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# Gauge-independent trace anomaly for gravitons

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We show that the trace anomaly for gravitons calculated using the usual effective action formalism depends on the choice of gauge when the background spacetime is not a solution of the classical equation of motion, that is, when off shell. We then use the gauge-independent Vilkovisky-DeWitt effective action to restore gauge independence to the off-shell case. Additionally we explicitly evaluate trace anomalies for some $N$-sphere background spacetimes.

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## I. INTRODUCTION

The usual effective action, in general, depends on the choice of quantum gauge fixing when the background field is not a solution of the classical equations of motion, that is, when the background is off shell [1]. This has posed a problem in the use of the effective action formalism to study, for example, the spontaneous compactification of Kaluza-Klein spaces [2,3] (see [4] for additional references). The same gauge dependence problem exists in the calculation of the trace anomaly for gravitons [5] because both the background spacetime and the graviton fields stem from the metric. When the background spacetime is not a solution of the Einstein equations, the trace anomaly calculated from the usual effective action may depend on the gauge choice. We illustrate this explicitly in Sec. II for the simple case of Einstein gravity with cosmological constant in a flat background.

To overcome the gauge dependence we use the Vilkovisky-DeWitt (VD) effective action formalism [6,7]. The VD formalism has been applied to spontaneous compactification of Kaluza-Klein spaces and unique answers which are independent of the choice of gauge have been obtained [8–11]. Recently, it has also been used to study two-dimensional (2D) quantum gravity [12] and even gravity–grand unified theory (-GUT) unifications [13]. In Sec. III we define the unique trace anomaly for gravitons using the one-loop VD effective action. We evaluate this VD trace anomaly for the case considered in Sec. II and show that it is indeed independent of gauge choice.

For most gauges the VD effective action involves evaluating the determinants or the $\zeta$ functions of complicated nonlocal operators. However, the calculation simplifies when the Landau-DeWitt gauge is used. Since the VD effective action is independent of gauge choice, one can of course choose whatever gauge is convenient without altering the final results. In this particular gauge the operators become local, but remain nonminimal [14]. In a previous paper [15] we devised a method to evaluate the $\zeta$ functions (at argument 0) of nonminimal vector and tensor operators on maximally symmetric spaces. In Sec. IV we use this method to calculate the VD trace anomalies for such background spaces. Explicit results for $N$-spheres and Euclidean spaces of dimensions 4, 6, 8, and 10 are given. Finally, conclusions are given in Sec. V.

## II. GAUGE DEPENDENCE OF THE TRACE ANOMALY

In this section we demonstrate the dependence of the trace anomaly for gravitons on the quantum gauge choice. To do so we consider the simple case of Einstein gravity with a cosmological constant in a flat background spacetime. The corresponding action is (in Euclidean signature)

$$S \equiv \int d^4x \mathcal{L},$$

where

$$\mathcal{L} = -\sqrt{\bar{g}}(\bar{R} - 2\Lambda) \tag{1}$$

[see Eqs. (72)–(74) for curvature conventions]. The metric is split into its background and quantum parts:

$$\bar{g}_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu} \ , \tag{2}$$

where $h_{\mu\nu}$ is the graviton field. To evaluate the trace anomaly for gravitons, we expand the Lagrangian in powers of $h_{\mu\nu}$ keeping only the quadratic part:

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$$\mathcal{L}_{\rm quad} = \tfrac{1}{4}h_{\mu\nu,\rho}h_{\mu\nu,\rho} - \tfrac{1}{8}h_{,\rho}h_{,\rho} - \tfrac{1}{2}(h_{\mu\rho,\mu} - \tfrac{1}{2}h_{,\rho})(h_{\rho\nu,\nu} - \tfrac{1}{2}h_{,\rho}) - \tfrac{1}{2}(h_{\mu\nu}^2 - \tfrac{1}{2}h^2)\Lambda \ , \tag{3}$$

where $h \equiv h_{\mu\mu}$. Next, we introduce the gauge-fixing term and the corresponding ghost term. We choose a one-parameter ($\alpha$) family of covariant gauges:

$$\mathcal{L}_{\rm GF} = \tfrac{1}{2}\alpha(h_{\mu\rho,\mu} - \tfrac{1}{2}h_{,\rho})(h_{\rho\nu,\nu} - \tfrac{1}{2}h_{,\rho}) \ . \tag{4}$$

The corresponding ghost Lagrangian is

$$\mathcal{L}_{\rm gh} = \overline{\eta}_\mu(-\partial^2)\eta_\mu \ , \tag{5}$$

where $\overline{\eta}_\mu$ and $\eta_\mu$ are vector ghosts. Putting the Lagrangians Eqs. (3)–(5) together, we have the quantum Lagrangian for gravitons:

$$\begin{aligned}\mathcal{L}_q = {}& \frac{1}{4}h_{\mu\nu}(-\partial^2 - 2\Lambda)h_{\mu\nu} - \frac{1}{4}h\left[\left(1 - \frac{1}{2\alpha}\right)(-\partial^2) - \Lambda\right]h \\ &+\frac{1}{2}h_{\mu\nu}\left(1 - \frac{1}{\alpha}\right)\partial_\nu\partial_\rho h_{\mu\rho} - \frac{1}{2}h\left(1 - \frac{1}{\alpha}\right)\partial_\mu\partial_\nu h_{\mu\nu} + \overline{\eta}_\mu(-\partial^2)\eta_\mu \ .\end{aligned} \tag{6}$$

We use $\zeta$-function regularization to evaluate the trace anomaly. The $\zeta$ function of an operator $M$ is defined by

$$\zeta_M(s) \equiv \sum_\lambda \lambda^{-s} \ , \tag{7}$$

where $\lambda$'s are the eigenvalues of the operator $M$. Thus we first have to find the eigenvalues of the operators acting on $h_{\mu\nu}$ and on the ghost fields from Eq. (6). To do so we rewrite this Lagrangian as

$$\mathcal{L}_q = \frac{1}{2}\psi_i\Theta_{ij}\psi_j + \overline{\eta}^\mu(-\partial^2)\eta_\mu \ , \tag{8}$$

where $\psi_i$, $i = 1, ..., 10$, are the ten independent components of $h_{\mu\nu}$. The eigenvalues of the matrix $\Theta_{ij}$ are [16]

$$\lambda_1 = \lambda_2 = \lambda_3 = k^2 - 2\Lambda \ , \tag{9}$$

$$\lambda_4 = \lambda_5 = \frac{1}{2}(k^2 - 2\Lambda) \ , \tag{10}$$

$$\lambda_6 = \lambda_7 = \lambda_8 = \frac{1}{\alpha}(k^2 - 2\alpha\Lambda) \ , \tag{11}$$

$$\lambda_9 = \frac{1}{2\alpha}\left\{(1-\alpha)k^2 + \sqrt{(1-\alpha+\alpha^2)k^4 - 2\alpha\Lambda(1+\alpha)k^2 + 4\alpha^2\Lambda^2}\right\} \ , \tag{12}$$

$$\lambda_{10} = \frac{1}{2\alpha}\left\{(1-\alpha)k^2 - \sqrt{(1-\alpha+\alpha^2)k^4 - 2\alpha\Lambda(1+\alpha)k^2 + 4\alpha^2\Lambda^2}\right\} \ , \tag{13}$$

having been written in momentum space. Hence, the $\zeta$ function for the graviton field $h_{\mu\nu}$ is

$$\zeta_{\rm gr}(s) = \sum_{i=1}^{10}\sum_k \lambda_i^{-s} \ , \tag{14}$$

and the $\zeta$ function for the ghost fields is,

$$\zeta_{\rm gh}(s) = \sum_k (k^2)^{-s} \ . \tag{15}$$

Following the arguments similar to [5] and [17], for example, it is easy to see that the trace anomaly for gravitons is given by

$$\langle T_\mu{}^\mu\rangle = \frac{1}{V_4}\left[\zeta_{\rm gr}(0) - 2\zeta_{\rm gh}(0)\right] \ , \tag{16}$$

where $V_4$ is the volume of the spacetime. $\zeta(0)$ can be calculated as follows. For a general eigenvalue $ak^2 + b$, where $a$ and $b$ are constants,

$$\begin{aligned}\zeta(0) &= \lim_{s\to 0}\sum_k (ak^2 + b)^{-s} \\ &= \lim_{s\to 0} V_4\int\frac{d^4k}{(2\pi)^4}\frac{1}{\Gamma(s)}\int_0^\infty d\tau\ \tau^{s-1}e^{-\tau(ak^2+b)} \ .\end{aligned} \tag{17}$$

Evaluating the $k$ integral first and then the $\tau$ integral,

$$\begin{aligned}\zeta(0) &= \lim_{s\to 0}\frac{V_4}{\Gamma(s)}\int_0^\infty d\tau\ \tau^{s-1}e^{-\tau b}\left(\frac{1}{4\pi a\tau}\right)^2 \\ &= \lim_{s\to 0}\frac{V_4}{(4\pi)^2}\frac{\Gamma(s-2)}{\Gamma(s)}\frac{b^{2-s}}{a^2} \\ &= \frac{V_4}{(4\pi)^2}\left(\frac{1}{2}\right)\left(\frac{b}{a}\right)^2 \ .\end{aligned} \tag{18}$$

Using this result, we see that, for $\lambda_1$–$\lambda_8$ in Eqs. (9)–(11),

$$\lim_{s\to 0}\sum_{i=1}^{8}\sum_{k}\lambda_i^{-s} = \frac{V_4}{(4\pi)^2}\left(\frac{1}{2}\right)\left[3(-2\Lambda)^2+2(-2\Lambda)^2 +3(-2\alpha\Lambda)^2\right]$$
$$= \frac{V_4}{(4\pi)^2}(5+3\alpha^2)(2\Lambda^2)\ , \quad (19)$$

and, for Eq. (15),

$$\zeta_{\rm gh}(0)=0\ . \quad (20)$$

For $\lambda_9$ and $\lambda_{10}$ it is more difficult to evaluate the $k$ integral because the eigenvalues are not polynomials in $k^2$. However, we are only interested in the $\zeta$ functions at $s=0$, and this depends only on the small $\tau$ behavior in the integrand, behavior which is much like the $\tau$ integral in Eq. (17). To extract the small $\tau$ behavior from the integral over $k$, we need only concentrate on the large $k$ behavior. Hence, we can expand $\lambda_9$ and $\lambda_{10}$ as power series in $1/k^2$, and then evaluate the integrals to obtain $\zeta(0)$. This is done in detail in [15]. Following the procedures there, we have

$$\lim_{s\to 0}\sum_{k}\left(\lambda_9^{-s}+\lambda_{10}^{-s}\right) = \frac{V_4}{(4\pi)^2}(1+\alpha^2)(2\Lambda^2)\ . \quad (21)$$

Combining the results in Eqs. (19) and (21), we have

$$\zeta_{\rm gr}(0) = \frac{V_4}{(4\pi)^2}(3+2\alpha^2)(4\Lambda^2)\ . \quad (22)$$

Then from Eq. (16) the trace anomaly for gravitons in a flat background spacetime with the one-parameter covariant gauge of Eq. (4) is

$$\langle T_\mu{}^\mu\rangle = \frac{1}{(4\pi)^2}(3+2\alpha^2)(4\Lambda^2)\ . \quad (23)$$

We see that this trace anomaly depends on the gauge parameter $\alpha$. For example, in the Landau-DeWitt gauge, $\alpha=0$,

$$\langle T_\mu{}^\mu\rangle = \frac{1}{(4\pi)^2}(12\Lambda^2)\ , \quad (24)$$

and in the Feynman gauge, $\alpha=1$,

$$\langle T_\mu{}^\mu\rangle = \frac{1}{(4\pi)^2}(20\Lambda^2)\ . \quad (25)$$

This happens because the flat background is not a classical solution of Einstein gravity with a cosmological constant. The usual effective action for an off-shell background is in general dependent on what quantum gauge fixing is used. It is therefore, not surprising to see that the trace anomaly calculated from this effective action also depends on the choice of gauge fixing. In the next section we remedy this by adopting the gauge-independent VD effective action formalism, and define the unique trace anomaly for the off-shell case.

## III. GAUGE INDEPENDENCE OF THE VILKOVISKY-DEWITT TRACE ANOMALY

In this section we introduce necessary elements of the VD effective action formalism. We then calculate the trace anomaly for the flat space case considered in the previous section and show that it is indeed independent of gauge choice.

To establish notation we write the conventional one-loop effective action as

$$\Gamma_1[\phi] = S[\phi] - \frac{1}{2}{\rm Tr}\ \ln S_{,ij}[\phi]\ , \quad (26)$$

where $\phi$ is now a general background field which may not be a solution of the classical equation of motion and $S$ is defined in Eq. (1). Note that we have used a condensed notation where $i$ represents both discrete and continuous indices and $S_{,i}$ denotes a functional derivative. The VD effective action can be obtained simply be replacing the ordinary derivative with the covariant functional derivative:

$$S_{,ij}\to S_{;ij} = S_{,ij} - \Gamma^k_{ij}S_{,k}\ , \quad (27)$$

where $\Gamma^k_{ij}$ is the connection of the field space. For non-gauge theories, the connection can be constructed from the metric $G_{ij}$ of this field space. It is just the Christoffel connection

$$\left\{ {k \atop i\ j} \right\} = \frac{1}{2}G^{kl}(G_{li,j}+G_{lj,i}-G_{ij,l})\ . \quad (28)$$

The prescription for defining $G_{ij}$ has been given by Vilkovisky [6].

For gauge theories, the connection on the physical field space is the Christoffel connection modulo local gauge transformations. Let $Q^i_\alpha$ be the generator of the gauge symmetry:

$$\delta\phi^i = Q^i_\alpha\epsilon^\alpha\ , \quad (29)$$

where $\epsilon^\alpha$ are parameters for the transformations. Then

$$\gamma_{\alpha\beta} = G_{ij}Q^i_\alpha Q^j_\beta \quad (30)$$

is the metric on that part of the field space orthogonal to the physical directions. The connection $\Gamma^k_{ij}$ for the physical field space is given by

$$\Gamma^k_{ij} = \left\{ {k \atop i\ j} \right\} + T^k_{ij}\ , \quad (31)$$

where

$$T^k_{ij} = -2Q^k_{\alpha;(i}Q^\alpha_{j)} + Q^l_\sigma Q^k_{\rho;l}Q^\sigma_{(i}Q^\rho_{j)}\ , \quad (32)$$
$$Q^\alpha_i = G_{ij}\gamma^{\alpha\beta}Q^j_\beta\ . \quad (33)$$

The derivation of Eq. (31) can be found in [1]. Note that there is a factor of $\frac{1}{2}$ in the symmetrization. The covariant derivatives in Eq. (32) are with respect to the Christoffel connection. The VD one-loop effective action is now given by

$$\Gamma_{\rm VD}[\phi] = S[\phi] - \frac{1}{2}{\rm Tr}\ \ln\left[G^{li}\left(S_{,ij} - \left\{ {k \atop i\ j} \right\} S_{,k} - T^k_{ij}S_{,k}\right)\right] \ , \tag{34}$$

plus contributions from the ghost determinant. In this definition $\Gamma_{\rm VD}$ is a scalar on the physical field space. A change of gauge conditions corresponds to just a coordinate transformation of the physical field space and leaves $\Gamma_{\rm VD}$ invariant.

We now return to the problem in Sec. II of calculating the trace anomaly for gravitons in a flat background spacetime. First we evaluate the Christoffel symbols and $T^k_{ij}$ in Eq. (31). Following Vilkovisky [6], we take the metric for the field space of metrics as,

$$G_{g_{\mu\nu}(x)g_{\alpha\beta}(y)} = \frac{1}{4}\sqrt{g}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta})\delta^4(x-y) \ . \tag{35}$$

The Christoffel symbol [see Eq. (28)] is thus [16],

$$\left\{ {g_{\rho\sigma}(z) \atop g_{\mu\nu}(x)g_{\alpha\beta}(y)} \right\}\Big|_{\rm back} = -\frac{1}{2}\delta^4(x-z)\delta^4(y-z)\Big[\delta_{\alpha(\mu}\delta_{\nu)(\rho}\delta_{\sigma)\beta} + \delta_{\beta(\mu}\delta_{\nu)(\rho}\delta_{\sigma)\alpha} - \frac{1}{2}\delta_{\alpha\beta}\delta_{\mu(\rho}\delta_{\sigma)\nu}$$
$$-\frac{1}{2}\delta_{\mu\nu}\delta_{\alpha(\rho}\delta_{\sigma)\beta} - \frac{1}{4}\delta_{\rho\sigma}(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\nu}\delta_{\alpha\beta})\Big] \ , \tag{36}$$

where $|_{\rm back}$ means that the quantity is evaluated at the background value. Now, from Eq. (1),

$$S_{,g_{\mu\nu}(x)}\big|_{\rm back} = \Lambda\delta_{\mu\nu} \ , \tag{37}$$

and combining with Eq. (36), we have

$$\left\{ {g_{\rho\sigma}(z) \atop g_{\mu\nu}(x)g_{\alpha\beta}(y)} \right\} S_{,g_{\rho\sigma}(z)}\Big|_{\rm back} = 0 \ . \tag{38}$$

Therefore in the case of a flat background the only possible VD correction comes from the $T^k_{ij}$ term.

To evaluate $T^k_{ij}$, we need to know the generators of the gauge symmetry. For metric fields gauge symmetry is general coordinate covariance:

$$\delta g_{\mu\nu} = -g_{\mu\alpha}\partial_\nu\epsilon^\alpha - g_{\alpha\nu}\partial_\mu\epsilon^\alpha - \epsilon^\alpha\partial_\alpha g_{\mu\nu} \ , \tag{39}$$

for some set of gauge parameters $\epsilon^\alpha$. The generators are, thus,

$$Q^{g_{\mu\nu}(x)}_{\alpha y} = -g_{\mu\alpha}\partial_\nu\delta^4(x-y) - g_{\alpha\nu}\partial_\mu\delta^4(x-y) - \delta^4(x-y)\partial_\alpha g_{\mu\nu} \ . \tag{40}$$

The gauge-space metric in Eq. (30) is then [18]

$$\gamma_{\alpha x,\beta y}|_{\rm back} = -2\delta_{\alpha\beta}\partial^2\delta^4(x-y) \ , \tag{41}$$

and its inverse is

$$\gamma^{\alpha x,\beta y}|_{\rm back} = -\frac{1}{2}\delta_{\alpha\beta}\left(\frac{1}{\partial^2}\right)\delta^4(x-y) \ . \tag{42}$$

Using these results in Eqs. (31)–(33), we have

$$Q^{\alpha x}_{g_{\mu\nu}(y)}\Big|_{\rm back} = -\frac{1}{2}\left(\frac{1}{\partial^2}\right)(\delta_{\mu\alpha}\partial_\nu + \delta_{\nu\alpha}\partial_\mu - \delta_{\mu\nu}\partial_\alpha)\delta^4(x-y) \tag{43}$$

and

$$T^{g_{\rho\sigma}(z)}_{g_{\mu\nu}(x)g_{\alpha\beta}(y)}S_{,g_{\rho\sigma}(z)}\Big|_{\rm back} = -\frac{1}{2}\Lambda(\delta_{\mu\eta}\partial_\nu + \delta_{\nu\eta}\partial_\mu - \delta_{\mu\nu}\partial_\eta)\frac{1}{\partial^2}(\delta_{\alpha\eta}\partial_\beta + \delta_{\beta\eta}\partial_\alpha - \delta_{\alpha\beta}\partial_\eta)\delta^4(x-y) \ . \tag{44}$$

By defining

$$\mathcal{L}' \equiv -\frac{1}{2}h_{\mu\nu}(x)\ (T^{g_{\rho\sigma}(z)}_{g_{\mu\nu}(x)g_{\alpha\beta}(y)}S_{,g_{\rho\sigma}(z)})\Big|_{\rm back} h_{\alpha\beta}(y)$$
$$= -\Lambda(h_{\mu\rho,\mu} - \tfrac{1}{2}h_{,\rho})\frac{1}{\partial^2}(h_{\nu\rho,\nu} - \tfrac{1}{2}h_{,\rho}) \ , \tag{45}$$

the VD corrections can be accounted for by adding $\mathcal{L}'$ to $\mathcal{L}_q$ in Eq. (6),

$$\begin{aligned}\mathcal{L}_{\rm VD} &\equiv \mathcal{L}_q + \mathcal{L}' \\ &= \frac{1}{4}h_{\mu\nu}(-\partial^2 - 2\Lambda)h_{\mu\nu} - \frac{1}{4}h\left[\left(1-\frac{1}{2\alpha}\right)(-\partial^2) - 2\Lambda\right]h \\ &\quad +\frac{1}{2}h_{\mu\nu}\left[\left(\frac{1}{\alpha}-1\right)(-\partial^2)+2\Lambda\right]\frac{\partial_\nu\partial_\rho}{\partial^2}h_{\mu\rho} - \frac{1}{2}h\left[\left(\frac{1}{\alpha}-1\right)(-\partial^2)+2\Lambda\right]\frac{\partial_\mu\partial_\nu}{\partial^2}h_{\mu\nu} + \bar{\eta}_\mu(-\partial^2)\eta_\mu \ . \end{aligned} \tag{46}$$

The corresponding ten eigenvalues for $\mathcal{L}_{\rm VD}$, as compared to Eqs. (9)–(13), are

$$\lambda_1 = \lambda_2 = \lambda_3 = k^2 - 2\Lambda \ , \tag{47}$$

$$\lambda_4 = \lambda_5 = \frac{1}{2}(k^2 - 2\Lambda) \ , \tag{48}$$

$$\lambda_6 = \lambda_7 = \lambda_8 = \left(\frac{1}{\alpha}\right)k^2 \ , \tag{49}$$

$$\lambda_9 = \frac{1}{2\alpha}\left\{\left[k^2(1-\alpha)+2\alpha\Lambda\right] + \sqrt{(1-\alpha+\alpha^2)k^4 + 2\alpha\Lambda k^2(1-2\alpha)+4\alpha^2\Lambda^2}\right\} \ , \tag{50}$$

$$\lambda_{10} = \frac{1}{2\alpha}\left\{\left[k^2(1-\alpha)+2\alpha\Lambda\right] - \sqrt{(1-\alpha+\alpha^2)k^4 + 2\alpha\Lambda k^2(1-2\alpha)+4\alpha^2\Lambda^2}\right\} \ . \tag{51}$$

Following the same procedures as in Sec. II, we obtain the necessary $\zeta$-function values for the graviton field in the VD formalism as

$$\zeta_{\rm gr}^{\rm VD}(0) = \frac{V_4}{(4\pi)^2}(12\Lambda^2) \ , \tag{52}$$

and for the ghost field we again have

$$\zeta_{\rm gh}^{\rm VD}(0) = 0 \ . \tag{53}$$

Therefore, the trace anomaly in the VD formalism is given by

$$\langle T_\mu{}^\mu\rangle_{\rm VD} = \frac{1}{(4\pi)^2}(12\Lambda^2) \ , \tag{54}$$

which is independent of the gauge parameter $\alpha$. We have thus confirmed explicitly that the trace anomaly in the VD formalism with a flat background is gauge independent even though this background is not a classical solution of Einstein gravity (with cosmological constant).

Note that the usual trace anomaly in Eq. (23) will be the same as the one in the VD formalism in Eq. (54) if the gauge parameter $\alpha$ is set to zero (Landau-DeWitt gauge). This is because in the Landau-DeWitt gauge the nonlocal $T^k_{ij}$ terms vanish [19]. This can be easily seen in the case that we are considering. For the Landau-DeWitt gauge, we have basically

$$h^{;\mu}_{\mu\rho} - \frac{1}{2}h_{,\rho} = 0 \ , \tag{55}$$

and the VD correction due to the $T^k_{ij}$ term in Eq. (45) clearly vanishes. Moreover, the Christoffel symbol term does not contribute in our case [Eq. (38)]. The usual trace anomaly in the Landau-DeWitt gauge is thus identical to the VD trace anomaly.

In the next section we shall choose the Landau-DeWitt gauge to avoid evaluating the complicated nonlocal $T^k_{ij}$ terms when evaluating trace anomalies in more general spacetimes. Although the operators whose $\zeta$ functions we need are simplified in this gauge, they remain nonminimal. The $\zeta$ functions for nonminimal operators have been discussed in some detail in [15]. In the next section we shall make use of those results to calculate the trace anomaly.

## IV. VILKOVISKY-DEWITT TRACE ANOMALIES ON $N$-SPHERES

In this section we show how to calculate the trace anomaly in the VD formalism for a general background spacetime, and then we consider explicitly the case of even-dimensional $N$-spheres. As discussed in the last section, we shall adopt the Landau-DeWitt gauge so that we can avoid calculating the nonlocal $T^k_{ij}$ terms.

We now return to the Lagrangian in Eq. (1) and consider a general $N$-dimensional background spacetime with metric $g_{\mu\nu}$. Instead of the splitting in Eq. (2), we have

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \ . \tag{56}$$

The quadratic part of the Lagrangian becomes

$$\mathcal{L}_{\rm quad} = \frac{1}{2}h_{\mu\nu}\gamma^{\mu\nu,\rho\sigma}\left[\delta^{\alpha\beta}_{\rho\sigma}(-\Box) + 2\delta^{(\alpha}_{(\rho}\nabla_{\sigma)}\nabla^{\beta)} - g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)} - K_{\rho\sigma}{}^{\alpha\beta}\right]h_{\alpha\beta} \ , \tag{57}$$

where

$$\gamma^{\mu\nu,\rho\sigma} = \frac{1}{4}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}) \ , \tag{58}$$

$$\delta^{\alpha\beta}_{\rho\sigma} = \delta^{(\alpha}_{\rho}\delta^{\beta)}_{\sigma} \ , \tag{59}$$

$$K_{\rho\sigma}{}^{\alpha\beta} = 2R_{\rho}{}^{(\alpha}{}_{\sigma}{}^{\beta)} + 2\delta^{(\alpha}_{(\rho}R^{\beta)}_{\sigma)} - g^{\alpha\beta}R_{\rho\sigma} - \frac{2}{N-2}g_{\rho\sigma}R^{\alpha\beta} + \frac{1}{N-2}g_{\rho\sigma}g^{\alpha\beta}R - (R-2\Lambda)\delta^{\alpha\beta}_{\rho\sigma} \ . \tag{60}$$

The gauge-fixing part of the Lagrangian is

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\alpha}h_{\mu\nu}\gamma^{\mu\nu,\rho\sigma}\left[2\delta^{(\alpha}_{(\rho}\nabla_{\sigma)}\nabla^{\beta)} - g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)}\right]h_{\alpha\beta} \ , \tag{61}$$

and the corresponding ghost Lagrangian is

$$\mathcal{L}_{\rm gh} = \bar{\eta}_{\alpha}g^{\alpha\gamma}\left[\delta^{\beta}_{\gamma}(-\Box) - R_{\gamma}{}^{\beta}\right]\eta_{\beta} \ . \tag{62}$$

Combining Eqs. (57)–(62), we obtain the quantum Lagrangian in a general background spacetime:

$$\begin{aligned}\mathcal{L}_q = {} & \frac{1}{2}h_{\mu\nu}\gamma^{\mu\nu,\rho\sigma}\left[\delta^{\alpha\beta}_{\rho\sigma}(-\Box) + 2\left(1-\frac{1}{\alpha}\right)\delta^{(\alpha}_{(\rho}\nabla_{\sigma)}\nabla^{\beta)} - \left(1-\frac{1}{\alpha}\right)g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)} - K_{\rho\sigma}{}^{\alpha\beta}\right]h_{\alpha\beta} \\ & +\bar{\eta}_{\alpha}g^{\alpha\gamma}\left[\delta^{\beta}_{\gamma}(-\Box) - R_{\gamma}{}^{\beta}\right]\eta_{\beta} \ . \end{aligned} \tag{63}$$

To calculate the VD corrections, we need to first evaluate the connection symbols in Eq. (31). Because we adopt the Landau-DeWitt gauge, the nonlocal $T^k_{ij}$ terms will not contribute, and we only have to concentrate on the Christoffel symbols. For a general background spacetime [11],

$$\begin{aligned}\left\{ \begin{matrix} g_{\rho\sigma}(z) \\ g_{\mu\nu}(x)g_{\alpha\beta}(y) \end{matrix} \right\} = {} & \delta(x-y)\delta(y-z)\left[\frac{1}{4}g^{\mu\nu}\delta^{\alpha\beta}_{\rho\sigma} + \frac{1}{4}g^{\alpha\beta}\delta^{\mu\nu}_{\rho\sigma} - \frac{1}{2}\delta^{\mu}_{(\rho}\delta^{(\alpha}_{\sigma)}g^{\beta)\nu} \right. \\ & \left. -\frac{1}{2}\delta^{\nu}_{(\rho}\delta^{(\alpha}_{\sigma)}g^{\beta)\mu} + \frac{1}{N-2}g_{\rho\sigma}\gamma^{\mu\nu,\alpha\beta}\right] \ . \end{aligned} \tag{64}$$

From Eq. (1),

$$S_{,g_{\mu\nu}(x)} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} \ . \tag{65}$$

The VD correction terms can be accounted for by adding to $\mathcal{L}_q$ the Lagrangian

$$\begin{aligned}\mathcal{L}' &= -\frac{1}{2}h_{\mu\nu}(x)\left\{ \begin{matrix} g_{\rho\sigma}(z) \\ g_{\mu\nu}(x)g_{\alpha\beta}(y) \end{matrix} \right\} S_{,g_{\rho\sigma}(z)}h_{\alpha\beta}(y) \ , \\ &= \frac{1}{2}h_{\mu\nu}\gamma^{\mu\nu,\rho\sigma}\left[2\delta^{(\alpha}_{(\rho}R^{\beta)}_{\sigma)} - \frac{1}{2}g^{\alpha\beta}R_{\rho\sigma} - \frac{1}{N-2}g_{\rho\sigma}R^{\alpha\beta} - \frac{1}{N-2}(\delta^{\alpha\beta}_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}g^{\alpha\beta})R \right. \\ & \quad \left. -\frac{N-4}{2(N-2)}\delta^{\alpha\beta}_{\rho\sigma}(R-2\Lambda)\right]h_{\alpha\beta} \ . \end{aligned} \tag{66}$$

The VD Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\rm VD} &= \mathcal{L}_q + \mathcal{L}' \\ &= \frac{1}{2}h_{\mu\nu}\gamma^{\mu\nu,\rho\sigma}\left[\delta^{\alpha\beta}_{\rho\sigma}(-\Box) + 2\left(1-\frac{1}{\alpha}\right)\delta^{(\alpha}_{(\rho}\nabla_{\sigma)}\nabla^{\beta)} - \left(1-\frac{1}{\alpha}\right)g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)} - P_{\rho\sigma}{}^{\alpha\beta}\right]h_{\alpha\beta} \\ & \quad +\bar{\eta}_{\alpha}g^{\alpha\gamma}\left[\delta^{\beta}_{\gamma}(-\Box) - R_{\gamma}{}^{\beta}\right]\eta_{\beta} \ , \end{aligned} \tag{67}$$

where

$$P_{\rho\sigma}{}^{\alpha\beta} = 2R_{\rho}{}^{(\alpha}{}_{\sigma}{}^{\beta)} - \frac{1}{2}g^{\alpha\beta}R_{\rho\sigma} - \frac{1}{N-2}g_{\rho\sigma}R^{\alpha\beta} + \frac{1}{2(N-2)}g_{\rho\sigma}g^{\alpha\beta}R - \frac{1}{2}\left[R - \frac{2N}{N-2}\Lambda\right]\delta^{\alpha\beta}_{\rho\sigma} \ . \tag{68}$$

As in Eq. (16), the trace anomaly for gravitons in a general $N$-dimensional background can now be written as

$$\langle T_{\mu}{}^{\mu}\rangle^{N}_{\rm VD} = \frac{1}{V_N}\left[\zeta^N_{M_T}(0) - 2\zeta^N_{M_V}(0)\right]\Big|_{\alpha\to 0} \ , \tag{69}$$

where $M_T$ is the tensor operator for $h_{\mu\nu}$ in $\mathcal{L}_{\rm VD}$,

$$M_{T\rho\sigma}{}^{\alpha\beta} = \delta^{\alpha\beta}_{\rho\sigma}(-\Box) + 2\left(1-\frac{1}{\alpha}\right)\delta^{(\alpha}_{(\rho}\nabla_{\sigma)}\nabla^{\beta)} - \left(1-\frac{1}{\alpha}\right)g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)} - P_{\rho\sigma}{}^{\alpha\beta} \ , \tag{70}$$

and $M_V$ is the ghost operator,

$$M_{V\alpha}{}^{\beta} = \delta^{\beta}_{\alpha}(-\Box) - R_{\alpha}{}^{\beta} \ . \tag{71}$$

Since $M_T$ is a nonminimal operator, it is quite difficult to evaluate its $\zeta$ function. However, we have devised a method in [15] to accomplish this in the case of maximally symmetric background spacetimes. In particular, we have explicitly given the $\zeta(0)$ values for nonminimal tensor and vector operators on $N$-spheres of even dimensions 2–10. In the following we use these results and evaluate the VD trace anomalies for gravitons.

On $N$-spheres the Riemann tensor, the Ricci tensor, and the scalar curvature are given by

$$R_{\mu\nu\alpha\beta} = \frac{1}{r^2}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \ , \tag{72}$$

$$R_{\mu\nu} = \frac{1}{r^2}(N-1)g_{\mu\nu} \ , \tag{73}$$

$$R = \frac{1}{r^2}N(N-1) \ , \tag{74}$$

where $r$ is the radius of the sphere. The operator $M_T$ in Eq. (70) becomes

$$\begin{aligned} M_{T\rho\sigma}{}^{\alpha\beta} = {} & \delta^{\alpha\beta}_{\rho\sigma}\left[-\Box + \frac{1}{2r^2}(N^2-N+4) - \frac{N}{N-2}\Lambda\right] - \frac{2}{r^2}g^{\alpha\beta}g_{\rho\sigma} \\ & +2\left(1-\frac{1}{\alpha}\right)\delta^{(\alpha}_{(\rho}\nabla_{\sigma)}\nabla^{\beta)} - \left(1-\frac{1}{\alpha}\right)g^{\alpha\beta}\nabla_{(\rho}\nabla_{\sigma)} \ , \end{aligned} \tag{75}$$

and the operator $M_V$ in Eq. (71) becomes

$$M_{V\alpha}{}^{\beta} = \delta^{\beta}_{\alpha}\left(-\Box - \frac{N-1}{r^2}\right) \ . \tag{76}$$

Using the results in the Appendix of [15] and taking the Landau-DeWitt gauge ($\alpha \to 0$), the $\zeta$ functions for $M_T$ on $N$-spheres in Eq. (75) are

$$\zeta^4_{M_T}(0) = 2(\Lambda r^2)^2 - 16(\Lambda r^2) + \frac{299}{9} \ , \tag{77}$$

$$\zeta^6_{M_T}(0) = \frac{9}{64}(\Lambda r^2)^3 - \frac{63}{16}(\Lambda r^2)^2 + \frac{529}{16}(\Lambda r^2) - \frac{2509}{36} \ , \tag{78}$$

$$\zeta^8_{M_T}(0) = \frac{16}{3645}(\Lambda r^2)^4 - \frac{1088}{3645}(\Lambda r^2)^3 + \frac{584}{81}(\Lambda r^2)^2 - \frac{2000}{27}(\Lambda r^2) + \frac{45097}{150} \ , \tag{79}$$

$$\zeta^{10}_{M_T}(0) = \frac{625}{8257536}(\Lambda r^2)^5 - \frac{10625}{1032192}(\Lambda r^2)^4 + \frac{419875}{774144}(\Lambda r^2)^3 - \frac{49855}{3584}(\Lambda r^2)^2 + \frac{5628229}{32256}(\Lambda r^2) - \frac{2242392227}{2721600} \ , \tag{80}$$

while the $\zeta(0)$ values of minimal operator $M_V$ in Eq. (76) are

$$\zeta^4_{M_V}(0) = \frac{358}{45} \ , \tag{81}$$

$$\zeta^6_{M_V}(0) = \frac{4808}{315} \ , \tag{82}$$

$$\zeta^8_{M_V}(0) = \frac{347857}{14175} \ , \tag{83}$$

$$\zeta^{10}_{M_V}(0) = \frac{66840359}{1871100} \ . \tag{84}$$

Putting these back into Eq. (69), and using the fact that the volume of a $N$-sphere is

$$V_N = \frac{2\pi^{(N+1)/2}}{\Gamma((N+1)/2)}\, r^N \ , \tag{85}$$

we obtain the VD trace anomalies for gravitons on $N$-spheres:

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=4} = \frac{1}{(4\pi)^2}\left[12\Lambda^2 - 96\frac{\Lambda}{r^2} + \frac{1558}{15}\frac{1}{r^4}\right] , \tag{86}$$

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=6} = \frac{1}{(4\pi)^3}\left[\frac{135}{16}\Lambda^3 - \frac{945}{4}\frac{\Lambda^2}{r^2} + \frac{7935}{4}\frac{\Lambda}{r^4} - \frac{42093}{7}\frac{1}{r^6}\right] , \tag{87}$$

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=8} = \frac{1}{(4\pi)^4}\left[\frac{896}{243}\Lambda^4 - \frac{60928}{243}\frac{\Lambda^3}{r^2} + \frac{163520}{27}\frac{\Lambda^2}{r^4} - \frac{560000}{9}\frac{\Lambda}{r^6} + \frac{5705524}{27}\frac{1}{r^8}\right] , \tag{88}$$

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=10} = \frac{1}{(4\pi)^5}\left[\frac{9375}{8192}\Lambda^5 - \frac{159375}{1024}\frac{\Lambda^4}{r^2} + \frac{2099375}{256}\frac{\Lambda^3}{r^4} - \frac{6730425}{32}\frac{\Lambda^2}{r^6} + \frac{84423435}{32}\frac{\Lambda}{r^8} - \frac{5361041197}{396}\frac{1}{r^{10}}\right] . \tag{89}$$

We can also obtain the trace anomalies in Euclidean spaces by taking $r \to \infty$ in Eqs. (86)–(89). Hence, in Euclidean spaces,

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=4} = \frac{1}{(4\pi)^2}(12\Lambda^2) , \tag{90}$$

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=6} = \frac{1}{(4\pi)^3}\left(\frac{135}{16}\Lambda^3\right) , \tag{91}$$

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=8} = \frac{1}{(4\pi)^4}\left(\frac{896}{243}\Lambda^4\right) , \tag{92}$$

$$\langle T_\mu{}^\mu\rangle_{\rm VD}^{N=10} = \frac{1}{(4\pi)^5}\left(\frac{9375}{8192}\Lambda^5\right) . \tag{93}$$

Note that the $N = 4$ result here agrees with the one in Eq. (54).

## V. CONCLUSIONS

We have confirmed that the trace anomaly for gravitons is gauge dependent if the background spacetime is not a solution of the classical equations of motion. By using the VD effective action formalism the gauge dependency was eliminated and a unique off shell trace anomaly for gravitons was obtain. Explicit evaluation of this VD trace anomaly involved the evaluation of $\zeta$ functions of nonminimal operators. The necessary $\zeta(0)$ values on maximally symmetric background spacetimes were given in our previous paper [15]. Using results obtained there we were able to evaluate gravitational trace anomalies on $N$-spheres and Euclidean spaces (for even dimensions from 4 to 10). The 4D result of Eq. (86) can be confirmed by [19]. However, the 6D result of Eq. (87) does not agree with [11]. A erratum for that paper is being prepared.

It should be straightforward to extend this calculation to other maximally symmetric spaces, notably Kaluza-Klein spacetimes such as $M^4 \times S^{N_1} \times S^{N_2} \times \cdots$. This consideration is important in the discussion of the cancellation of trace anomalies between different species of particles [20] in these spacetimes. We hope to address this and related problems in a future publication.

Although the VD effective action is manifestly gauge independent, it possesses, as pointed out by Odintsov [21], an ambiguity with respect to the choice of the field space metric. In this paper, since we are working exclusively with Einstein gravity, we have chosen to stay with the field metric in Eq. (35). This particular field metric comes out quite naturally from the Einstein action, as derived by Vilkovisky [6].

The method described in this paper can also be applied to the evaluation of the Casimir energies or the one-loop effective potentials in Kaluza-Klein spacetimes. In [11], we were able to obtain the VD effective action for a general background spacetime using a method of Barvinsky and Vilkovisky [14]. However, because of the complexity of that calculation, it seems quite impossible to push the method to higher dimensions. On the other hand, the procedures in this paper are much more manageable and they can be implemented by computer code. There should be no major difficulty in extending them to dimensions higher than 10.

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