigma_0) = \frac{|\Delta S_1|}{S_{1,\rm I}} = \frac{2}{\pi} \left(\frac{1}{\sigma_0} + \frac{1}{2\pi - \sigma_0} \right) \sin^2 \left(\frac{\sigma_0}{2} \right), \quad (2.48)$$

which reaches its maximum value for $\sigma_0 = \pi$,

$$\eta_{1,\rm max} = \frac{4}{\pi^2} \simeq 0.405. \quad (2.49)$$

Thus, for the string configuration that we have chosen [Eq. (2.15)] the area swept by the strings that splits into two equally long pieces reduces to 60% of the area swept by the string that does not split.

III. STRING SPLITTING IN A SINGULAR PLANE WAVE BACKGROUND

We discuss now the splitting solution for a closed string in a strong gravitational field. We consider a D-dimensional singular gravitational plane wave described by the metric

$$ds^2 = \frac{\alpha}{U^2}(X^2 - Y^2) dU^2 - dU\, dV + \sum_{j=2}^{D-1} (dX^j)^2, \quad (3.1)$$

where $U = X^0 - X^1$, and $V = X^0 + X^1$ are light cone coordinates, $X^2 \equiv X$, $X^3 \equiv Y$, and α is a constant. This space-time is sourceless and has curvature on the null plane $U = 0$. In this space-time, the classical string equations of motion and constraints have been solved for the ordinary nonsplitting string in [5]. In this metric, the equation for U is simply $\partial^2 U = 0$. This allows us to take the light cone gauge exactly for all τ:

$$U = p\tau. \quad (3.2)$$

In this gauge the string equations for X and Y reduce to the linear equations

$$\left(-\partial_\tau^2 + \partial_\sigma^2 + \frac{\alpha}{\tau^2} \right) X = 0, \qquad \left(-\partial_\tau^2 + \partial_\sigma^2 - \frac{\alpha}{\tau^2} \right) Y = 0, \quad (3.3)$$

which can be solved by Fourier expanding $X(\sigma,\tau)$ and $Y(\sigma,\tau)$ in σ. Then, the τ-dependent Fourier coefficients $X_n(\tau)$ and $Y_n(\tau)$ express in terms of Bessel functions.

The remaining transverse coordinates $j = 4, \dots, D-1$ satisfy the flat space-time equations

$$\left(-\partial_\tau^2 + \partial_\sigma^2 \right) X^j = 0; \quad j = 4, \dots, D-1. \quad (3.4)$$

Finally, the longitudinal coordinate V is determined through the constraints

$$G_{AB} \partial_\pm X^A \partial_\pm X^B = 0, \quad (3.5)$$

which yield

$$p\partial_\sigma V = 2 \sum_{j=2}^{D-1} \partial_\tau X^j \partial_\sigma X^j, \qquad p\partial_\tau V = \frac{\alpha}{\tau^2}(X^2 - Y^2) + \sum_{j=2}^{D-1} \left\{ (\partial_\tau X^j)^2 + (\partial_\sigma X^j)^2 \right\}. \quad (3.6)$$

Let us describe now the splitting solution. We consider a generic string configuration evolving in the region of negative τ, i.e., before the string reaches the singularity at U=0, and splitting at a certain point (σ_0, τ_0) with $\tau_0 < 0$. We choose for the three strings $\mathcal{J}$ = I, II, III, a solution of the form

$$U^{(\mathcal{J})} = p\tau, \qquad X^{(\mathcal{J})} = \sum_{n=-\infty}^{\infty} X_n^{(\mathcal{J})}(\tau) \exp\left(i \frac{2\pi}{\lambda_{\mathcal{J}}} n\sigma \right), \qquad Y^{(\mathcal{J})} = 0, \qquad X^{j(\mathcal{J})} = 0, \quad j = 4, \dots, D-1, \quad (3.7)$$

with the string coordinate V determined through the constraints

$$p\partial_\sigma V^{(\mathcal{J})} = 2\partial_\tau X^{(\mathcal{J})} \partial_\sigma X^{(\mathcal{J})}, \qquad p\partial_\tau V^{(\mathcal{J})} = \frac{\alpha}{\tau^2}(X^{(\mathcal{J})})^2 + (\partial_\tau X^{(\mathcal{J})})^2 + (\partial_\sigma X^{(\mathcal{J})})^2. \quad (3.8)$$

The Fourier coefficients $X_n^{(\mathcal{J})}(\tau)$ are solutions of the equations

$$\ddot{X}_n^{(\mathcal{J})} + \left[\left(\frac{2\pi n}{\lambda_{\mathcal{J}}} \right)^2 - \frac{\alpha}{\tau^2} \right] X_n^{(\mathcal{J})} = 0, \quad (3.9)$$

which can be written in the form

$$X_n^{(\mathcal{J})}(\tau) = C_n^{(\mathcal{J})} \sqrt{-\tau} J_{-\nu}\left(-\frac{2\pi}{\lambda_{\mathcal{J}}} |n| \tau \right) + D_n^{(\mathcal{J})} \sqrt{-\tau} J_\nu \left(-\frac{2\pi}{\lambda_{\mathcal{J}}} |n| \tau \right) (n \neq 0),$$

$$X_0^{(\mathcal{J})}(\tau) = C_0^{(\mathcal{J})} (-\tau)^{1/2-\nu} + D_0^{(\mathcal{J})} (-\tau)^{1/2+\nu}, \quad (3.10)$$

where $J_{-\nu}$ and J_ν are Bessel functions with the index

$$\nu = \sqrt{\frac{1}{4} + \alpha} \tag{3.11}$$

and $\mathcal{J}$=I,II,III as defined above.

As in the flat space-time case, the functions $(U,V,X)^{(\mathrm{II})}$ and $(U,V,X)^{(\mathrm{III})}$ which describe the evolution of strings II and III, are fixed by the initial string $(U,V,X)^{(\mathrm{I})}$, and the matching conditions (2.2) and (2.3). However, since we are working now in the light cone gauge, the following two remarks are in order. First, the choice of the light cone gauge for string I, and the matching conditions (2.2) and (2.3) for U, imply that the light cone gauge holds for the pieces II and III as well, as stated in Eqs. (3.7). Second, the string coordinates $V^{(\mathrm{II})}$ and $V^{(\mathrm{III})}$ are determined through the constraints (3.8), instead of through the matching conditions (2.2) and (2.3). However, this is consistent because the matching conditions for the string coordinate X, together with the constraints (2.2) and (2.3), imply that $V^{(\mathcal{J})}$ (J=I,II,III) also satisfy the matching conditions (2.2) and (2.3).

Let us choose now a specific initial configuration for the string coordinate $X^{(\mathrm{I})}$ given by

$$X^{(\mathrm{I})} = \frac{k}{r} f_\nu(\tau) \cos r\sigma, \tag{3.12}$$

where

$$f_\nu(\tau) = r\sqrt{\tau\tau_0}[J_\nu(-r\tau_0)J_{-\nu}(-r\tau) - J_{-\nu}(-r\tau_0)J_\nu(r\tau)]. \tag{3.13}$$

This describes a straight string along the X axis. According to Eq. (3.10) we have chosen the constants $C_n^{(\mathrm{I})}$ and $D_N^{(\mathrm{I})}$ in the form

$$\begin{aligned} C_r^{(\mathrm{I})} &= C_{-r}^{(\mathrm{I})} = \frac{k}{2}\sqrt{-\tau_0}J_\nu(-r\tau_0), \\ D_r^{(\mathrm{I})} &= D_{-r}^{(\mathrm{I})} = \frac{k}{2}\sqrt{-\tau_0}J_{-\nu}(-r\tau_0), \\ C_n^{(\mathrm{I})} &= D_n^{(\mathrm{I})} = 0 \quad \text{for } n \neq \pm r. \end{aligned} \tag{3.14}$$

This initial string configuration $(U,V,X)^{(\mathrm{I})}$ that we have chosen must satisfy the splitting consistency condition (2.8). For U [Eq.(3.2)] this condition is trivially satisfied. On the other hand, Eqs. (3.12) and (3.13) yield

$$f_\nu(\tau_0) = 0. \tag{3.15}$$

Thus

$$X^{(\mathrm{I})}(\sigma,\tau_0) = 0 \tag{3.16}$$

and so the condition (2.8) holds for the X string coordinate. Finally, using Eqs. (3.12) and (3.15) in the constraints (3.8), we obtain

$$\begin{aligned} p\partial_\sigma V^{(\mathrm{I})}(\sigma,\tau) &= -k\dot{f}_\nu(\tau)f_\nu(\tau)\sin(2r\sigma), \\ p\partial_\sigma V^{(\mathrm{I})}(\sigma,\tau_0) &= 0 , \end{aligned} \tag{3.17}$$

i.e., $V^{(\mathrm{I})}(\sigma,\tau_0)$ is independent of σ and also satisfies the consistency condition (2.8).

We can determine now the constants $(C_n^{(\mathrm{II})}, D_n^{(\mathrm{II})})$ and $(C_n^{(\mathrm{III})}, D_n^{(\mathrm{III})})$ from the initial data $(C_n^{(\mathrm{I})}, D_n^{(\mathrm{I})})$, by using the matching conditions (2.2) and (2.3). These are

$$\begin{aligned} C_0^{(\mathrm{II})} &= K\frac{\sin\nu\pi}{\nu\pi}(-\tau_0)^{1/2+\nu}\frac{\sin r\sigma_0}{r\sigma_0}, \\ C_n^{(\mathrm{II})} &= K\sqrt{-\tau_0}J_\nu\left(-\frac{2\pi}{\sigma_0}|n|\tau_0\right)\phi_n(\sigma_0) \quad \text{for } n\neq 0, \\ D_0^{(\mathrm{II})} &= -K\frac{\sin\nu\pi}{\nu\pi}(-\tau_0)^{1/2-\nu}\frac{\sin r\sigma_0}{r\sigma_0}, \\ D_n^{(\mathrm{II})} &= -K\sqrt{-\tau_0}J_{-\nu}\left(-\frac{2\pi}{\sigma_0}|n|\tau_0\right)\phi_n(\sigma_0) \quad \text{for } n\neq 0, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} C_0^{(\mathrm{III})} &= -K\frac{\sin\nu\pi}{\nu\pi}(-\tau_0)^{1/2+\nu}\frac{\sin r\sigma_0}{r\sigma_0}, \\ C_n^{(\mathrm{III})} &= K\exp\left(-i\frac{2\pi}{2\pi-\sigma_0}n\sigma_0\right)\sqrt{-\tau_0}J_\nu\left(-\frac{2\pi}{2\pi-\sigma_0}|n|\tau_0\right)\phi_n^*(2\pi-\sigma_0) \quad \text{for } n\neq 0, \\ D_0^{(\mathrm{III})} &= K\frac{\sin\nu\pi}{\nu\pi}(-\tau_0)^{1/2-\nu}\frac{\sin r\sigma_0}{r\sigma_0}, \\ D_n^{(\mathrm{III})} &= -K\exp\left(-i\frac{2\pi}{2\pi-\sigma_0}n\sigma_0\right)\sqrt{-\tau_0}J_{-\nu}\left(-\frac{2\pi}{2\pi-\sigma_0}|n|\tau_0\right)\phi_n^*(2\pi-\sigma_0) \quad \text{for } n\neq 0. \end{aligned} \tag{3.19}$$

Hence, the Fourier expansions (3.7) for the X coordinates of strings II and III read

$$X^{(\mathrm{II})}(\sigma,\tau) = X_0^{(\mathrm{II})}(\tau) + \frac{\sigma_0}{2\pi}k\sum_{n\neq 0}\left\{\frac{1}{|n|}F_\nu\left(\frac{2\pi}{\sigma_0}|n|\tau, \frac{2\pi}{\sigma_0}|n|\tau_0\right)\phi_n(\sigma_0)\exp\left(i\frac{2\pi}{\sigma_0}n\sigma\right)\right\} \tag{3.20}$$

and

$$X^{(\mathrm{III})}(\sigma,\tau) = X_0^{(\mathrm{III})}(\tau) + \frac{2\pi-\sigma_0}{2\pi}k\sum_{n\neq 0}\left\{\frac{1}{|n|}F_\nu\left(\frac{2\pi}{2\pi-\sigma_0}|n|\tau, \frac{2\pi}{2\pi-\sigma_0}|n|\tau_0\right)\phi_n^*(2\pi-\sigma_0)\right.$$
$$\left.\times\exp\left(i\frac{2\pi}{2\pi-\sigma_0}n(\sigma-\sigma_0)\right)\right\}, \tag{3.21}$$

where

$$F_\nu(u,v) = \sqrt{uv}[J_\nu(-v)J_{-\nu}(-u) - J_{-\nu}(-v)J_\nu(-u)] \tag{3.22}$$

and

$$X_0^{(\mathrm{II})}(\tau) = -X_0^{(\mathrm{III})}(\tau) = k\frac{\sin\nu\pi}{\nu\pi}\frac{\sin r\sigma_0}{r\sigma_0}\sqrt{\tau_0\tau}\left[\left(\frac{\tau_0}{\tau}\right)^\nu - \left(\frac{\tau}{\tau_0}\right)^\nu\right]. \tag{3.23}$$

Let us discuss now the properties of the splitting string solution given by (3.20) and (3.21). First, we notice that, from Eq. (2.26),

$$X^{(\mathrm{II})}(\sigma,\tau) \underset{\sigma_0\to 2\pi}{\sim} X^{(\mathrm{I})}(\sigma,\tau),$$
$$X^{(\mathrm{III})}(\sigma,\tau) \underset{\sigma_0\to 0}{\sim} X^{(\mathrm{I})}(\sigma,\tau). \tag{3.24}$$

That is, the splitting solution with strings II and III contains the one string nonsplitting solution as a particular case. In addition, for

$$r\sigma_0 = 2l\pi;\quad l = 1,\ldots,r-1 \tag{3.25}$$

the Fourier series (3.20) and (3.21) sum up to $X^{(\mathrm{I})}(\sigma,\tau)$. Thus again, cutting the string by an integer number of windings is equivalent to not cutting it at all.

Let us study now the string action. We want to compare the area S_{I} swept by the string without splitting, with the areas S_{II} and S_{III} swept by the strings II and III. We consider the evolution of the three strings for the same τ interval

$$\tau_0 \le \tau \le \tau_f < 0 \tag{3.26}$$

in the ingoing region, where the string has not yet reached the singularity at $\tau = 0$. We shall compute the action for a long τ interval, i.e., $\tau_f - \tau_0 \gg 2\pi$, as we did in flat space-time. However, in this case we shall implement this approximation by letting $\tau_0 \to -\infty$, and allowing $\tau_f \to 0^-$ in order to incorporate the effect of the space-time singularity at $\tau = 0$.

In the space-time (3.1) the string action is

$$S = \int\int_{\mathrm{WS}} d\tau\, d\sigma\left\{\frac{\alpha}{U^2}(X^2 - Y^2)\partial_a U\partial^a U - \partial_a U\partial^a V + \sum_{j=2}^{D-1}\partial_a X^j\partial^a X^j\right\}. \tag{3.27}$$

In the light cone gauge, using the constraints (3.8) and for the particular string configuration (3.7), the action for the three strings takes the form

$$S_{\mathcal{J}} = \frac{1}{\pi}\int_{\tau_0}^{\tau_f} d\tau\int_0^{\lambda_{\mathcal{J}}} d\sigma(\partial_\sigma X^{(\mathcal{J})})^2, \tag{3.28}$$

where $\mathcal{J}$ =I,II,III. Then, using Eqs. (3.12), (3.20), and (3.22), we have

$$S_{\mathrm{I}} = k^2\int_{\tau_0}^{\tau_f} d\tau\, F_\nu^2(r\tau, r\tau_0) \tag{3.29}$$

and

$$S_{\mathrm{II}} = k^2\frac{\sigma_0}{\pi}\int_{\tau_0}^{\tau_f} d\tau\sum_{n\neq 0}\left|F_\nu\left(\frac{2\pi}{\sigma_0}|n|\tau, \frac{2\pi}{\sigma_0}|n|\tau_0\right)\right|^2 \times|\phi_n(\sigma_0)|^2. \tag{3.30}$$

From Eq. (2.27) we see that $S_{\mathrm{II}} \to S_{\mathrm{I}}$ when $\sigma_0 \to 2\pi$ as it should be.

We shall do the comparison of the two areas S_{I} and $S_{\mathrm{II}}+S_{\mathrm{III}}$ in two regimes: first for α=0 which corresponds to flat space-time, and then for $\alpha \ge \frac{3}{4}$ which corresponds to a strong gravitational wave.

For α=0, the index ν is $\frac{1}{2}$ and the Bessel functions reduce to circular functions:

$$F_{1/2}\left(\frac{2\pi}{\sigma_0}|n|\tau, \frac{2\pi}{\sigma_0}|n|\tau_0\right) = \frac{2}{\pi}\sin\left(\frac{2\pi}{\sigma_0}|n|(\tau-\tau_0)\right). \tag{3.31}$$

Then, for $\tau_0 \to -\infty$, Eqs. (3.29) and (3.30) yield

$$S_{\rm I}^{(\alpha=0)} = \frac{4k^2}{\pi^2}\int_{\tau_0}^{\tau_f} d\tau \, \sin^2 r(\tau-\tau_0) \sim \frac{2k^2}{\pi^2}(\tau_f-\tau_0) \tag{3.32}$$

and

$$S_{\rm II}^{(\alpha=0)} = \frac{4k^2}{\pi^3}\sigma_0 \int_{\tau_0}^{\tau_f} d\tau \sum_{n\neq 0} \sin^2\left(\frac{2\pi}{\sigma_0}|n|(\tau-\tau_0)\right) \times |\phi_n(\sigma_0)|^2$$
$$\sim \frac{2k^2}{\pi^2}(\tau_f-\tau_0)\frac{\sigma_0}{\pi}\sum_{n\neq 0}|\phi_n(\sigma_0)|^2 \tag{3.33}$$
$$= \frac{2k^2}{\pi^2}(\tau_f-\tau_0)\frac{\sigma_0}{\pi}\left[1+\frac{\sin 2r\sigma_0}{2r\sigma_0}-\frac{2\sin^2 r\sigma_0}{r^2-\sigma_0^2}\right].$$

In addition, replacing $\sigma_0 \to 2\pi - \sigma_0$, we have

$$S_{\rm III}^{(\alpha=0)} \sim \frac{2k^2}{\pi^2}(\tau_f-\tau_0)\frac{2\pi-\sigma_0}{2\pi}\left[1-\frac{\sin 2r\sigma_0}{2r(2\pi-\sigma_0)} -\frac{2\sin^2 r\sigma_0}{r^2(2\pi-\sigma_0)^2}\right]. \tag{3.34}$$

Thus

$$\Delta S^{(\alpha=0)} = (S_{\rm II}+S_{\rm III}) - S_{\rm I} = -\frac{1}{\pi r^2}\left(\frac{1}{\sigma_0}+\frac{1}{2\pi-\sigma_0}\right)\sin^2 r\sigma_0 S_{\rm I}^{(\alpha=0)}, \tag{3.35}$$

which is a negative quantity. Again, the string that splits has smaller action than the nonsplitting one. However, the result (3.35) is different from (2.47), because we have here a different string configuration. In particular, the action difference (3.35) vanishes in the present case for

$$r\sigma_0 = (2s+1)\pi, \quad s = 0,1\ldots,\left[\frac{r-1}{2}\right]; \tag{3.36}$$

i.e., it vanishes not only for an integer number of windings $r\sigma_0 = 2l\pi$, $l = 0,1\ldots,r$, but also for a half-integer number of windings. This is so because the barycentric term [Eq. (3.23)] vanish for an integer or half-integer number of windings. However, the Fourier expansions (3.20) and (3.21) sum up to $X^{(I)}(\sigma,\tau)$ for an integer number of windings, but do not for a half-integer number of windings. This happens here because when σ_0 corresponds to a half-integer number of windings, we have a straight string configuration with $X'(0,\tau_0) = X'(\sigma_0,\tau_0) = 0$. Hence the initial closed string may split into two open strings. Thus, in this case the strings II and III are open strings that stay together and change their respective shapes compared with string I.

For the fundamental case r=1, the relative decrease in area is

$$\eta_1^{(\alpha=0)}(\sigma_0) = \frac{|\Delta S_1|}{S_{1,\rm I}} = \frac{1}{\pi}\left(\frac{1}{\sigma_0}+\frac{1}{2\pi-\sigma_0}\right)\sin^2\sigma_0, \tag{3.37}$$

which has two symmetric maxima at the points $\sigma_0 = 1.291$ and $\sigma_0 = 2\pi - 1.291$ with a height

$$\eta_{1,\rm max} = 0.287. \tag{3.38}$$

Notice that in this case $\sigma_0 = \pi$ gives a minimum with $\eta_1(\pi)$=0, corresponding to the splitting of the closed string into two open strings.

Let us turn now to the discussion of the regime $\alpha \geq \frac{3}{4}$ ($\nu \geq 1$). In this case, the integral (3.29) diverges in the limit $\tau_f \to 0^-$. Thus, for $\nu \geq 1$ the behavior of $S_{\rm I}$ for $\tau_f \to 0^-$ is dominated by the upper limit in the integral (3.29). That is, the most important contribution to $S_{\rm I}$ comes from the region near the singularity. It is in this sense that we talk of a strong enough gravitational wave for $\alpha \geq \frac{3}{4}$. For $\tau_f \to 0^-$ we have

$$S_{\rm I} \sim k^2 \frac{2^{2\nu}\tau_0}{\Gamma^2(1-\nu)(2-2\nu)r^{2\nu-2}} J_\nu^2(-r\tau_0)\left(\frac{\tau_f}{\tau_1}\right)^{2-2\nu}, \tag{3.39}$$

where τ_1 is an intermediate point in the interval $\tau_0 < \tau_1 < \tau_f < 0$, and we assume that τ_0 is such that

$$J_\nu^2(-r\tau_0) \neq 0. \tag{3.40}$$

Let us consider the series in the integrand of Eq. (3.30). The terms with $|n\tau| \ll 1$ behave as

$$\left|F_\nu\left(\frac{2\pi}{\sigma_0}|n|\tau, \frac{2\pi}{\sigma_0}|n|\tau_0\right)\right|^2 \sim \frac{2^{2\nu}}{\Gamma^2(1-\nu)}\left(\frac{2\pi|n|}{\sigma_0}\right)^{2-2\nu}(-\tau_0)J_\nu^2\left(-\frac{2\pi}{\sigma_0}|n|\tau_0\right)(-\tau)^{1-2\nu}. \tag{3.41}$$

For $|n\tau| \geq 1$, and small enough $|-\tau|$(i.e., $|n| \gg 1$), the terms in the series of Eq. (3.30) are very much suppressed because for $n \to \infty$, $|\phi_n(\sigma_0)|^2 \sim 1/n^2$. Therefore, the behavior of the series for $\tau \to 0_-$ is

$$\sum_{n\neq 0}\left|F_\nu\left(\frac{2\pi}{\sigma_0}|n|\tau,\frac{2\pi}{\sigma_0}|n|\tau_0\right)\right|^2|\phi_n(\sigma_0)|^2\sim\frac{2^{2\nu}}{\Gamma^2(1-\nu)}\left(\frac{2\pi}{\sigma_0}\right)^{2-2\nu}(-\tau_0)-\tau^{1-2\nu}\sum_{n\neq 0}\frac{J_\nu^2\left(-\frac{2\pi}{\sigma_0}|n|\tau_0\right)}{|n|^{2\nu-2}}|\phi_n(\sigma_0)|^2. \quad (3.42)$$

Then, for $\nu\geq 1$ and $\tau_0\to 0_-$ the behavior of $S_{\rm II}$ is dominated by the upper limit of the integral. So, inserting Eq. (3.42) into Eq. (3.30) and taking into account Eq. (3.39) we get

$$S_{\rm II}\sim\frac{\sigma_0}{\pi}\left(\frac{2\pi}{\sigma_0}\right)^{2-2\nu}\frac{r^{2\nu-2}}{J_\nu^2(-r\tau_0)}\sum_{n\neq 0}\frac{J_\nu^2\left(-\frac{2\pi}{\sigma_0}|n|\tau_0\right)}{|n|^{2\nu-2}}|\phi_n(\sigma_0)|^2S_{\rm I}. \quad (3.43)$$

Now we take the long τ interval approximation by doing $\tau_0\to-\infty$ in (3.43). Thus

$$S_{\rm II}\sim\frac{\sigma_0}{\pi}\left(\frac{\sigma_0}{2\pi}\right)^{2\nu-1}\sum_{n\neq 0}\left(\frac{r}{|n|}\right)^{2\nu-1}\frac{\cos^2\left(-\frac{2\pi}{\sigma_0}|n|\tau_0-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}{\cos^2\left(-r\tau_0-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)}|\phi_n(\sigma_0)|^2S_{\rm I}. \quad (3.44)$$

The factor

$$\frac{1}{\cos^2\left(-r\tau_0-\frac{\pi}{2}\nu-\frac{\pi}{4}\right)} \quad (3.45)$$

comes from the reciprocal of the Bessel functions $J_\nu^2(-r\tau_0)$ for $\tau_0\to-\infty$, entering in (3.43), and which was assumed not to vanish. In particular we can choose τ_0 in such a way that the factor (3.45) is 1. This yields an upper bound estimate of the action $S_{\rm II}$ of the form

$$S_{\rm II}\leq\left(\frac{\sigma_0}{2\pi}\right)^{2\nu-1}\frac{\sigma_0}{\pi}\sum_{n\neq 0}\left(\frac{r}{|n|}\right)^{2\nu-1}|\phi_n(\sigma_0)|^2S_{\rm I} \quad (3.46)$$

and for string III we have

$$S_{\rm III}\leq\left(\frac{2\pi-\sigma_0}{2\pi}\right)^{2\nu-1}\frac{2\pi-\sigma_0}{\pi}\sum_{n\neq 0}\left(\frac{r}{|n|}\right)^{2\nu-1}|\phi_n(2\pi-\sigma_0)|^2S_{\rm I}. \quad (3.47)$$

Notice that this upper bound becomes exact for $\sigma_0=0$ and $\sigma_0=2\pi$ (the nonsplitting solution).

In order to get a better insight on the behavior of ΔS, we choose $\alpha=2$ ($\nu=\frac{3}{2}$), in which case, the series (3.46) and (3.47) can be summed in closed form. For α=2 we have

$$S_{\rm II}^{(\alpha=2)}\leq\frac{\sigma_0}{2\pi}\left(1+\frac{\sin 2r\sigma_0}{2r\sigma_0}-\frac{2\sin^2 r\sigma_0}{r^2\sigma_0^2}+J+\frac{\sin^2 r\sigma_0}{6}+\frac{2\sin r\sigma_0}{r\sigma_0}-\frac{8\sin^2\frac{1}{2}r\sigma_0}{r^2\sigma_0^2}\right)S_{\rm I}. \quad (3.48)$$

Thus

$$\begin{aligned}\Delta S^{(\alpha=2)}&=(S_{\rm II}^{(\alpha=2)}+S_{\rm III}^{(\alpha=2)})-S_{\rm I}^{(\alpha=2)}\\&\leq\left\{-\frac{1}{\pi r^2}\left(\frac{1}{\sigma_0}+\frac{1}{2\pi-\sigma_0}\right)\sin^2 r\sigma_0+\frac{\sin^2 r\sigma_0}{6}-\frac{4}{\pi r^2}\left(\frac{1}{\sigma_0}+\frac{1}{2\pi-\sigma_0}\right)\sin^2\frac{r\sigma_0}{2}\right\}S_{\rm I}. \quad (3.49)\end{aligned}$$

For the fundamental case $r=1$, the action difference takes the form

$$\Delta S_1^{(\alpha=2)}=\left\{\left[\frac{1}{6}-\frac{1}{\pi}\left(\frac{1}{\sigma_0}+\frac{1}{2\pi-\sigma_0}\right)\right]\sin^2\sigma_0-\frac{4}{\pi}\left(\frac{1}{\sigma_0}+\frac{1}{2\pi-\sigma_0}\right)\sin^2\frac{\sigma_0}{2}\right\}S_{1,{\rm I}}, \quad (3.50)$$

which is a negative quantity for all values of σ_0 in the interval $[0,2\pi]$. Thus, in the background of the singular gravitational plane wave ($\alpha=2$), the action of the splitting solution is smaller than the action of the nonsplitting one. Moreover, this effect is magnified by an overall divergent τ_f dependent factor when we approach the singularity at $\tau=0$, because the action $S_{\rm I}^{(\alpha=2)}$ is multiplied by such a factor in this limit [Eq. (3.39)]. In addition, the effect of smaller action for the splitting solution is also increased in relative terms, as a consequence of the new terms appearing in (3.50). This is easily seen in terms of the lower bound that we have for the relative decrease in area. According to (3.50),

$$\eta_1^{(\alpha=2)}(\sigma_0) = \frac{|\Delta S_1^{(\alpha=2)}|}{S_{1,\mathrm{I}}^{(\alpha=2)}} \geq h(\sigma_0), \tag{3.51}$$

where

$$h(\sigma_0) = \left[\frac{1}{\pi}\left(\frac{1}{\sigma_0} + \frac{1}{2\pi - \sigma_0}\right) - \frac{1}{6}\right]\sin^2\sigma_0 + \frac{4}{\pi}\left(\frac{1}{\sigma_0} + \frac{1}{2\pi - \sigma_0}\right)\sin^2\frac{\sigma_0}{2}. \tag{3.52}$$

The lower bound $h(\sigma_0)$ has its maximum at $\sigma_0 = \pi$ with a height h_{max}=0.811. Therefore

$$\eta_{1,\mathrm{max}}^{(\alpha=2)} > 0.811 \tag{3.53}$$

which is much larger than the relative decrease in area for the same string configuration in flat space-time, given in (3.38).

IV. CONCLUDING REMARKS

In this paper we have explored a new type of solutions of the string equations of motion and constraints. We are mainly thinking in the theory of fundamental strings. We show that the solutions in which the string splits into two are perfectly natural within the classical theory of strings based on the Polyakov action. There is no need of extra interactions; i.e., extra terms in the action to produce the splitting. The only difference with the nonsplitting solutions are the boundary conditions. In this sense the choice of a splitting or a nonsplitting solution is somehow similar to the choice between an open or closed string. Of course, open and closed strings can be used to describe different physics and this can also be true for the splitting and nonsplitting string solutions. In particular, these solutions could be relevant for the description of cosmic strings breaking.

The solutions that we have constructed describe the splitting of strings as a natural decay process that takes place in real (Lorentzian signature) space-time. This process occurs at the classical level and this is natural because the string is an extended object. The splitting solutions are already present in flat space-time, and they correspond to stationary points of the action (area) with lower value than the nonsplitting strings. In order to explore the effect of a gravitational field on the splitting solutions, we have considered a gravitational singular plane wave background. In the case that we analyze, the gravitational field produces an enhancement of the effect of smaller area for the splitting solution. It would be interesting to settle to what extent these results are universal, and work in this direction is now in progress by the present authors.

On the other hand, string splitting is usually considered and discussed within a quantum formulation, namely, the Euclidean path integral functional approach to the quantum string scattering amplitudes in the light-cone gauge [7]. In this context, the stationary points of the Euclidean action correspond to solutions of Dirichlet or Neumann boundary value problems for elliptic operators (Laplacians) on bordered Riemann surfaces [8]; i.e., the classical string equations of motion and constraints solve the boundary value problems for elliptic operators with Dirichlet or Neumann boundary conditions. Of course, these are different from the solutions considered in this paper, in which we solve the hyperbolic (Lorentzian) evolution equations for the Cauchy data $X^A(\sigma,\tau_0)$ and $\dot{X}^A(\sigma,\tau_0)$ with some fixed topology. This topology should be viewed as enforced by the world-sheet metric used to construct the D'Alambertian operator. Notice that quantum mechanically the initial data $X^A(\sigma,\tau_0)$ and $\dot{X}^A(\sigma,\tau_0)$ cannot be given simultaneously. Instead, one gives the initial and final string shapes to compute a transition amplitude between them.

Although our splitting solutions are purely classical, string splitting for massive strings is also present at the quantum level. The relevant magnitude to be computed in that case is the probability amplitude for such a process. In fact, such probability has been computed in [6] for flat space-time. It would be very interesting to explore the relationships between the classical splitting solutions and the quantum probability for string disintegration, and also the effect of a gravitational field on such probability. The quantum probability amplitude for string splitting in a singular plane wave will be discussed by the present authors in a forthcoming paper.

We conclude with a final remark concerning the classical solutions of the string equations of motion in curved space-times. It has been established (see, for instance, [9] and references therein) that these solutions present a phenomenon of indefinitely string stretching near space-time singularities, due to the absorption by the string of energy from the background gravitational field. Of course, there must be a mechanism that avoids this indefinite string growing, and indeed the strings can radiate away energy by emitting gravitons or other particlelike excitations. However, another natural mechanism to avoid string growing is string splitting, and it would be interesting to elucidate its quantitative relevance to avoid the indefinite stretching of strings in strong gravitational fields.

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